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CONSUMER'S SURPLUS, PRICE INSTABILITY, AND CONSUMER WELFARE

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This paper evaluates the benefits to consumers from price stabilization in terms of the convexity-concavity properties of the consumer's indirect utility function. It is shown that in the case where only a single commodity price is stabilized, the consumer's preference for price instability depends upon four parameters: the income elasticity of demand for the commodity, the price elasticity of demand, the share of the budget spent on the commodity, and the coefficient of relative risk aversion. All of these parameters enter in an intuitive way and the analysis includes the conventional consumer's surplus approach as a special case. The analysis is extended to consider the benefits of stabilizing an arbitrary number of commodity prices. Finally, some issues related to the choice of numeraire and certainty price in this context are discussed.

1. INTRODUCTION AND SUMMARY

BECAUSE OF THE PHENOMENAL increase in price instability for many products produced and consumed around the world, it is of increasing importance to understand the welfare effects of price stabilization brought about by government stock activities. As discussed in a recent survey paper by Turnovsky [**23**], a considerable theoretical literature has developed which focuses on this issue. However, most of the studies base their analysis on the classic concept of economic surplus. This approach was first used by Waugh [**24**] to shown that consumers facing exogenous random prices are better off than if these prices were stabilized at their arithmetic means through a buffer stock activity. Later, Oi [**13**] demonstrated that, if prices are exogenous, producers also prefer price instability. Massell [**11**] then showed—also using the surplus approach—that, if both producers and consumers are considered in a closed model, society as a whole prefers price stability, with the gainers in principle being able to compensate the losers. However, in the absence of compensation, price stability is only potentially desirable, since one group may lose. This same general conclusion was later reached by Samuelson [**18**].

The concept of consumer's surplus as a measure of economic welfare has never been fully accepted by the economics profession.[2] Hence in this paper the question of whether or not consumers benefit from price stability is re-examined using a more general utility criterion which relaxes the stringent assumptions upon which consumer's surplus is based.[3] Specifically, we evaluate the benefits from

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[2] For a survey on the concept of economic surplus, see Currie, Murphy, and Schmitz [**4**]. Recently, however, this criticism has been tempered somewhat and consumer's surplus measure is regaining some credibility as a useful criterion for applied work; see, e.g., Glaister [**7**], Willig [**25**].

[3] The advantages of the indirect utility function approach are by now widely recognized in the closely related trade and uncertainty literature; see, e.g., Ruffin [**16**], Turnovsky [**21**], and the other references cited by Flemming, Turnovsky, and Kemp [**6**].

price stabilization in terms of the consumer's indirect utility function. At the same time, we interpret the consumer's surplus measure within this more general framework, thereby enabling us to see more precisely how its welfare implications for price stabilization relate to those of the more general criterion.

An immediate consequence of adopting this more general approach is that the Arrow–Pratt coefficient of relative risk aversion comes to play a critical role in the analysis. The relationship between different measures of risk aversion and conventional demand functions has recently been studied by a number of authors; see Stiglitz [**20**], Deschamps [**5**], Hanoch [**9**]. Whereas demand functions are derived from ordinal utility theory, measures of risk aversion are defined cardinally in terms of underlying risk experiments. Consequently, as Hanoch has stressed, demand functions obtained under certainty imply relatively little information about attitudes to risk taking. On the other hand assumptions about risk attitudes, e.g. the constancy of the coefficient of relative risk aversion, do yield strong conclusions regarding the form of corresponding demand functions.

Conventional demand theory begins with a *quasi-concave* utility function $U(x_1, \ldots, x_n)$ defined in terms of commodities x_i. Constrained maximization of this function yields an indirect utility function $V(p_1, \ldots, p_n, Y)$ in terms of prices p_i and income Y, which from duality theory is *quasi-convex* in $p_1, \ldots, p_n$; see, e.g., Lau [**10**]. Risk analysis on the other hand, and in particular the analysis of price stabilization, turns on the stronger property of *concavity* or *convexity* of V in the relevant prices. But because of the quasi-convexity of V in the p_i, V can never be concave in *all* p_i. It is therefore impossible to establish conditions under which the simultaneous stabilization of all prices will definitely be beneficial. At the same time, the demonstration we shall give that variable prices may be preferable, involves strengthening the quasi-convexity of V in p to convexity, and the conditions introduced in Propositions 1, 3, 4, 5, 6, 7 below suffice to achieve this. Broadly speaking, if the number of commodities whose prices are being stabilized (denoted by m) exceeds 2, these conditions have the effect of imposing homotheticity on the direct utility function (or some separable subfunction if $m < n$), the effect of which is also to validate the use of consumer's surplus as a measure of changes in utility. Moreover, Newman [**12**] has established conditions under which concavity and quasi-concavity (and likewise convexity and quasi-convexity) are equivalent. These conditions also in effect impose homotheticity, and do not appear to yield any significant generalizations of our analysis.

