THE JOURNAL OF FINANCE • VOL. XXXIX, NO. 5 • DECEMBER 1984

Mean-Gini, Portfolio Theory, and the Pricing of Risky Assets

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ABSTRACT

This paper presents the mean-Gini (MG) approach to analyze risky prospects and construct optimum portfolios. The proposed method has the simplicity of a mean-variance model and the main features of stochastic dominance efficiency. Since mean-Gini is consistent with investor behavior under uncertainty for a wide class of probability distributions, Gini's mean difference is shown to be more adequate than the variance for evaluating the variability of a prospect. The MG approach is then applied to capital markets and the security valuation theorem is derived as a general relationship between average return and risk. This is further extended to include a degree of risk aversion that can be estimated from capital market data. The analysis is concluded with the concentration ratio to allow for the classification of different securities with respect to their relative riskiness.

MEAN-VARIANCE ANALYSIS, which forms the single most striking advance in finance theory, is based on the assumption that either prospects are normally distributed or the utility function is quadratic. Because of these restrictions, alternative versions of the fundamental model have been developed as in the case of lognormal distributions by Elton and Gruber [5] and Levy [11] and for paretian distributions by Bawa et al. [2]. In this paper, we present the mean-Gini approach to analyze risky investments and circumvent the problems of the mean-variance analysis that are specific to the choice of probability distributions. In addition, the proposed method has the attractiveness and simplicity of a two-parameter model and the main features of stochastic dominance efficiency.

Gini's mean difference is a statistic extensively used in measuring income inequality. The formal similarity between modelling decision-making under uncertainty and income inequality has, however, been established by several economists. Samuelson [19, 20] used the same method to prove both that it pays to diversify risky investments and that equal distribution of income among identical individuals maximizes the Benthamite social welfare function. Atkinson [1] showed that the rules for stochastic dominance, which were developed as criteria for evaluating risky prospects, can be translated into Lorenz curve terms in evaluating income inequality. In the present paper, we interpret some of the recent results on income inequality and apply them to portfolio analysis. In

* Shalit from the University of Maryland and the Hebrew University of Jerusalem. Yitzhaki from the Hebrew University of Jerusalem. This research was supported by a grant from the Government of Israel, the National Committee for Research and Development, Grant No. 5107. We are grateful to Haim Levy and the late Vijay Bawa, who as anonymous referees provided useful comments. All the remaining errors are, of course, ours.

particular, we use Gini's mean difference, the Lorenz curve, and the concentration ratio to characterize risky prospects and construct optimum portfolios.

The advantage of Gini's mean difference over the variance as a measure of dispersion has been demonstrated by Yitzhaki [26]. In particular, it was shown that using the mean and Gini's mean difference as summary statistics of a risky investment allows the derivation of necessary conditions for stochastic dominance, enabling the investigator to discard from the efficient set prospects that are stochastically dominated by others. Therefore, we argue that Gini's mean difference can replace the variance and that the concentration ratio based on it can substitute for the covariance needed in portfolio theory whenever mean-variance analysis fails to produce consistent results.

By providing necessary conditions for stochastic dominance, the mean-Gini portfolio selection rule is appealing to investigators since it prevents them from choosing a portfolio which can be considered inferior by all individuals, i.e., no other portfolio exists which is preferred by all other investors and, for at least one of them, the chosen portfolio is optimal. In general, the probability distribution of a portfolio is unknown even if *all* prospects are derived from a certain distribution. For example, if all prospects are lognormally distributed, the portfolio distribution *is not* lognormal. Then, if the optimal portfolio is chosen according to the mean-variance criterion, assuming lognormally distributed prospects, an alternative portfolio might be found that will be preferred by all risk-averse investors. Mean-variance analysis requires the perfect knowledge of all prospects' probability distributions; hence, it might fail to rank portfolios consistently to individual preferences when some prospect distribution is not well known. For example, take two risky investments with bounded and nonoverlapping distributions with the mean *and* the variance of the first prospect smaller than the mean *and* the variance of the second prospect. Clearly all investors will prefer the second investment although according to mean-variance analysis, the two prospects are in the efficient set. The use of the mean-Gini criterion and its necessary condition for stochastic dominance allows us to discard the first prospect from the efficient set and thus obtain a consistent ranking of risky alternatives.

In addition, if we consider families of cumulative probability distributions that intersect at most once, for example, the lognormal, uniform, exponential, and normal distributions among others, the mean-Gini necessary conditions are then sufficient for stochastic dominance and the efficient set of stochastic dominance portfolios is obtained. The investigator who is interested in isolating the stochastic dominance efficient set must simply check whether or not cumulative distributions intersect more than once. For a wide range of continuous and discrete distributions and their combinations, the sufficient conditions are satisfied. Finally, as with the mean-variance model, the mean-Gini approach can be used to derive the pricing of risky assets in a market equilibrium. Since the proposed method is independent of the distribution chosen, the pricing model provides a consistent evaluation of systematic risk, a feature obtained by mean-variance only if all the securities are either normally or lognormally distributed.

In the present paper, Gini's mean difference is extended into a family of coefficients differing from each other by a parameter which expresses some measure of risk aversion and permits the construction of different capital asset pricing models (CAPM). The mean-extended Gini method provides a wide range of CAPMs that cannot be differentiated without some empirical test. Only when *all* the securities are normally distributed will the different models converge to the CAPM derived from mean-variance analysis. Hence, it is sufficient that one prospect differs in probability distribution to obtain biased results when evaluating the securities' systematic risk by mean-variance analysis. Furthermore, some important risk behavior information is ignored since for a given security it is possible to obtain a positive beta for some risk parameter and a negative one for another. This implies that, depending on the degree of risk aversion used by the investigator, the security is either aggressive or defensive. In a forthcoming empirical work [23], we derive from the securities market data the value of the risk parameter used by most investors.

The plan of the paper is as follows: In the first section, we define Gini's mean difference and motivate its use in finance theory. In the second section, the properties of a portfolio, which are specified by its mean and its Gini coefficient, are developed. The third section derives the mean-Gini capital asset pricing model and discusses the properties of the approach, whereas the fourth section presents the extended Gini's mean difference, applies it to portfolio analysis, and extends the CAPM. The last section presents some extensions of the Gini coefficient to portfolio theory, such as the concentration ratio.

Much of the discussion in this paper is based on a reinterpretation of results on income inequality in finance theory. Thus, whenever possible we do not prove the propositions but refer the reader to the original papers.

I. Gini's Mean Difference

Gini's mean difference is an index of the variability of a random variable. It is based on the expected value of the absolute difference between every pair of realizations of the random variable. That is, let $F(R)$ and $f(R)$, respectively, represent the cumulative distribution and the density function of prospect R, and assume that there exist $a \geqq -\infty$ and $b \leqq \infty$ such that $F(a) = 0$ and $F(b) = 1$, then Gini's mean difference is defined as follows:[1]

$$\Gamma = \tfrac{1}{2} \int_a^b \int_a^b |R - r| f(r) f(R)\, dr\, dR, \tag{1}$$

where R and r are realization pairs of prospect R.

