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"Impossibility of Speculation" Theorems with Noisy Information*

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The "impossibility of speculations" result implied by some models of information economics seems to follow directly from the strong assumptions concerning the information structure in the relevant models. This paper investigates whether the impossibility result hold when some "noise" is introduced into the information system in a "no trade" model. It is proved that when the degree of noise approaches zero, the noisy model converges to the impossibility result of the no-noise model. Yet it is shown that the introduction of some positive noise into the information system can disrupt the impossibility result in many applications. *Journal of Economic Literature* Classification Numbers: 024, 026. © 1995 Academic Press, Inc.

INTRODUCTION

Should we believe in the "impossibility of speculations"?

One stream in the existing theory of information economics implies that purely speculative trades are impossible, that zero sum gambles proposed to risk-averse Bayesian rational agents will frequently be rejected, and that agents who have agreed to participate in a zero sum gamble will eventually cancel the agreement if we allow them enough time to renegotiate.

The common feature of the different models generating the impossibility of speculations results is a highly stylized information structure. The impossibility results seem to follow directly from the extremely strong assumptions concerning the information in the relevant models.

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The basic intuition behind the impossibility results is that reasoning of the type "if she still wants a bet although she knows I want to bet even when I know that she wants to bet when I want to bet . . ." is strong enough to kill a speculation under the appropriate assumptions. This intuitive argument seems quite appealing, per se, but the strong impossibility results still seem to contradict reality in many applications. Zero sum gambles are actually quite prevalent, and one can easily imagine a situation in which players stick to a zero sum gamble even when each of them has many opportunities to withdraw.

The suggested conflict between the models' prediction and reality provides a motivation for analyzing the sensitivity of the impossibility result to the strong assumptions concerning the agents' information. This paper focuses on studying the impact of "noisy" information on a "no-trade" result presented by Milgrom and Stokey (1982). The main assumption used to derive this result is that once trade occurs, it is common knowledge among the participating agents that all of them expect some gain from trading. In this paper we extend Milgrom and Stokey's result for the case in which common knowledge is approximated by almost common knowledge. Section 1 presents formal definitions of common knowledge and two approximations of common knowledge: almost common knowledge and common p-beliefs. In Section 2 Milgrom and Stokey's theorem and the "impossibility of speculative trades" result are presented and discussed. Section 3 provides a counterexample demonstrating that speculations may be possible if we replace the common knowledge assumption with an almost common knowledge assumption. Section 4 presents an extension of the impossibility of speculative trades result to the case where common knowledge is replaced by common p-beliefs. We prove that, in ex-ante efficient markets, if an agreement to trade implies that it is commonly p-believed, with high enough probability p, that all agents expect some positive gain ε from trading, then trade is impossible. Still, we demonstrate that if p is not that high (relative to ε) then a priori possibility of speculative trades can be close to 1. Section 5 presents an application demonstrating the impossibility of gambling in the common p-beliefs' case.

All through the paper we use card(A) to denote the cardinality of a set A. The expected value of a random variable x will be denoted by a bold $\mathbf{E}(x)$.

1. Preliminaries: Common Knowledge and Almost Common Knowledge

1.1. *Common Knowledge*

Let I be a finite set of agents and let (Ω, Σ, μ) be a probability space, where Ω is the space of states, Σ is a σ-field of events, and μ is a probability function on Σ. For each $i \in I$, Π_i is a partition of Ω into measurable sets with positive probabilities. For every $\omega \in \Omega$, denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i is interpreted as the information partition of agent i, and $\Pi_i(\omega)$ is the set of all states which are indistinguishable to i when ω occurs. Denote by $\mathcal{F}_i$ the σ-field generated by Π_i.

We say that agent i knows event A at ω iff $\Pi_i(\omega) \subseteq A$.

Let $K_i(A)$ be the event "i knows A." That is, $K_i(A) = \{\omega \mid \Pi_i(\omega) \subseteq A\}$.

The following properties of the knowledge operators follow immediately from the definition:

(1) $\forall A \subseteq B \subseteq \Omega,\ K_i(A) \subseteq K_i(B)$

(2) $\forall A \subseteq \Omega,\ K_i(A) \subseteq A$

(3) $K_i(A) = A$ iff $A \in \mathcal{F}_i$.

We say that an event A is common knowledge at ω iff for every integer $n \geq 1$ and any sequence of agents $i_1, i_2, \ldots, i_n$, $\omega \in K_{i_1}K_{i_2} \cdots K_{i_n}(A)$. That is, when ω occurs, all the agents know that everybody knows that everybody knows ... that everybody knows A.

Using this natural definition of common knowledge we have to check an infinte number of conditions in order to identify common knowledge. In his famous work "Agreeing to disagree," Aumann (1976) presents an equivalent concise definition of common knowledge:

An event A is common knowledge at ω iff A includes that member of the meet (finest common coarsening) $\bigwedge_{i\in I}\Pi_i$ that contains ω. That is, A is common knowledge at ω iff there exists an event E such that $\omega \in E \subseteq A$ and $E \in \mathcal{F}_i\ \forall i \in I$.

Using the first, natural, definition given above, we can define the event "A is common knowledge":

$$\text{Let } E(A) = \bigcap_{n\geq 1} A^n, \qquad \text{where } A^0 = A \text{ and } A^n = \bigcap_{i\in I} K_i(A^{n-1}) \text{ for } n \geq 1.$$

It can easily be proved that A is common knowledge at ω iff $\omega \in E(A)$.