This definition is not easy to handle and one finds at least eight different formulations of Gini's mean difference in the literature. For our purposes it will be useful to deal with two of them. The first is

[1] For simplicity of presentation, continuous and bounded random variables will be used throughout this paper, keeping in mind that most of the results can be applied to discontinuous and unbounded distributions. The index i will be omitted whenever it is not necessary to distinguish between two variables. Note that we use half of Gini's original mean difference. Note too that we use here the absolute forms of Gini's mean difference and the concentration ratio; that is, we do not follow the more usual practice of dividing by the mean (see Kendall and Stuart [9]).

$$\Gamma = \int_a^b [1 - F(R)]\, dR - \int_a^b [1 - F(R)]^2\, dR, \tag{2}$$

and

$$\Gamma = \bar{R} - a - \int_a^b [1 - F(R)]^2\, dR, \qquad \text{for finite values of } a. \tag{2a}$$

Equation (2a) presents Gini's mean difference in terms of the expected value of the distribution, $\bar{R}$, and its cumulative distribution function, $F(R)$.[2] This equation is important when dealing with stochastic dominance criteria.[3]

The other Gini formula which is useful for analyzing portfolios is[4]

$$\Gamma = 2 \int_a^b R[F(R) - ½]f(R)\, dR; \tag{3}$$

that is, Gini's mean difference is twice the covariance of variable R and its cumulative distribution. For completeness, we state the following properties of Gini's mean difference:

1. The Gini coefficient is nonnegative and bounded from above by $\bar{R} - a$. If R is a given constant, $\Gamma = 0$. Furthermore, as is shown in Yitzhaki [27], its maximum value is reached for the distribution

$$f(R) = \begin{cases} ½ & R = a \\ ½ & R = b \\ 0 & \text{otherwise.} \end{cases}$$

2. The Gini coefficient is sensitive to mean-preserving spreads (Atkinson [1]).
3. Let $R_2 = \alpha R_1$, where α is a constant; then $\Gamma_2 = |\alpha|\Gamma_1$.
4. Let $R_2 = R_1 + c$ where c is a constant; then $\Gamma_2 = \Gamma_1$.
5. Let $R_3 = R_1 + R_2$; then $\Gamma_3 \leq \Gamma_1 + \Gamma_2$.
6. Let R be a normally distributed variable with $\bar{R}$ and σ^2; then $\Gamma = \sigma/\sqrt{\Pi}$ (Nair [15]).

Properties 2–5 are similar to those attributed to the standard deviation. Hence, it is not surprising that the Gini coefficient is used to derive the efficient set of uncertain prospects in the same way it is done using the mean-variance criterion. This feature is implied by the behavior of investors who rank uncertain prospects by their mean and the dispersion of their returns. Efficient sets of uncertain prospects are constructed such that no other feasible prospect is included in the set unless it has a lower dispersion for a given mean or a higher mean for a given

[2] For a derivation of Equation (2) for continuous, discrete, and unbounded distributions see Dorfman [4].

[3] For the definitions of the stochastic dominance rules see Hanoch and Levy [7] and Rothschild and Stiglitz [18].

[4] For a derivation of Equation (3) see Kendall and Stuart [9] for the continuous case and Pyatt et al. [16] for the case of discrete distributions. See also Lerman and Yitzhaki [10] for a calculation of the Gini with a regression program.

dispersion. Usually the standard deviation is used as the measure of dispersion; we propose to use instead Gini's mean difference. Hence, the efficient set according to the mean-Gini criterion is obtained by discarding, for each given mean $\bar{R}_0$, all prospects with a larger Gini coefficient. If combinations of alternatives are allowed to be held, the efficient set is obtained by finding, for each given mean, the mix of prospects minimizing the Gini coefficient of the portfolio.

We advocate the use of the mean-Gini (MG) method over the mean-variance (MV) analysis for the following reasons. First, if prospects are normally distributed, the efficient MG set is identical to the efficient MV set. This feature is justified by Property 6 above, implying that MV analysis is a special case of MG. Second, for a wide class of probability distributions, mean-Gini analysis provides a consistent ranking of uncertain alternatives whenever mean-variance fails. This advantage of the proposed method prevails since MG ranking is identical to stochastic dominance (SD) for cumulative distributions that intersect at most once. The link of MG analysis with SD rules is provided by the two following propositions that present the necessary and sufficient conditions for stochastic dominance.

PROPOSITION 1. *Let R_1 and R_2 be two uncertain prospects. The conditions $\bar{R}_1 \geqq \bar{R}_2$ and $\bar{R}_1 - \Gamma_1 \geq \bar{R}_2 - \Gamma_2$ are necessary conditions for R_1 to dominate R_2 according to first and second stochastic dominance rules* (Yitzhaki [26]).

The implications of Proposition 1 are important since they provide us with a two-parameter instrument that can be used to discard from the efficient set the stochastically inferior alternatives.

The efficient set S_i is defined as the set of all prospects that are not dominated by applying the appropriate rule i. S_{MV} is the efficient set obtained by mean-variance rules, S_{SD} the efficient set obeying the stochastic dominance first and second rules, S_{MG} the set obtained by minimizing the Gini subject to a given mean, and S_{MG1} the set obeying Proposition 1. Then if we restrict the set of distributions to the normal distribution, we can assert that S_{SD}, S_{MV}, and S_{MG} are identical and S_{MG1} is contained in the other sets. However, only in that precise case are the MV, MG, and SD rankings identical and consistent with risk-averse behavior.

Whereas the necessary conditions for SD rules can be used for any distribution, the following sufficient conditions are weaker in the sense that they can be applied to families of cumulative distributions that intersect at most once;[5] such as normal, lognormal, uniform, the exponential, and Gamma distributions, among the continuous distributions and a wide range of other discrete distributions.

PROPOSITION 2. *Let R_1 and R_2 be two prospects with equal expected return. Assume also that the cumulative distributions $F_1(R)$ and $F_2(R)$ intersect at most once. Then $\bar{R}_1 - \Gamma_1 > \bar{R}_2 - \Gamma_2$ is a sufficient condition for R_1 to dominate R_2 according to SD rules.*

[5] F_1 intersects F_2 from above if for every $R_0 < R_1$ such that $F_1(R_1) > F_2(R_1)$ and $F_1(R_0) < F_2(R_0)$ then $F_1(R) < F_2(R)$ for all $R < R_0$ and $F_1(R) \geq F_2(R)$ for all $R > R_1$ (*mutatis mutandis* if F_2 intersects F_1 from above).

Hence, for families of nonnormal distributions that intersect at most once, only S_{SD} and S_{MG} are identical implying that mean-variance analysis fails to provide consistent ranking of risky alternatives when distributions are not normal (see Rothschild and Stiglitz [18], Hanoch and Levy [7]). Furthermore, the mean-variance approach requires a knowledge of the type of distributions used in the analysis. Assume that prospects are lognormally distributed but the investigator ignores it; using the return variance as a risk parameter instead of the variance of the return logarithm might lead to paradoxical results (Levy [11]).

In general, S_{MG1} is a subset of S_{MG} and S_{SD}. By using that subset we are able to solve the following problem which was raised by Whitmore [25, p. 248].

"For instance, the following situation frequently arises in practice. A prospect $G(R)$ appears attractive relative to a second prospect $F(R)$ except that one (or several) outcomes of $G(R)$ fall below all the possible outcomes of $F(R)$. Regardless of how attractive $G(R)$ is and how small the probabilities of the adverse outcomes of $G(R)$ are (provided they are not zero), the rule of third-degree stochastic dominance will not indicate that $G(R)$ is at least as preferred as $F(R)$."

By using S_{MG1} instead of S_{MG}, we can discard from the efficient set prospects such as $F(R)$ provided that the mean return is high enough. Hence, applying Proposition 1 to the efficient set constructed by the MG method enables us to obtain *an efficient set which is a subset of the efficient set* according to first and second SD rules. This subset is not liable to the criticism usually advanced against the efficient set constructed by the MV rule [7, 11, 18].

Apparently, stochastic dominance analysis has not been successfully applied to finance practical work mainly because of the complexity involved in computing efficient portfolios. However, stochastic dominance theory provides us with the only ranking of uncertain prospects and combinations of prospects that is consistent with risk-averse behavior. Mean-variance analysis fails to pass the consistent test whenever prospects are not normally distributed. The proposed MG method and the use of Proposition 1 can be demonstrated using two simple examples for prospects with different expected returns.[6]

First assume two investments R_1 and R_2 that are uniformly distributed, the first between 0 and 1 and the second between 2 and 4. The two prospects have nonoverlapping distributions and clearly all investors will prefer R_2 over R_1 by stochastic dominance rules. However, according to mean-variance and mean-Gini analysis, the two prospects are in the efficient set with $\bar{R}_1 = 1/2$, $\Gamma_1 = 1/6$, $\sigma_1 = 1/\sqrt{12}$ and $\bar{R}_2 = 3$, $\Gamma_2 = 1/3$, $\sigma_2 = 1/\sqrt{3}$. Applying Proposition 1 to the set obtained by the mean-Gini criterion enables us to discard R_1 from the efficient set. The second example is depicted in Figure 1. Let R_1 and R_2 be two prospects with Gamma density distributions as shown in Figure 1. Clearly R_2 is preferred over R_1 by all investors. However, since $\sigma_1 < \sigma_2$ and $\bar{R}_1 < \bar{R}_2$, the two prospects are in the mean-variance efficient set and, since $\Gamma_1 < \Gamma_2$, they are also the mean-Gini efficient set. Applying Proposition 1 to the MG efficient set removes the first alternative from the efficient set and forces us to consider R_2 only. In these two examples, we have ruled out the possibility of combining the two prospects to emphasize the use of the method and its power.

[6] In that case, Proposition 2 cannot be used as a sufficient condition for stochastic dominance. However, it is shown later that sufficient conditions can be constructed by the extended Gini.

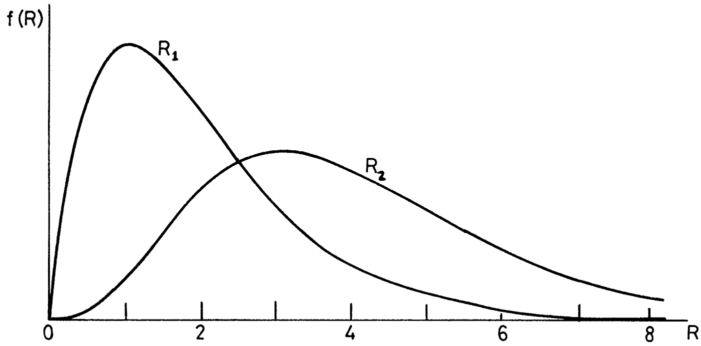

Figure 1. The Evaluation of Two Gamma Distributed Prospects. R is a distributed Gamma with $f(R) = \frac{1}{\Gamma(\alpha)} \cdot R^{\alpha-1} \cdot e^{-R}$ where the mean $= \alpha$, the variance $= \alpha$, the Gini $= \frac{\alpha \cdot \Gamma(2\alpha + 1)}{2^{2\alpha}[\Gamma(\alpha + 1)]^2}$, and $\Gamma(\cdot)$ is the gamma function $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z}\, dz$. For R_1, $\alpha = 2$, $\mu_1 = 2$, $\sigma_1^2 = 2$, and $\Gamma_1 = 0.75$. For R_2, $\alpha = 4$, $\mu_2 = 4$, $\sigma_2^2 = 4$, and $\Gamma_2 = 1.09375$.

The advantage of the mean-Gini approach over the stochastic dominance criteria results from its similarity to the MV method. As far as we know, there is no easy method for constructing optimum *portfolios* by SD rules. Users of the mean-Gini method may minimize the Gini coefficient of a linear combination of prospects subject to a given required mean return. Changing the mean permits the user to construct the efficient set corresponding to the mean-Gini criterion. This set can be used for portfolio analysis and capital asset pricing equilibrium according to MG rules.[7]

II. Portfolio Analysis

The properties of a portfolio whose performance is summarized by the mean and the Gini coefficient are similar to those of the regular mean-standard deviation model. These properties can be illustrated by the familiar textbook diagram in which Γ, the Gini coefficient of the return on the portfolio, is depicted on the horizontal axis, while $\bar{R}$, the mean of the one-period return, is on the vertical axis, as shown in Figure 2.

The performance of two prospects, A and B, is denoted by A and B in the mean-Gini space. The return on portfolio R_p is given by the convex combination of the return on A and B. Hence, $R_p = xR_A + (1 - x)R_B$ where x is the share of wealth invested in A, and R_A and R_B are the one-period returns.

As in the ordinary mean-standard deviation model, the performance of the portfolio R_p depends also on the correlation between A and B and on x. To show this, consider the three special cases as depicted in Figure 2. First, if prospects A and B are linearly dependent, the coefficient of correlation (ρ_{AB}) is equal to unity and, following from the properties of Gini's mean difference, the line ACB represents all the possibilities of a portfolio mix composed of A and B. Second,

[7] An algorithm for minimizing the Gini of a linear combination of prospects, subject to a given expected return by the method of piecewise linear programming, has been developed and is available from the authors upon request.