Alternatively, define an event E to be evident knowledge iff $E \subseteq K_i(E)\ \forall i \in I$. That is, once E occurs, everybody knows it. Monderer and Samet (1988) show that an event A is common knowledge at ω iff there exists an evident knowledge event E such that $\omega \in E$ and $E \subseteq K_i(A)\ \forall i \in I$.

Following this second definition of common knowledge, we can establish that once an agreement is reached by a group of agents the subject of the agreement is common knowledge among the agents. Even if the agents cannot directly validate the infinitely many conditions imposed by the first iterative definition because of "rationality limitations" (Lewis, 1969), signing the agreement, per se, creates common knowledge about the subject of agreement.

1.2. *Almost Common Knowledge*

A basic assumption in most studies of game theory is that the game played is common knowledge among the players. In his "Electronic Mail Game" Rubinstein (1989) presents an example in which the game played is almost common knowledge: everybody knows that everybody knows ... that everybody knows the game played, but the knowledge hierarchy is cut off after a finite number of iterations.

Specifically, Rubinstein considers a case of Bayesian games with two possible games; A and B. The players do not know exactly which game is actually played. A communication mechanism between the two players creates a state of almost common knowledge that the game played is game B. Rubinstein proves that the Nash equilibrium when B is almost common knowledge can be different from the Nash equilibrium played when B is common knowledge. Although everybody knows that everybody knows that ... the game played is game B, when the knowledge hierarchy is cut off after a finite number of iterations, the players will choose equilibrium strategies different from the equilibrium strategies they would have played if B were common knowledge.

Rubinstein concludes that a game with common knowledge is not a limit case of a game with almost common knowledge, and that "players' strategic behavior under almost common knowledge may be very different from that under common knowledge" (1989).

1.3. *Common p-Beliefs*

An alternative way of approximating common knowledge was suggested by Monderer and Samet (1988). Instead of an infinite hierarchy of knowledge, Monderer and Samet suggest an infinite hierarchy of beliefs in probability p—common p-beliefs.

Rubinstein's example and the "common p-beliefs" model are two natural approximations to the common knowledge case. While Rubinstein truncates the knowledge hierarchy after a large, but finite, number of iterations, Monderer and Samet leave intact the infinite hierarchy, but change "knowledge" to "belief in probability p." Thus, the common p-belief model considers the "common *almost-knowledge*" case, while the electronic mail game example describes the "*almost-common* knowledge" situation.

As before, (Ω, Σ, μ) is a probability space and I is a finite set of agents whose available information is given by partitions Π_i of Ω into measurable sets with positive probabilities.

Let $0 \leq p \leq 1, \quad i \in I$.

Say that agent i p-believes an event A at ω iff $\mu(A \mid \Pi_i(\omega)) \geq p$.

Let $B_i^p(A)$ be the event "i p-believes A." That is,

$$B_i^p(A) = \{\omega \in \Omega \mid \mu(A \mid \Pi_i(\omega)) \geq p\}.$$

B_i^p will be referred to henceforth as the "belief operator." The following properties of B_i^p follow immediately from the definition:

For every $0 < p \leq 1$, $i \in I$, and $A, B \in \Sigma$,

(1) $B_i^p(A) \in \mathcal{F}_i$
(2) if $A \in \mathcal{F}_i$ then $B_i^p(A) = A$
(3) $B_i^p(B_i^p(A)) = B_i^p(A)$
(4) if $A \subseteq B$ then $B_i^p(A) \subseteq B_i^p(B)$
(5) $\mu(A \mid B_i^p(A)) \geq p$.

Say that an event E is "evident p-belief" iff $E \subseteq B_i^p(E)$ $\forall i \in I$. That is, once the event occurs, everybody p-believes it occurred.

Following the two basic definitions of common knowledge presented above, Monderer and Samet introduce two equivalent definitions of common p-beliefs:

(1) THE ITERATIVE DEFINITION. Let $E^p(A) = \bigcap_{n \geq 1} A^n$ where $A^0 = A$ and $A^n = \bigcap_{i \in I} B_i^p(A^{n-1})$ for $n \geq 1$. A is common p-believed at $\omega \in \Omega$ iff $\omega \in E^p(A)$.

(2) THE "EVIDENT p-BELIEF" DEFINITION. A is a common p-belief at $\omega \in \Omega$ iff there exists an evident p-belief event E such that $\omega \in E$ and $E \subseteq B_i^p(A)$ $\forall i \in I$.

Monderer and Samet prove that in the case of Bayesian games, when the game played is commonly p-believed by the agents in a subset of Ω with measure δ, when p and δ approach 1—that is, when the game played is commonly p-believed by the agents in high probability p, in a close to 1 set of measure δ—then (for any selection of Nash equilibria in the complete information games) there exist equilibrium strategies in the incomplete information game that are very close to the equilibrium strategies that would have been played if the games played were common knowledge in every state of nature. Moreover, given these strategies, the expected payments to the players in each state of nature (ex-post) approach the payments the players would have received if the games played were common knowledge. In this sense, the results obtained under common p-beliefs approximate the results obtained under common knowledge when p approaches 1.

2. Impossibility of Speculative Trades—The Common Knowledge Case

Consider a general market environment as follows:

Ω is a set of possible states of nature,

(Ω, Σ, μ) a probability space,

$L = \{1, 2, \ldots, l\}$ a finite set of commodities, and

$I = \{1, 2, \ldots, n\}$ a finite set of agents whose available information is given by partitions Π_i of Ω into measurable sets with positive probabilities.

As before, let $\mathcal{F}_i$ be the σ-field generated by Π_i, and denote by R the meet of $\Pi_1, \Pi_2, \ldots, \Pi_n$. That is,

$$R = \Pi_1 \wedge \Pi_2 \wedge \cdots \wedge \Pi_n.$$

Let

$e^i: \Omega \to \mathfrak{R}^l_+$ be the state contingent initial endowment of agent i,

$e = (e^1, e^2, \ldots, e^n)$ the initial allocation in the market, and

$x^i: \Omega \to \mathfrak{R}^l_+$ a state-contingent consumption bundle proposed to agent i.

Say that $x = (x^1, x^2, \ldots, x^n)$ is a feasible allocation (given the initial allocation e) iff $\Sigma_{i \in I} x^i(\omega) \le \Sigma_{i \in I} e^i(\omega)\ \forall \omega \in \Omega$. Note that each feasible allocation defines a feasible trade $t = (t^1, \ldots, t^n)$, where $t^i: \Omega \to \mathfrak{R}^l$ is defined by $t^i(\omega) = e^i(\omega) - x^i(\omega)\ \forall \omega \in \Omega$, and

(1) $\Sigma_{i \in I} t^i(\omega) \ge 0\ \forall \omega \in \Omega$.

(2) $t^i(\omega) \le e^i(\omega)\ \forall i \in I, \omega \in \Omega$.

For each $i \in I$ and every state of nature $\omega \in \Omega$, let $U^i_\omega: \mathfrak{R}^l_+ \to \mathfrak{R}$ be the utility function of agent i at ω. All through the paper we will assume that, for every agent i, x^i, e^i, and $U^i(x^i)$ are integrable functions.

Denote by $V^i(x^i)$ the ex-ante expected utility of agent i from allocation x^i. That is, $V^i(x^i) = \mathbf{E}[U^i_\omega(x^i(\omega))]$. Let $\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)]$ be the ex-post expected utility of agent i from allocation x^i at ω. Note that since $\mu(\Pi_i(\omega)) > 0\ \forall i \in I, \forall \omega \in \Omega$, conditional expectations are well defined.

Say that the initial allocation e is ex-ante Pareto optimal iff there is no other feasible allocation x such that

(1) $V^i(x^i) \ge V^i(e^i)\ \forall i \in I$, and

(2) $\exists j \in I: V^j(x^j) > V^j(e^j)$.

We say that agent i weakly prefers x^i to e^i at ω and write $x^i \succeq_\omega e^i$ iff

$$\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] \geq \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)].$$

We say that the agents prefer x to e at ω and write $x >_\omega e$ iff

(1) $\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] \geq \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)]\ \forall i \in I$, and

(2) $\exists j \in I$ s.t. $\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] > \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)]$.

THEOREM (Milgrom and Stokey, 1982). *If e is ex-ante Pareto optimal and it is common knowledge at ω that the agents weakly prefer a feasible allocation x to the initial allocation e, then:*

(1) *The agents are indifferent between x and e. That is,*

$$\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] = \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)] \qquad \forall i \in I.$$

(2) *If all utility functions are strictly concave then $x = e$ and $t = 0$.*

Milgrom and Stokey's theorem is often interpreted as referring to the "impossibility of speculative trades" (Rubinstein and Wolinsky, 1988). Since this result is the main subject of the three following sections we will present this result formally:

COROLLARY: Impossibility of Speculative Trades. *Let e be an ex-ante Pareto optimal allocation. For every feasible allocation x,*

$$\{\omega \in \Omega \mid \textit{it is common knowledge at } \omega \textit{ that } x > e\} = \varnothing.$$

Milgrom and Stokey's central result is that in ex-ante Pareto optimal markets the receipt of new information cannot create any incentive to trade.

The ex-ante Pareto optimality condition can be looked upon as an initial market efficiency assumption. Ex-ante Pareto optimality is immediate if, for example, the initial allocation is the outcome of trading in complete competitive markets.

Once we accept the ex-ante efficiency assumption, the only reason for trade is the asymmetric information of the different agents. Thus, trades in an ex-ante efficient market can be referred to as "speculative trades" or gambles. Now, say that a trade is acceptable if it is (weakly) preferred by the agents. We can claim that the occurrence of trade, per se, implies that it is common knowledge that the trade is acceptable. Milgrom and Stokey used rational expectations to establish this claim: in rational expectations economies, it is assumed that each agent infers whatever information he can from the market variables he observes. Furthermore, each agent believes that all other agents make full use of the information available to them. Thus, at an equilibrium of any voluntary trade process, each agent knows the trade is acceptable, each agent knows that the other agents know that the trade is acceptable, each agent knows that the others know that the others know that the trade is acceptable, etc. In other words, in a rational expectations economy, existence of voluntary trade immediately implies that it is common knowledge that the trade is acceptable. Alternatively, we can use the second definition of common knowledge as presented in Section 1.1 to derive the same observation. Assume that trade is an evident knowledge event: once trade occurs everybody knows about it. Assume also that once trade occurs everybody knows it is acceptable. These two assumptions establish the "common knowledge of trade's acceptability" result immediately, bypassing the infinite number of knowledge iterations derived from the rational expectations assumption.

Now, if we adopt the stronger definition of trade's acceptability, requiring that at least one agent must expect to strictly improve his ex-post utility as a result of trading, we conclude that speculative trades are impossible in ex-ante efficient markets.