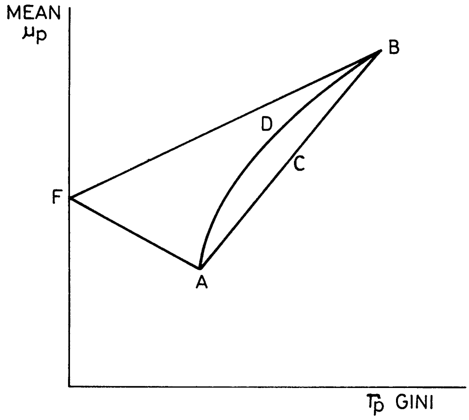

Figure 2. Portfolio Construction in the MG Space

if A and B are independent ($\rho_{AB} = 0$), the curve ADB expresses the performance of the portfolio, R_p, showing intuitively that diversification improves performance. Furthermore, the portfolio returns would be much improved if A and B were negatively correlated as shown by the broken line AFB for the extreme cases of $\rho_{AB} = -1$.

The effect of the variability of a prospect on the variability of the portfolio can be presented, much as it is in the mean-standard deviation model. By Equation (3), the Gini coefficient of a portfolio is

$$\Gamma_p = 2 \int_a^b R[F_p(R) - ½]\, dF_p(R) = 2 \operatorname{cov}[R_p, F_p(R_p)], \tag{4}$$

where F_p is the cumulative distribution of the portfolio. Since

$$R_p = \textstyle\sum_{i=1}^N x_i R_i, \qquad \text{for} \quad \sum_{i=1}^N x_i = 1 \tag{5}$$

where R_i is the return on prospect i, we obtain

$$\Gamma_p = 2 \textstyle\sum_{i=1}^N x_i \operatorname{cov}[R_i, F_p(R_p)]; \tag{6}$$

that is, the risk of the portfolio can be decomposed into a weighted sum of the covariance between the variables R_i and the cumulative distribution of the portfolio p.

It is worth mentioning that the variance of the portfolio can be written as

$$\operatorname{var}(R_p) = \textstyle\sum_{i=1}^N x_i \operatorname{cov}(R_i, R_p), \tag{7}$$

and the difference in the decomposition of the nondiversifiable risk by the two methods is that in (6) the portfolio is represented by the cumulative distribution of its returns, F_p, while in (7) it is represented by its returns, R_p.

By multiplying and dividing each component of (6) by $\Gamma_i = 2 \operatorname{cov}[R_i, F_i(R_i)]$ where $F_i(R_i)$ is the cumulative distribution of prospect i, we obtain

$$\Gamma_p = \textstyle\sum_{i=1}^N x_i \theta_i \Gamma_i, \tag{8}$$

where $\theta_i = \operatorname{cov}[R_i, F_p(R_p)]/\operatorname{cov}[R_i, F_i(R_i)]$ which represents the ratio of nondiversifiable risk to the variability of prospect i.

III. The Pricing of Risky Assets

In this section, we develop the security valuation theorem for investors holding mean-Gini efficient portfolios. The CAPM market-equilibrium relationship has been formulated for MV efficient portfolios by Treynor [24], Sharpe [22], Lintner [12], and Mossin [14]. The theorem states that for any security, the higher its nondiversifiable risk, the higher will be its expected return. Nondiversifiable (or systematic) risk is that part of the security's total risk that cannot be reduced by diversifying a portfolio without reducing its expected rate of return. The theorem is stated in the context of competitive financial markets without taxes and without restrictions on short selling and borrowing. In these markets, investors trade risky assets whose quantities are known and fixed, to build efficient portfolios that answer their preferences. By doing so, they act in the securities market, building forces that influence and determine the value of these securities. Assuming that investors build their portfolios according to an MV utility, the familiar CAPM relationship between expected return and risk is expressed as

$$\bar{R}_i = r_f + [\bar{R}_m - r_f][\mathrm{cov}(R_i, R_m)]/\sigma_m^2, \tag{9}$$

where

$\bar{R}_i = E(R_i)$ is the expected rate of return on security i,
$r_f =$ is the rate of return on a risk-free security,
$\bar{R}_m = E(R_m)$ is the expected rate of return on the market portfolio, and
$\sigma_m^2 =$ variance of the rate of return on the market portfolio.

Equation (9) was derived under several assumptions of which the most important are (a) single-period analysis, (b) the existence of a risk-free asset, and (c) perfect competition in the securities market.

The valuation theorem proposed here retains the main assumptions of the classical CAPM. However, instead of holding MV efficient portfolios, investors construct market portfolios which are subsets of the SD efficient set. This is done by using the mean-Gini method. To justify that approach, we advance two avenues of consideration to derive the investor's behavior in the capital assets market. The first assumes that investors have a specific utility function that weights the mean against the measure of dispersion represented by the Gini. Then as investors minimize the measure of dispersion for a given mean, they maximize that utility function and the mean-Gini efficient set is obtained as the result.

The second approach is based on the use of Proposition 1. It assumes that investors maximize their expected utility of returns but that the utility function is unknown to the analyst. Therefore, the investigator uses the mean-Gini approach to characterize the investors' decisions. Minimizing the portfolio's Gini for each given mean determines the MG efficient set. Applying Proposition 1 now defines an efficient subset of the SD efficient set. Hence, every member of that set satisfies the conditions for expected utility maximization.

Each investor determines his optimum portfolio by choosing a securities mix that minimizes the Gini's mean difference of the portfolio given its expected rate of return. Investors are permitted to borrow and lend at the riskless rate, r_f. The rate of return of investor j's portfolio is given by

$$R_j = \sum_{i=1}^{N} x_i^j R_i + (1 - \sum_{i=1}^{N} x_i^j) r_f, \qquad \sum x_i^j \leq 1, \tag{10}$$

where R_i is the rate of return on security i ($i = 1, \cdots, N$) and x_i^j is the share of investor j's wealth invested in security i. Hence, the investors' problem is to minimize Γ_j, subject to

$$E(R_j) = \sum_{i=1}^{N} x_i^j \bar{R}_i + (1 - \sum_{i=1}^{N} x_i^j) r_f. \tag{11}$$

Recall that the Gini coefficient of the portfolio can be written as

$$\Gamma_j = 2 \operatorname{cov}[R_j, F_j(R_j)] = 2 \sum_{i=1}^{N} x_i^j \operatorname{cov}[R_i, F_j(R_j)]. \tag{12}$$

Thus, the necessary conditions for a minimum are simply:

$$2 \operatorname{cov}[R_i, F_j(R_j)] + 2 \sum_{k=1}^{N} x_k^j \frac{\partial \operatorname{cov}[R_k, F_j(R_j)]}{\partial x_i} = \lambda_j(\bar{R}_i - r_f), \qquad \text{for all} \quad i = 1, \cdots, N \tag{13}$$

and Equation (11), where λ_j is the Lagrange multiplier associated with investor j. The second-order conditions are insured by quasi-convexity of the Gini with respect to the weights of x.[8]

As we will show below, $\operatorname{cov}[R_i, F_j(R_j)]$ is the concentration ratio[9] between security i and the portfolio of individual j. It represents the degree by which the risk of security i cannot be diversified by including it in j's portfolio.

By Property 3 of the Gini coefficient and Equation (6), we know that Γ_j is homogeneous of degree one in x_i^j.[10] Therefore, by Euler's theorem,

$$\Gamma_j = \sum_{i=1}^{N} x_i^j \frac{\partial \Gamma_j}{\partial x_i} \tag{14}$$

or

$$\Gamma_j = 2 \sum_{i=1}^{N} x_i^j \operatorname{cov}[R_i, F_j(R_j)] + 2 \sum_{i=1}^{N} \sum_{k=1}^{N} x_i^j x_k^j \frac{\partial \operatorname{cov}[R_k, F_j(R_j)]}{\partial x_i}, \tag{14a}$$

[8] To see that, define from Equation (10) two different portfolios, R_p and R_q, with weights $\{x_i^1\}$ and $\{x_i^2\}$, respectively. The strictly convex combination of the two portfolios is defined as

$$R_\gamma = \gamma R_p + (1 - \gamma) R_q, \qquad \text{for} \quad 0 < \gamma < 1,$$

or

$$R_\gamma = \sum_{i=1}^{N} [\gamma x_i^1 + (1 - \gamma) x_i^2] R_i + \{1 - \sum_{i=1}^{N} [\gamma x_i^1 + (1 - \gamma) x_i^2]\} r_f.$$

From Properties 3 and 5 of the Gini,

$$\Gamma(R_\gamma) \leqq \Gamma(\gamma R_p) + \Gamma[(1 - \gamma) R_q] = \gamma \Gamma_1 + (1 - \gamma) \Gamma_2$$

with strict subadditivity holding whenever R_p and R_q are not collinear (i.e., $|\rho_{12}| < 1$). Assume $\Gamma_1 = \Gamma_2 = \Gamma_0$. Then every R_γ lying on the line connecting R_p to R_q in the space defined by the x's satisfies $\Gamma_\gamma \leqq \Gamma_0$. Hence the set $\{x$, subject to $\Gamma(x) \leqq \Gamma_0\}$ is convex and Γ is quasi-convex.

[9] Note that to obtain the concentration ratio as defined by Pyatt et al. [16], $\operatorname{cov}[R_i, F(R_j)]$ should be divided by $E_j(R_i)$.

[10] Multiplying R_j and $F_j(R_j)$ by a constant δ will not change the cumulative distribution. Hence $\Gamma_j(\delta) = 2 \operatorname{cov}[\delta R_j, \delta F_j(R_j)] = 2\delta \operatorname{cov}[R_j, F_j(R_j)] = \delta \Gamma_j$.

implying by Equation (12) that the double sum on the right-hand side vanishes.

By multiplying each of the conditions in (13) by its share x_i^j and summing over all the securities N, we obtain for every investor

$$\Gamma_j = \lambda_j \textstyle\sum_{i=1}^N x_i^j(\bar{R}_i - r_f) \tag{15}$$

or

$$\Gamma_j = \lambda_j[\textstyle\sum_{i=1}^N x_i^j\bar{R}_i - r_f + (1 - \textstyle\sum_{i=1}^N x_i^j)r_f] \tag{16}$$

and we have obtained the relation between risk (expressed by the Gini) of the portfolio and its expected return as

$$\Gamma_j = \lambda_j[E(R_j) - r_f]. \tag{17}$$

Equation (17) represents the highest feasible straight line in the $(\bar{R}, \Gamma)$ space given an efficiency frontier constructed by portfolio combinations of risky assets. Since the efficiency frontier is concave within that space due to Properties 3 and 5, the second-order conditions for a minimum are satisfied. In that context, $1/\lambda_j$ is the investor subjective price of risk since it relates the expected rate of return of the chosen portfolio to its risk. The investor will choose a portfolio mix along this line that maximizes his utility.

Now assume a market of similar investors who are risk averse, have identical investment opportunities, and minimize the Gini coefficients of their portfolios subject to their expected rates of return. In that case, Equation (17) will be identical for all the investors in that market.

For a given risk-free rate of return, the unit price of risk will be equal and determined by the slope of the market line in the $(\bar{R}, \Gamma)$ space (see Figure 3).

For an investor who does not borrow or lend, all his wealth will be invested in risky securities, whose portfolio is the market portfolio pictured by m in Figure 3. Thus, for all investors, the price of risk will be

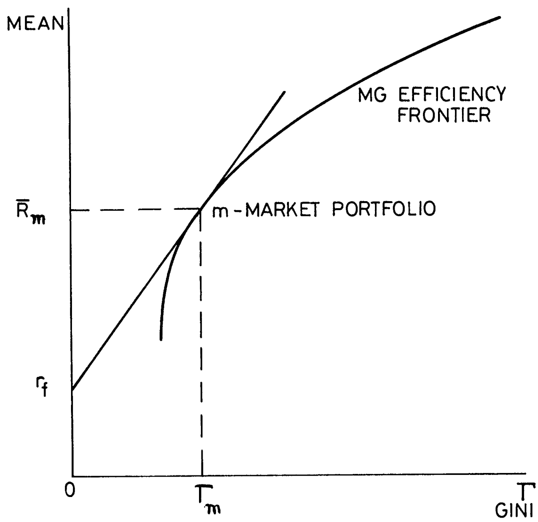

Figure 3. The MG Efficiency Frontier

$$1/\lambda = (\bar{R}_m - r_f)/\Gamma_m \tag{18}$$

where $\bar{R}_m$ is the expected return on the market portfolio and Γ_m is the Gini coefficient. In that case, the Gini coefficient of the portfolio held by investor j is equal to Γ_m since the optimum ranking, $F_j(R_j)$ of investor j, remains unchanged whether or not a risk-free security is added and thus is equal to $F_m(R_m)$ for all investors. Thus, from (13)

$$2 \sum_{k=1}^{N} x_k^j \frac{\partial \operatorname{cov}[R_k, F_m(R_m)]}{\partial x_i} = 0 \tag{19}$$

and the equilibrium condition for every security i and investor j becomes

$$\bar{R}_i = r_f + (\bar{R}_m - r_f)\cdot 2 \operatorname{cov}[R_i, F_m(R_m)]/\Gamma_m. \tag{20}$$

This is essentially the CAPM valuation relationship for a market of investors using the mean-Gini approach. To understand the dependence between systematic risk and expected return, let us rewrite $2 \operatorname{cov}[R_i, F_m(R_m)]$ as $\theta_{im}\Gamma_i$, where

$$\theta_{im} = \frac{\operatorname{cov}[R_i, F_m(R_m)]}{\operatorname{cov}[R_i, F_i(R_i)]}$$

is a sort of rank correlation coefficient[11] and $\Gamma_i = 2 \operatorname{cov}[R_i, F_i(R_i)]$. Thus,

$$\bar{R}_i = r_f + (\bar{R}_m - r_f)\theta_{im}(\Gamma_i/\Gamma_m). \tag{21}$$

Therefore, security i's beta is simply

$$\beta_i = \theta_{im}(\Gamma_i/\Gamma_m). \tag{22}$$

As is well known, β_i represents the degree of responsiveness of the rate of return of security i to changes in the market, and θ_{im} is the proportion of total risk (expressed by the Gini coefficients of security i, Γ_i) that cannot be eliminated by the market without reducing the expected rate of return.