3. A Counterexample for the Almost Common Knowledge Case

Following Rubinstein's example (1989) closely, we describe a two-agent gambling game as follows: Agent 1 drops a fair coin. If the coin shows "heads" the agent sends a signal to agent 2 through an electronic system. The system's technical design allows some small probability $\varepsilon > 0$ that a message transmitted will not reach its destination. If agent 2 receives the message he sends a confirmation message to agent 1, agent 1 confirms the confirmation, etc. The probability of the system's failure ε is fixed for each of the transmissions. Once the communication system fails, a screen displays to each of the agents separately the number of messages he sent until failure. Each agent then has to decide whether he accepts the following bet: If the total number of messages sent until failure is odd (1, 3, 5, . . .), agent 2 pays x dollars to agent 1. Otherwise agent 1 pays x dollars to agent 2. If one of the agents refuses the bet, the payoffs to both agents are 0.

This gambling situation illustrates that Milgrom and Stokey's theorem does not hold if we replace "common knowledge" with "almost common knowledge." Let ω be the total number of messages sent until failure, $\Omega = \{0, 1, 2, \ldots\}$. Clearly, the information partition for agent 1 consists of the elements $\{0\}$ and $\{2n - 1, 2n\}$ for $n = 1, 2, \ldots$, while the information partition for agent 2 is of the form $\{2n, 2n + 1\}$ for $n = 0, 1, \ldots$. Given the information $\omega \in \{2n - 1, 2n\}$, agent 1 assigns probability $\varepsilon/[\varepsilon + (1 - \varepsilon)\varepsilon] > 0.5$ to $\omega = 2n - 1$, and since agent 1 wins when ω is

odd he will expect a gain from betting whenever $\omega \geq 1$. Similar calculations show that the second agent will expect a gain from betting in every ω. Thus if we denote by A the event "both agents expect (ex-post) a gain from the proposed bet," then $A = \{1, 2, \ldots\}$. Now note that if $\omega = 4$ then $\omega \in K_1(A)$, $\omega \in K_2(A)$, $\omega \in K_2K_1(A)$, $\omega \in K_1K_2(A)$, $\omega \in K_1K_2K_1(A)$, $\omega \in K_2K_1K_2(A)$, and $\omega \in K_2K_1K_2K_1(A)$, but $\omega \notin K_1K_2K_1K_2(A)$, and if $\omega = 7$ then $\omega \in K_1(A)$, $\omega \in K_2(A), \ldots,$
$\omega \in K_1K_2K_1K_2K_1K_2K_1(A)$, but $\omega \notin K_2K_1K_2K_1K_2K_1K_2(A)$.

Thus, as ω increases, we climb higher and higher in the knowledge hierarchy about the strict ex-post attractiveness of the proposed bet to both agents. Still, no matter how big ω is, although both agents know that both know ... that they both know that the proposed bet is ex-post attractive to them both, each of the agents will still be ex-post expecting some positive gain from betting. In this sense, the results under almost-common knowledge do not approximate Milgrom and Stokey's common knowledge results: Speculations may seem ex-post profitable even if it is almost common knowledge that the agents strictly prefer the after-trade allocation.

Now, if we assume that each agent accepts a bet iff his ex-post expected payoff is positive, then we may also conclude that the impossibility corollary is disrupted if we replace the common knowledge assumption with almost common knowledge. Yet one should be careful with this further interpretation: if we analyze the gamble as a game with incomplete information, then it is easy to verify that, in every Nash equilibrium in which agent 1 rejects the bet when the coin shows "tails" (i.e., when $\omega = 0$), at least one of the agents refuses to bet in every state of nature. The proof is by induction: If $\omega = 1$ then agent 2 knows that $\omega \in \{0, 1\}$. Since agent 1 rejects the bet at $\omega = 0$, agent 2 must reject the bet if agent 1 accepts the bet at $\omega = 1$ with a positive probability. Thus, it must hold that either agent 2 rejects the bet given the information $\omega \in \{0, 1\}$, or agent 1 rejects the bet given the information $\omega \in \{1, 2\}$, and one of the agents will refuse the bet at $\omega = 1$. Now assume that for $\omega = n - 1$ one of the agents refuses to bet. Consider the case where n is odd. Then it is either the case that it is agent 2 who refuses to bet when $\omega = n - 1$, in which case he will also refuse to bet at $\omega = n$ (which is indistinguishable to that player from the case $\omega = n - 1$), or it is the case that agent 1 refuses to bet given the information $\omega \in \{n - 2, n - 1\}$. But in this last case, if agent 1 accepts the bet with a positive probability when $\omega \in \{n, n + 1\}$, then agent 2 will refuse the bet when $\omega \in \{n - 1, n\}$. Thus, one of the agents will refuse the bet at $\omega = n$. The proof for an even n is similar.

4. IMPOSSIBILITY OF SPECULATIVE TRADES— THE COMMON BELIEFS CASE

Recall that I is a set of agents, e is an initial efficient allocation, and x is some (fixed) feasible allocation.

We say that agent i weakly ε prefers x^i to e^i at ω and denote $x^i \succeq_{\omega,\varepsilon} e^i$ iff

$$\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] \geq \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)] + \varepsilon.$$

We say that the agents ε prefer x to e at ω and write $x \succ_{\omega,\varepsilon} e$ iff

(1) $\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] \gtrsim \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)] + \varepsilon \; \forall i \in I$ and

(2) $\exists j \in I$ s.t. $\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] > \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)] + \varepsilon$.

Define $\overline{M} = \max_{i \in I} \operatorname{esssup}_\mu (U^i_\omega(x^i(\omega)) - U^i_\omega(e^i(\omega))$ (where $\operatorname{esssup}_\mu$ denotes the μ-essential supremum), and assume that $\overline{M} < \infty$.

THEOREM 1. *Let e be an ex-ante Pareto optimal allocation, let x be a feasible allocation, and let* $0.5 < p \leq 1$. *For every* $\varepsilon \geq (1 - p)\overline{M}$,

$$\{\omega \mid \textit{it is common p-believed at } \omega \textit{ that } x \succ_{\omega,\varepsilon} e\} = \varnothing.$$

Proof. Let $A(\varepsilon)$ be the event "the agents ε prefer x to e." That is, $A(\varepsilon) = \{\omega \mid x \succ_{\omega,\varepsilon} e\}$.

For each $i \in I$, let $A_i(\varepsilon) = \{\omega \mid \Pi_i(\omega) \cap A(\varepsilon) \neq \varnothing\}$. By definition,

$$A_i(\varepsilon) \text{ is } \Pi_i\text{-measurable} \qquad \text{and} \qquad \forall \omega \in A_i(\varepsilon),\ x^i \succeq_{\omega,\varepsilon} e^i.$$

Recall that $\mathbf{E}^p[A(\varepsilon)]$ is the event "it is common p-belief that the agents ε prefer x to e." To simplify notation we write henceforth $E = E^p[A(\varepsilon)]$ and $A = A(\varepsilon)$.

Assume that $E \neq \varnothing$.

Monderer and Samet (1988, proposition 3) prove that

$$(1) \quad E \subseteq B^p_i(A)\, \forall i \in I, \qquad \text{and} \qquad E \subseteq B^p_i(E)\, \forall i \in I.$$

Thus, $\forall \omega \in E$, $\mu(E \mid \Pi_i(\omega)) \geq p > 0$, so $\mu(E \cap \Pi_i(\omega)) > 0$ and $\mu(E) > 0$. (Recall that we assume $\mu(\Pi_i(\omega)) > 0$ for every agent i, for every $\omega \in \Omega$.)

Now, using the belief operator's properties:

For each $i \in I$ and every $\omega \in B^p_i(A)$,

$$\mu(A \mid \Pi_i(\omega)) \geq p > 0, \qquad \text{so } A \cap \Pi_i(\omega) \neq \emptyset \text{ and } \omega \in A_i.$$

Thus, $B_i^p(A) \subseteq A_i \ \forall i \in I$, and from (1) $E \subseteq \bigcap_{i \in I} A_i$. So,

$$\text{(2)} \quad \forall \omega \in E, \forall i \in I, \qquad x^i \geq_{\omega,\varepsilon} e^i.$$

By properties of the belief operator, for every $\omega \in E \subseteq B_i^p(E)$, $\mu(\overline{E} \mid \Pi_i(\omega)) \leq 1 - p$. Thus, $\forall i \in I$, for every $\omega \in E$ such that $\mu(\Pi_i(\omega) \backslash E) > 0$,

$$\text{(3)} \quad [\mathbf{E}[U^i(x^i) \mid (\Pi_i(\omega) \backslash E)] - \mathbf{E}[U^i(e^i) \mid (\Pi_i(\omega) \backslash E)]] \times \frac{\mu(\Pi_i(\omega) \backslash E)}{\mu(\Pi_i(\omega))} \leq (1 - p)\overline{M}.$$

But from (2), $\forall i \in I$,

$$\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] - \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)] \geq \varepsilon \geq (1 - p)\overline{M}.$$

So with (3),

$$\text{(4)} \quad \mathbf{E}[U^i(x^i) \mid \Pi_i(\omega) \cap E] \geq \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega) \cap E] \qquad \forall i \in I, \omega \in E,$$

and

$$\text{(5)} \quad \mathbf{E}[U^i(x^i) \mid E] \geq \mathbf{E}[U^i(e^i) \mid E] \qquad \forall i \in I.$$

Finally, note that if $\mu(A \cap E) = 0$, then for every $\omega \in E \subseteq B_i^p(A)$,

$$\mu(A \mid \Pi_i(\omega)) \leq \mu(\overline{E} \mid \Pi_i(\omega)) \leq 1 - p < 0.5,$$

a contradiction.

Thus, $\mu(A \cap E) > 0$, $A \cap E \neq \emptyset$, and by the definition of A, $\exists j \in I$ and $\exists \omega \in E$ s.t. inequality (4) holds strictly for j at ω, and inequality (5) holds strictly for j—a contradiction to the Pareto optimality of e. ∎

$*$When $p \leq 0.5$ it is possible that $A \cap E = \emptyset$ and the theorem, as presented, does not hold. Still, we can slightly change the theorem in one of the following ways:

(1) assume that $\varepsilon > (1 - p)\overline{M}$, or

(2) say that the agents strongly ε prefer x to e at ω iff

$$\mathbf{E}[U^i(X^i) \mid \Pi_i(\omega)] > \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)] + \varepsilon \qquad \forall i \in I,$$

and replace "ε preference" in the theorem as presented above with "strong ε preference."