At this point, it is important to draw the analogy between the β_i in (24) and the β_i derived from the MV-CAPM. It can be shown that for normally distributed securities, $\Gamma_i = \sigma_i/\sqrt{\Pi}$ and $\Gamma_m = \sigma_m/\sqrt{\Pi}$, where σ_i and σ_m are the standard deviations of i and m, respectively.

Therefore, θ_{im} does converge to the Pearson correlation coefficient between security i and the market m. This assertion is intuitively deduced since, in the case of normally distributed prospects, the MV and MG "betas" coincide for the same set of observations. However, this is not always true for prospects that are *not* normally distributed. In general, MV and MG betas will be different, with the MG betas corresponding to SD efficient securities markets. However, the question arises as to whether the Gini is the only index of variability that can be related to stochastic dominance and, at the same time, be used to derive "betas."

The next section shows that there is an infinite number of indices of variability with similar properties to the Gini. This is the extended Gini family. As we will show, all CAPMs derived using the extended Gini approach will coincide with the MV-CAPM only if all prospects are normally distributed.

[11] For a description of the statistical properties of θ_{im} and a suggested method of testing, see Lerman and Yitzhaki [10] and Schechtman and Yitzhaki [21].

IV. The Extended Gini Coefficient

In this section, we develop the extended Gini coefficient and apply it to portfolio analysis and the CAPM. Gini's mean difference may be extended into a family of coefficients of variability differing from each other in the decision-maker's degree of risk aversion, which is reflected by the parameter ν. Rewrite Equation (2) as

$$\Gamma(\nu) = \int_a^b [1 - F(R)]\, dR - \int_a^b [1 - F(R)]^\nu\, dR$$

$$= \bar{R} - a - \int_a^b [1 - F(R)]^\nu\, dR, \qquad \text{for finite values of } a \tag{23}$$

where $1 < \nu < \infty$ is a parameter chosen by the user. We define (23) as the extended Gini coefficient and intend to use it instead of the simple Gini in portfolio risk valuation. The need for the mean-extended Gini analysis is motivated by its potential use as follows. First, the investigator can obtain the SD efficient set as the union of all efficient sets of prospects obtained by the mean-extended Gini analyses performed for the various ν's. Second, as the extended Gini is a measure of prospects' variability related, on one hand, to stochastic dominance and reflecting, on the other hand, the investor's attitude toward risk, different risk pricing of assets can be evaluated. For nonnormally distributed returns, the CAPMs associated with the various extended Gini analyses are not necessarily equal to each other as they incorporate some measure of risk aversion. Therefore, the *market's* attitude toward risk as represented by ν can be estimated by looking at the market composition data. Finally the extended Gini can be used to classify the relative riskiness of prospects according to different risk-averse investors.

The properties of the extended Gini are as follows (Yitzhaki [27]):

1. The extended Gini is nonnegative, bounded from above and nondecreasing in ν.
2. For a positive integer value of ν, the extended Gini is $\Gamma(\nu) = \bar{R} - E[\min(R_1, R_2, \cdots, R_\nu)]$.
3. The extended Gini coefficient is sensitive to mean-preserving spreads.
4. If $R_2 = \alpha R_1$, then $\Gamma_2(\nu) = |\alpha|\Gamma_1(\nu)$.
5. Let $R_2 = R_1 + c$; then $\Gamma_2(\nu) = \Gamma_1(\nu)$.
6. Let R_1, R_2 be two prospects; then $[\bar{R}_1 - \Gamma_1(\nu)] = [\bar{R}_2 - \Gamma_2(\nu)] \geq 0$, for $\nu = 1, 2, 3, \cdots$, is the necessary condition for first- and second-degree stochastic dominance (Yitzhaki [26]).
7. Let R_1, R_2 be two prospects with cumulative distributions that intersect at most once. Then $[\bar{R}_1 - \Gamma_1(\nu)] - [\bar{R}_2 - \Gamma_2(\nu)] \geq 0$, for all $\nu \geq 1$, is a sufficient condition for R_1 to dominate R_2 according to SD. If, on the other hand, $\bar{R}_1 = \bar{R}_2$, sufficient conditions for SD are $[\bar{R}_1 - \Gamma_1(\nu)] - [\bar{R}_2 - \Gamma_2(\nu)] > 0$ for *any* ν. The proof is available from the authors.

From Equation (23), one can represent the extended Gini as (see the Appendix):

$$\Gamma(\nu) = -\nu \operatorname{cov}\{R, [1 - F(R)]^{\nu-1}\}. \tag{24}$$

The extended Gini is simply a "weighted" covariance between the variate and 1 minus its cumulative distribution raised to the power $\nu - 1$. The higher ν becomes, the more risk averse is the investor since he or she is attributing a larger weight to the worst realizations of the distribution.

The interpretation of ν can also be seen by looking at $\bar{R} - \Gamma(\nu)$ which, for different values of ν, becomes

$$\bar{R} - \Gamma(\nu) = \begin{cases} \bar{R} & \nu = 1 \\ \bar{R} - \Gamma & \nu = 2 \\ a & \nu \rightarrow \infty \end{cases}$$

and is a nonincreasing function of ν. Thus, one can view $\Gamma(\nu)$ as the risk premium that should be subtracted from the expected value of the distribution. The case $\nu = 1$ represents the risk-neutral investor while $\nu \rightarrow \infty$ represents the investor who is interested in the minimum value of the distribution and maximizes this minimum.[12]

The extended Gini coefficient of a portfolio can be decomposed into an equation similar to (6), where the extended Gini is a weighted sum of the covariances of the individual securities with portfolio distribution raised to the power $\nu - 1$ (see the Appendix for the proof):

$$\Gamma_p(\nu) = -\nu \textstyle\sum_{i=1}^{N} x_i \operatorname{cov}\{R_i, [1 - F_p(R_p)]^{\nu-1}\}. \tag{25}$$

That is, the nondiversified risk of a prospect is its covariance with the portfolio raised to the power $\nu - 1$. The higher ν becomes, the greater the weight given to the performance of the prospect when the yield of the portfolio is low. Note that if $\nu = 2$, then we have the simple Gini coefficient for the portfolio.