From the proof above it is obvious that each of the modified theorms holds for every $0 < p \leq 1$. These modified theorems immediately imply that if $\varepsilon > 0$ and $p > p^0 = 1 - (\varepsilon/\overline{M})$, then it cannot be p-believed that the agents ε prefer x to e. (Note that $p^0 \in (0, 1)$ since $\varepsilon < \overline{M}$.) From Theorem 1 we also have that, for $\varepsilon = 0$,

$$\{\omega \mid \text{it is common 1-belief at } \omega \text{ that } x >_0 e\} = \emptyset.$$

Thus we get the following alternative presentation of the theorem:

THEOREM 1 (Alternative Presentation). *Let e be an ex-ante Pareto optimal allocation, and let x be a feasible allocation. For every* $\varepsilon > 0$, $\exists p^0 \in (0, 1)$ *such that for every* $p > p^0$,

$$\{\omega \mid \textit{it is common p-believed at } \omega \textit{ that } x >_\varepsilon e\} = \emptyset.$$

Moreover,

$$\{\omega \mid \textit{it is common } 1 \textit{ believed at } \omega \textit{ that } x >_0 e\} = \emptyset.$$

Theorem 1 can be interpreted as a generalization of Milgrom and Stokey's model. We introduce two basic changes in the market of the original theorem. First, we presume the agents "bother" to trade only if they expect some gain from trading. Second, we allow some "noise" in the information system and replace common knowledge with common p-beliefs. If the "noise" in the market is not too strong (relative to $\varepsilon/\overline{M}$), then "impossibility of speculative trades" holds in the modified market. The quotient $\varepsilon/\overline{M}$ can be loosely interpreted as a measure of the "intensity" of a speculation: when ε is much smaller then $\overline{M}$, the speculation involves a higher degree of uncertainty. Thus, Theorem 1 implies that "intense" speculations are more sensitive to noisy information in the sense that a higher degree of common p-beliefs (in the ε-acceptability of the speculation) is required to sustain the impossibility result.

As p approaches 1, the result of Theorem 1 approaches the impossibility of speculative trades result presented in Section 2. The result under common 1-beliefs is identical to the result under common knowledge. Yet the introduction of the ε noise into the model enables us to significantly restrict the domain of speculative trades situations for which the impossibility result is valid. The following two examples demonstrate that when the p, ε condition of Theorem 1 is not satisfied: (1) the a priori probability of speculative trades, the acceptability of which is commonly p-believed with p close to 1, can be close to 1 as well; (2) both agents may find the bet very attractive even when the acceptability of the trade is commonly p-believed with probability p close to 1.

EXAMPLE 1. Consider the following market:

$\Omega = \{a, b, c\}$;

$\mu(a) = \mu(c) = (1 - p)/(2 - p)$, $\mu(b) = p/(2 - p)$ for some $p \in (0, 1)$;

$\text{card}(L) = 1$, $I = \{1, 2\}$, $\Pi_1 = \{(a, b), (c)\}$, $\Pi_2 = \{(a), (b, c)\}$;

$U^i_\omega(x^i) = x^i$, $\forall i, \omega$, $e^i(\omega) = 1$, $\forall i, \omega$.

Note that Pareto optimality holds immediately in this case since the utility functions are linear and $\text{card}(L) = 1$, and consider the following feasible allocation:

$$x^1(a) = 2, x^1(b) = 1, x^1(c) = 0, \qquad x^2(a) = 0, x^2(b) = 1, x^2(c) = 2.$$

Clearly, for each $p \in (0, 1)$ it is common p-believed at $\omega = b$ that x improves both agents' ex-post utility by exactly $(1 - p)\overline{M}$. (But note that x is not $(1 - p)\overline{M}$ preferred by the agents.) Thus, as $p \to 1$, probability (it is commonly p-believed that both agents improve utility by exactly $(1 - p)\overline{M} = p/(2 - p) \to 1$.

Recall that for every $0.5 < p \leq 1$, if we slightly change the example presented and examine the case where it is commonly p-believed that one of the agents strictly improves his ex-post utility by $(1 - p)\overline{M}$ and the other agent improves his utility by exactly $(1 - p)\overline{M}$, then, from Theorem 1, probability(it is commonly p-believed that the agents $(1 - p)\overline{M}$ prefer x to e) $= 0$. Thus, this example demonstrates some discontinuity in the possibility of speculative trades under common p-beliefs: Fix $0.5 < p \leq 1$ and assume that (for every ε considered) once trade occurs the agents common p-believe it is ε-preferred. Then, if $\varepsilon \geq (1 - p)\overline{M}$ speculative trade is impossible, whereas if $\varepsilon < (1 - p)\overline{M}$ the a priori probability of speculative trade can be close to 1.

EXAMPLE 2. Let

$\Omega = \{a, b, c, d\}$, $I = \{1, 2\}$, $\text{card}(L) = 1$;

$\mu(a) = k(1 - p)/(1 + k)$, $\mu(b) = kp/(1 + k)$, $\mu(c) = p/(1 + k)$, and $\mu(d) = (1 - p)/(1 + k)$ where $k > 2p/(1 - p)$;

$\Pi_1 = \{(a, b), (c, d)\}$, $\Pi_2 = \{(a), (b, c), (d)\}$;
$U^i_\omega(x^i) = x^i \; \forall i, \omega$, $e^1(\omega) = 1 \; \forall \omega$, and $e^2(\omega) = 2p/(1 - p) \; \forall \omega$.

Consider the following feasible allocation x: if ω is a, c, or d, agent 2 gives his initial endowment to agent 1. If ω is b then agent 1 transfers his initial endownment to agent 2. Simple calcualtions show that, for every $0 < p < 1$, it is commonly p-believed at states b and c that the agents prefer x to e (to settle this result with Theorem 1 note that at $\omega = b$ agent 1 improves his ex-post utility by p, while $(1 - p)\overline{M} = 2p$) and that $\mu(b)/(\mu(b) + \mu(c)) \to 1$ as $p \to 1$.

Choose p close to 1 to obtain that at $\omega = c$ it is commonly p-believed, in very high probability, that both agents prefer x to e. Still, each of the agents expects to gain almost all of the other agents' initial endowment. Thus, speculative trades may seem very tempting to all agents, although they commonly p-believe in the trade's acceptability in a close to 1 probability. Recall that when the acceptability of some speculative trade is common knowledge, the agents cannot improve their ex-post utilities by trading. In this aspect the results under common p-beliefs can be very different from the results under common knowledge.

The only general result we get for the case where the acceptability of a speculative trade is commonly p-believed at some ω is that by choosing p close enough to 1, we can push the expected gain for one of the agents as close to 0 as we wish:

PROPOSITION 2. *Let e be an ex-ante Pareto optimal allocation, and let x be a feasible allocation.*

For every $\varepsilon > 0$, $\exists p^0$, $0 < p^0 < 1$, such that for every $p \geq p^0$, if it is commonly p-believed at ω that the agents weakly prefer x to e, then $\exists j \in I$ s.t.

$$\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] - \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)] \leq \varepsilon.$$

Moreover, if it is commonly 1-believed at ω that the agents weakly prefer x to e, than $\forall i \in I$,

$$\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] = \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega)].$$

Proof. Let $A = \{\omega \mid x^i \geq_\omega e^i \; \forall i \in I\}$, $0 < p \leq 1$, and let E be the event "it is commonly p-believed that the agents weakly prefer x to e."

We first prove that if $\omega \in E$, then $\exists j \in I$ s.t.

$$\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] - \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)] \leq (1 - p)\frac{\mu(B^p_j(E))}{\mu(\Pi_j(\omega))}\overline{M}.$$

Using the same arguments as in the first part of the Proof of Theorem 1, it is easy to verify that $E \subseteq A$ so that $\Pi_i(\omega') \cap E \neq \emptyset \Rightarrow x^i \succeq_{\omega'} e^i$.

Note that $\forall i \in I$, $\forall \omega \in B_i^p(E)$,

$$\mu(E \mid \Pi_i(\omega)) \geq p > 0, \qquad \text{so } E \cap \Pi_i(\omega) \neq \emptyset \text{ and } x^i \succeq_\omega e^i.$$

Recall (from the Proof of Theorem 1) that if $E \neq \emptyset$ then $\mu(E) > 0$. Thus, from the Pareto optimality of e, $\exists j \in I$ s.t. $\mathbf{E}[U^j(x^j) \mid E] - \mathbf{E}[U^j(e^j) \mid E] \leq 0$.

Recall also from the properties of the belief operator that $\mu((B_j^p(E) \backslash E) \mid B_j^p(E)) \leq 1 - p$, so

$$\mathbf{E}[U^j(x^j) \mid B_j^p(E)] - \mathbf{E}[U^j(e^j) \mid B_j^p(E)] \leq (1 - p)\overline{M},$$

and since $\forall \omega \in B_j^p(E)$, $x^j \succeq_\omega e^j$,

$$[\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] - \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)]] \frac{\mu(\Pi_j(\omega))}{\mu(B_j^p(E))} \leq (1 - p)\overline{M}$$

and

$$\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] - \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)] \leq (1 - p) \frac{\mu(B_j^p(E))}{\mu(\Pi_j(\omega))} \overline{M},$$

which proves the above claim.

Now let $\delta = \min_{i \in I}\{\mu(\Pi_i(\omega)\}$. By the assumptions $\delta > 0$.

For any $\varepsilon > 0$ let

$$p^0(\varepsilon) = 1 - \frac{\delta}{\overline{M}} \varepsilon.$$

Obviously $0 < p^0 < 1$ and $\forall p \geq p^0$,

$$\mathbf{E}[U^j(x^j) \mid \Pi_j(\omega)] - \mathbf{E}[U^j(e^j) \mid \Pi_j(\omega)] \leq (1 - p) \frac{\mu(B_j^p(E))}{\mu(\Pi_j(\omega))} \overline{M} \leq \varepsilon,$$

which proves the first part of the proposition.

Now note that for $p = 1$, $\forall \omega \in E$, $\mu(E \mid \Pi_i(\omega)) = 1$.

Thus, $x^i \succeq_\omega e^i$ $\forall i \in I$ implies that

$$\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega) \cap E] - \mathbf{E}[U^i(e^i) \mid \Pi_i(\omega) \cap E] \geq 0 \qquad \forall i \in I, \omega \in E.$$

The ex-ante optimality of e implies that all these equalities must hold with equality, and the second part of the proposition follows. ■

5. An Application: Impossibility of Gambling

As discussed in Section 2, speculative trades are essentially a specific type of gambling. The only problem which arises if we try to derive the impossibility result for more general gambling models is that the ex-ante efficiency assumption seems inappropriate in some of these models: initial allocations might be inefficient, especially when the agents have concave utility functions or when the cardinality of the consumption/betting set is greater than 1. The following example describes a specific gambling environment in which the initial allocations must be efficient. Proposition 3 proves an impossibility of betting result for this specific model.

Consider the following gambling model:

Assume that Ω is finite, $L = \{\text{money}\}$, and for each $i \in I$ and every $\omega \in \Omega$, $U^i_\omega : \Re \to \Re$ is continuous, strictly concave, and monotonically increasing. Assume U^i_ω is Π_i-measurable. Assume also that the initial wealth of each agent $e^i : \Omega \to \Re$ is Π_i-measurable, and that x is a feasible gamble, i.e., $\sum_{i \in I} x^i(\omega) \leq \sum_{i \in I} e^i(\omega)\ \forall \omega \in \Omega$.

We adopt a stronger definition of attractiveness than the one used before and say that a gamble is attractive at ω iff $\mathbf{E}[x^i \mid \Pi_i(\omega)] > \mathbf{E}[e^i \mid \Pi_i(\omega)]$ $\forall i \in I$.