We derive the CAPM, using the extended Gini of degree ν. For investor j, the extended Gini of his portfolio is given by

$$\Gamma_j(\nu) = -\nu \textstyle\sum_{i=1}^{N} x_i^j \operatorname{cov}\{[1 - F_j(R_j)]^{\nu-1}, R_i\}. \tag{25a}$$

Investor j chooses the x_i^j that minimize $\Gamma_j(\nu)$, subject to the expected rate of return on the portfolio given by (11). As shown, ν reflects the degree of risk aversion. Hence, it is possible to model a securities market that will exhibit different relations between prospects and the market portfolio because of different degrees of risk aversion. For the present, we require that investors are similar and that they display identical ν. Therefore, for all j, the necessary conditions for a minimum are given similarly to (13) and (18) by:

$$-\nu \operatorname{cov}\{[1 - F_j(R_j)]^{\nu-1}, R_i\} = \lambda_j(\bar{R}_i - r_f)$$

$$1/\lambda_j = (\bar{R}_m - r_f)/\Gamma_m(\nu),$$

where $\Gamma_m(\nu)$ is the market portfolio's extended Gini coefficient of degree ν, and the market-equilibrium relation similar to (20) becomes

$$\bar{R}_i = r_f - [(\bar{R}_m - r_f)/\Gamma_m(\nu)]\nu \operatorname{cov}\{[1 - F_m(R_m)]^{\nu-1}, R_i\}. \tag{26}$$

[12] Values of $0 \leq \nu < 1$ represent the case of the risk lover, the extreme case $\nu = 0$ representing the investor interested in the maximum value of the distribution, b (the max-max investor). In this paper, we restrict ourselves to risk-averse investors.

If the "rank" correlation ratio of degree ν between security i and the market is

$$\theta_{im}(\nu) = \frac{\operatorname{cov}\{[1 - F_m(R_m)]^{\nu-1}, R_i\}}{\operatorname{cov}\{[1 - F_i(R_i)]^{\nu-1}, R_i\}},$$

then the market-equilibrium valuation for every i on any ν is given by

$$\bar{R}_i = r_f + (\bar{R}_m - r_f)\theta_{im}(\nu)\,\frac{\Gamma_i(\nu)}{\Gamma_m(\nu)}. \tag{27}$$

Thus,

$$\beta_i(\nu) = \theta_{im}(\nu)\,\frac{\Gamma_i(\nu)}{\Gamma_m(\nu)}. \tag{28}$$

Thus, even if investors have the same attitude towards this risk as expressed by ν, different systematic risks for each security will be obtained. This feature must be borne in mind when estimating betas. Minimizing the extended Gini of a portfolio subject to its expected return provides different efficient portfolios for different ν. Comparing these portfolios with the market enables us to estimate the parameter of risk aversion that provides the best fit for the data. Hence, the parameter ν that is chosen by this procedure reflects the aversion towards risk expressed by the securities market. It must be added that if securities are normally distributed, the MEG betas will be identical to the MV betas, independent of ν. But our concern for the existence for different systematic risks is principally directed towards distributions other than normal such as the lognormal and the uniform distribution. For these distributions, the MV approach is not consistent with expected utility maximization and stochastic dominance, whereas the mean-Gini is.[13] The intention of the next section is to establish the conditions for which all mean-extended Gini investors agree on the classification of the securities' riskiness.

V. The Classification of Prospects by Relative Risk

In this section, we apply the mean-extended Gini analysis to classify prospects according to their relative risk. One way of doing that for security i is simply to use $\operatorname{cov}[R_i, F_p(R_p)]$ as a measure of its undiversified risk since one can always include that security in a portfolio yielding R_p. The undiversified risk is the share of total risk as expressed by $\operatorname{cov}[R_i, F_i(R_i)]$ that cannot be reduced by incorporating security i in the portfolio p. That feature has been investigated in Section III. However, this type of classification is silent about the possibility that security i can be riskier than security j in one situation and less risky in another.

One method of improving that classification is to use concentration curves enabling the investigator to determine the security's risk according to the return on the portfolio.[14] This arrangement permits a check for any two prospects A and B within portfolio p, for instance, whether all investors agree that prospect

[13] See Yitzhaki [26] for a consistency analysis of the mean-Gini approach for different distributions.

[14] For a discussion of the properties of the concentration curve see Kakwani [8] and Pyatt et al. [16].

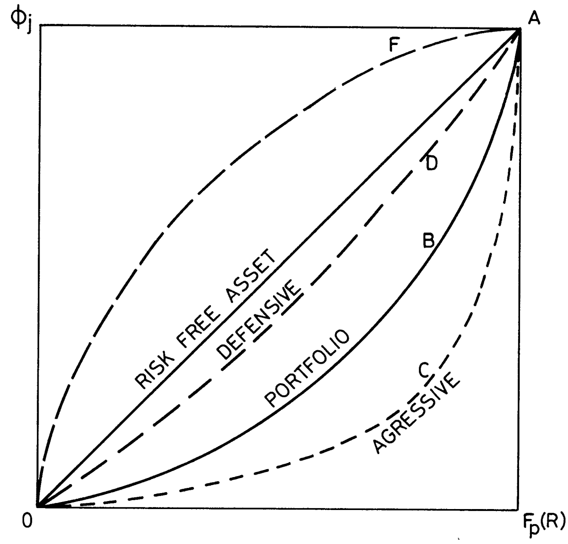

Figure 4. Portfolios' Lorenz Curves and Prospects' Concentration Ratios

A is riskier than B with respect to portfolio p or that they disagree about the relative riskiness of A and B. Assume two securities with identical positive expected rates of return; i.e., $E_i(R_i) = E_j(R_j) = \bar{R}$. Now define the function $R_j(R_p)$ as the conditional expectation of rate of return R_j, given the portfolio R_p; that is,

$$g_j(R_p) = E_j(R_j \mid R_p). \tag{29}$$

We assume that $R_p \geq 0$, $g_j(R_p) \geq 0$, and that the first derivative of $g(\cdot)$ exists.[15] If $E[g_j(R_p)] = \bar{R}$, one defines the concentration curve of security j as the relationship between

$$\phi_j[g_j(R_p)] = \frac{1}{\bar{R}} \int_0^{R_p} g_j(R) f_p(R)\, dR$$

and

$$F_p(R) = \int_0^{R_p} f_p(R)\, dR. \tag{30}$$

Concentration curves are plotted in Figure 4. It is easily seen that if $R_j = R_p$, Equation (30) represents the Lorenz curve of portfolio R_p (curve B in the figure). The relative riskiness of the securities in a portfolio can be compared according to the following proposition:

PROPOSITION 3.[16] *The concentration curve for the function $g_j(R_p)$ will be above (below) the concentration curve for the function $g_i(R_p)$ if $\eta_j(R_p)$ is smaller (greater) than $\eta_i(R_p)$, for all R_p, where η is the elasticity of g with respect to R as defined by* $(\partial g/\partial R)\cdot(R/g)$.

[15] The restriction $g_j(\ ) \geq 0$ and $R_p \geq 0$ should be interpreted as a shift of the origin. They are needed because some of the propositions use the elasticity concept, which is meaningful only if variables are positive.