Proposition 3. *Let e be an initial wealth and let x be a feasible gamble. Then $\exists\, p^0, 0 < p^0 < 1$, such that for every $p \geq p_0$, if it is commonly p-believed at ω that the gamble is attractive then one of the agents will refuse to gamble.*

Proof. Let $\Omega' = \{\omega \in \Omega | x \text{ is attractive at } \omega\}$.

If $\Omega' = \varnothing$ then the proposition holds (trivially) for any $p_0 \in (0, 1)$.

Assume $\Omega' \neq \varnothing$ and let $\omega \in \Omega'$.

Let $J(\omega) = \{i \in I \mid x^i(\omega') = x^i(\omega'')\ \forall \omega', \omega'' \in \Pi_i(\omega)\}$.

For each $j \in J(\omega)$ let $k^j(\omega) = x^j(\omega) - e^j(\omega)$. Obviously, $k^j(\omega) > 0$ $\forall j \in J(\omega)$, and since x is feasible $\text{card}(J(\omega)) < \text{card}(I)$.

Let $I'(\omega) = I \setminus J(\omega)$. From the strict concavity of the utility functions, for each $i \in I'(\omega)$: $\mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] < U^i[\,\mathbf{E}(x^i \mid \Pi_i(\omega))]$ and $\exists \xi(\omega) > 0$ s.t.

$$(1)\quad \mathbf{E}[U^i(x^i) \mid \Pi_i(\omega)] < U^i[\mathbf{E}(x^i \mid \Pi_i(\omega)) - \xi(\omega)] \qquad \forall i \in I'(\omega).$$

Let $k(\omega) = \min_{j \in J(\omega)} k^j(\omega)$ if $J(\omega) \neq \varnothing$, $k(\omega) = \infty$ otherwise, and let

$\delta(\omega) = \min\{\xi(\omega), k(\omega)\}$. By construction, $\delta(\omega) > 0$. Choose $\varepsilon(\omega) \in (0, \delta(\omega))$.

It is straightforward to verify that under the above assumptions e is immediately Pareto optimal, thus from Proposition 2 $\exists p^0(\omega) \in (0, 1)$ s.t., $\forall p \geq p^0(\omega)$, if it is common p-believed at ω that the gamble is attractive then $\exists \ell \in I$ s.t.

$$\text{(2)} \quad \mathbf{E}[x^\ell \mid \Pi_\ell(\omega)] - \mathbf{E}[e^\ell \mid \Pi_\ell(\omega)] \leq \varepsilon(\omega).$$

Note that since $\varepsilon(\omega) < k(\omega) \leq k^j(\omega)\ \forall j \in J(\omega)$, $\ell \in I'(\omega)$. Thus, since utility functions are monotonically increasing, and from (1) and (2),

$$\begin{aligned}\mathbf{E}[U^\ell(e^\ell) \mid \Pi_\ell(\omega)] &= U^\ell[\mathbf{E}(e^\ell \mid \Pi_\ell(\omega))] \geq U^\ell[\mathbf{E}(x^\ell \mid \Pi_\ell(\omega)) - \varepsilon(\omega)] \\ &\geq U^\ell[\mathbf{E}(x^\ell \mid \Pi_\ell(\omega)) - \xi(\omega)] > \mathbf{E}[U^\ell(x^\ell) \mid \Pi_\ell(\omega)],\end{aligned}$$

and gambler ℓ will refuse to gamble at ω.

Take $p^0 = \max_{\omega \in \Omega'} p^0(\omega)$ to complete the proof. ■

Finally note that by the concavity assumption, $\{\omega \mid x >_\omega e\} \subseteq \{\omega \mid x$ is attractive at $\omega\}$. It follows from the monotonicity of the belief operator that $\forall p \in (0, 1]$, $\{\omega |$ it is commonly p-believed at ω that $x > e\} \subseteq \{\omega \mid x$ it is commonly p-believed at ω that x is attractive$\}$. As shown in the proof of Proposition 1, $\{\omega \mid$ it is commonly p-believed at ω that $x > e\} \subseteq \{\omega \mid x^i \geq_\omega e^i\ \forall i \in I\}$. Thus, $\{\omega \mid$ it is commonly p-believed at ω that $x > e\} \subseteq \{\omega \mid$ it is commonly p-believed at ω that x is attractive, and $x^i \geq_\omega e^i\ \forall i \in I\}$, and Proposition 3 implies Theorem 1 for this specific application. Clearly, the proposition may apply to impossibility cases that are not covered by the general theorem.

6. Conclusion

The results presented in this paper demonstrate, on the one hand, some robustness of the "Impossibility of speculations" theorems to a small amount of noise in the information systems: if we allow a "high enough" degree of common p-beliefs, speculative trades are still impossible in ex ante efficient markets. At the same time, the results demonstrate the sensitivity of "Impossibility of speculation" theorems to noisy information: purely speculative trades may seem to appear very tempting to all of the participating agents—even when the acceptability of the trade is common p-believed with probability close to 1.

We believe that these results support the "possibility of speculations" in reality. The models considered in this work are still highly structured:

agents are asumed to be highly rational, information is assumed to be given as a partition of the space of states, and we assume that the information partitions, the noise mechanism, and the degree of noise (ε) are common knowledge among the agents. Yet, even in such stylized structures, taking a first step toward a more realistic model by introducing a small (positive) amount of noise into the information system shows that the original impossibility result might be disrupted in many applications. The strong, general impossibility results thus seem to be relevant only in the limit, zero-noise case.

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