[16] This proposition and the two corollaries were proved by Kakwani [8].

Two corollaries follow from Proposition 3:

COROLLARY 1. *The concentration curve for the function $g_j(R)$ will be above (below) the egalitarian line (45° line) if $\eta_j(R)$ is less (greater) than zero, for all R.*

That is, in Figure 4, the 45° line represents the risk-free asset. Stocks which are *always* negatively correlated with the portfolio have a concentration curve which is above the 45° line, as in the curve *OFA*.

The second corollary permits the comparison between the individual securities and the portfolio they compose.

COROLLARY 2. *The concentration curve for the function $g_j(R)$ lies above (below) the Lorenz curve for the distribution $F_p(R_p)$ if $\eta_j(R_p)$ is less (greater) than unity for all $R_p \geq 0$.*

Corollary 2 permits us to distinguish between two kinds of securities in a portfolio. First, we observe *aggressive* securities, with concentration curves below the Lorenz curve of the portfolio, whose high degree of responsiveness to the portfolio leads to considerable instability. Second, we have *defensive* securities, with concentration curves above the Lorenz curve of the portfolio, which reduce the instability because they are less responsive. These results are summarized in Figure 4.

Let *OAB* be the Lorenz curve of the portfolio. The *aggressive* stock will be represented by *OCA* while the *defensive* stock is represented by the concentration curve *ODA*. The 45° line portrays the risk-free asset (if it exists) while *OFA* represents a stock which is negatively correlated with the portfolio.

By definition and construction, the relative Gini is equal to twice the area between the 45° line and the Lorenz curve, while the concentration ratio for security j is equal to twice the area between the 45° line and the concentration curve for security j. Similarly, the relative extended Gini is equal to a weighted integration of the difference between the 45° line and the Lorenz curve (or concentration curve for security j). Formally it can be shown that[17]

$$\begin{aligned}\operatorname{cov}\{R_j, [1 - F_p(R)]^{\nu-1}\} &- \operatorname{cov}\{R_i, [1 - F_p(R)]^{\nu-1}\} \\ &= (\nu - 1)\bar{R} \int_0^1 (1 - F)^{\nu-2}[\phi_j(F) - \phi_i(F)]\, dF, \qquad (31)\end{aligned}$$

where on the left-hand side of (31), we have the difference between the nondiversified risk of securities i and j and, on the right-hand side, we have the weighted difference between the concentration curves of securities i and j.

Equation (31) enables us to see when the classification of risk by $\operatorname{cov}\{R_j, [1 - F_p(R)]^{\nu-1}\}$ is independent of the choice of ν.

Consider first the case where concentration curves do not intersect. Then, the ranking of securities by their riskiness, as defined by $\operatorname{cov}(R, F_p)$, (i.e., the case where $\nu = 2$) will be the same for all ν. That is, if $\operatorname{cov}[R_i, F_p(R_p)] > \operatorname{cov}[R_p, F_p(R_p)]$, then security i will be considered as *aggressive* by all investors, and it will be agreed that it is *defensive* if the inequality is reversed. However, when concentration curves intersect, the relative riskiness of security i depends on the

[17] For a derivation of a similar equation, see Yitzhaki [27, p. 621].

investor aversion to risk as weighted by ν. Thus, different investors may disagree about the relative riskiness of securities and rank them differently.

VI. Conclusion

We have presented a new approach to analyze risky prospects and construct optimal portfolios. This method has the simplicity of a two-parameter model with the efficiency of stochastic dominance. To that extent, we claim that Gini's mean difference is superior to the variance for evaluating the variability of a security. Furthermore, the concentration ratio based on the Gini coefficient permits us to classify different securities with respect to their relative riskiness. Finally we have applied the MG approach to capital markets and the security valuation theorem was derived on a general relationship between average return and risk. By extending the analysis with the mean-extended Gini method, we have explicitly introduced the degree of risk aversion as a parameter that can be determined by the specific composition of the market portfolio.

The main implications of the model reside in whether investors, in general, behave more in an MG or MEG framework rather than follow the MV approach. These implications can be empirically tested by estimating the performance of CAPM for different degrees of ν, and comparing these with the results obtained from MV-CAPM. Two-parameter portfolio models were tested using regression techniques. This was notably exemplified by Fama and MacBeth [6] who supported the hypothesis that risk-averse investors hold efficient portfolios in terms of the mean and the standard deviation of the returns. Recently, certain doubts were raised as to whether the different regression procedures were valid to test the CAPM (see Ross [17]). By proposing the mean-Gini and mean-extended Gini models as a means of evaluating capital market data, we add a new dimension to modern finance theory, suggesting that we should return to the drawing board.

Appendix

The extended Gini coefficient of a portfolio is decomposed as follows:

$$\Gamma_p(\nu) = -\nu \sum_{i=1}^{N} x_i \mathrm{cov}\{R_i, [1 - F_p(R_p)]^{\nu-1}\}.$$

Proof: By Exposition (23), the extended Gini coefficient is equal to

$$\Gamma_p(\nu) = \bar{R}_p - a - \int_a^b [1 - F_p(R)]^\nu \, dR.$$

Let

$$v = [1 - F(R)]^\nu; \qquad \frac{\partial v}{\partial R} = -\nu[1 - F(R)]^{\nu-1} f(R)$$

$$u = R; \qquad \frac{\partial u}{\partial R} = 1$$

and applying integration by parts, we get

$$\Gamma_p(\nu) = \bar{R}_p - a - [1 - F(R)]^\nu R \,|_a^b$$
$$- \nu \int_a^b [1 - F(R)]^{\nu-1} Rf(R)\, dR$$
$$= \bar{R}_p - \nu \int_a^b [1 - F_p(R)]^{\nu-1} Rf_p(R)\, dR;$$

but

$$\int_a^b [1 - F_p(R)]^{\nu-1} f_p(R)\, dR = -[1 - F_p(R)]^\nu/\nu \,|_a^b = \frac{1}{\nu}.$$

Thus,

$$\Gamma_p(\nu) = \bar{R}_p - \nu \int_a^b \left\{[1 - F_p(R)]^{\nu-1} - \frac{1}{\nu}\right\} Rf_p(R)\, dR$$
$$- \int_a^b Rf_p(R)\, dR$$
$$= -\nu \int_a^b \left\{[1 - F_p(R)]^{\nu-1} - \frac{1}{\nu}\right\} Rf_p(R)\, dR$$
$$= -\nu \operatorname{cov}\{R_p, [1 - F_p(R)]^{\nu-1}\}.$$

Therefore,

$$\Gamma_p(\nu) = -\nu \ \textstyle\sum_{i=1}^N x_i \operatorname{cov}\{R_i, [1 - F_p(R)]^{\nu-1}\},$$

since $R_p = \sum_i x_i R_i$. Q.E.D.

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