In view of the fact that so much of the price stabilization literature focuses on the welfare gains resulting from stabilizing a single commodity price, this case is clearly of importance and is analyzed in some detail. It is shown that once the surplus measure is abandoned, the consumer's preference for price instability depends upon four parameters: (i) the income elasticity of demand for the commodity, (ii) the price elasticity of demand for the commodity, (iii) the share of the budget spent on the commodity in question, and (iv) the coefficient of relative risk aversion. These parameters all enter in an intuitive way. The desirability of price instability increases with the magnitudes of the two elasticities; it decreases with the degree of risk aversion; its response to an increase in the budget share is indeterminate. But while it is possible for the Waugh proposition concerning the desirability of price instability no longer to hold, for plausible parameter values one would still expect price instability to be preferred. A *sufficient* condition for this to be so is that the income elasticity of demand exceed at least half the coefficient of relative risk aversion. On the other hand, a *necessary* condition for consumers to prefer stability is that the coefficient of relative risk aversion exceed twice the income elasticity of demand. Moreover, as well as deriving these qualitative effects, we are able to derive a simple *quantitative* relationship explaining the degree of relative risk aversion with respect to price risk in terms of the degree of relative risk aversion with respect to income risk, together with the parameters in (1)–(3) above which characterize risk-free demand behavior.

The analysis is then extended to consider the benefits of stabilizing an arbitrary number of commodity prices. The analysis hinges on the convexity/concavity of the indirect utility function in terms of the prices being stabilized. Sufficient conditions for consumers to lose from the stabilization of a subgroup of prices are obtained. These are rather stringent and include the condition under which consumer's surplus is proportional to changes in utility as a special case. In particular, a sufficient condition for consumers to lose from the stabilization of all commodity prices is that the direct utility function be homothetic and that the coefficient of relative risk aversion be less than two. At the same time, because of the quasi-convexity of the indirect utility function it is impossible to establish conditions (independent of the probability distributions for prices) under which the stabilization of *all* commodity prices will definitely be beneficial.

In some respects our analysis parallels that of Hanoch [**8**, **9**], although our emphasis and many of our results are somewhat different from his, particularly his latter paper.[4] One further aspect we consider is the choice of numeraire and certainty price, an issue recently discussed by Flemming, Turnovsky, and Kemp [**6**]. These authors show how for an objective function homogeneous of degree zero, the conventional use of the arithmetic mean as the certainty price can lead to certainty-uncertainty comparisons which depend critically upon the (arbitrary) choice of numeraire. This turns out to be relevant in the present context and is illustrated by a proposition originally due to Hanoch (Proposition 7 below) in which he shows how the gains or losses from price stabilization for the case of a single composite commodity depends upon whether or not the coefficient of relative risk aversion exceeds two. In Section 4 below, we reconsider this proposition, following the procedure suggested by Flemming, Turnovsky, and Kemp, of using the geometric mean as the certainty price. We show how the comparison is now numeraire invariant, and in this case depends upon whether or not the coefficient of relative risk aversion exceeds unity. The same kinds of issues apply with respect to the other propositions. While we do not discuss these in detail, we briefly show how price stabilization at the geometric mean increases the desirability for price stability.

[4] In his first paper in particular, Hanoch develops an extensive range of propositions regarding the benefits to consumers from price stabilization under a variety of assumptions. These include the stabilization at weighted and unweighted mean prices, whether the decisions are made ex ante or ex post, the nature of the budget constraint.

Like Hanoch, as well as the original Waugh paper, our analysis is a partial equilibrium one, in the sense that we take the prices as being exogenous to consumers. Moreover, in cases where consumers are shown to prefer price instability, we recognize that society as a whole is likely to prefer price stability, if appropriate compensation is made.

2. CONSUMER'S SURPLUS AND THE INDIRECT UTILITY FUNCTION

In view of the important historical role played by the consumer's surplus measure in the stabilization literature, it is illuminating to begin by reviewing this concept and relating it to the more general notion of the indirect utility function, which forms the basis for our subsequent analysis.

Let $U(x_1, \ldots, x_n)$ be the direct utility function which an individual maximizes subject to his budget constraint, $\Sigma_{i=1}^{n} p_i x_i = Y$. $U(x)$ is assumed to be cardinal, twice differentiable, and quasi-concave.

The first-order conditions are simply:

$$
\begin{aligned}
&U_i = \lambda p_i \qquad (i = 1, \ldots, n), \\
&\sum_{i=1}^{n} p_i x_i = Y,
\end{aligned} \tag{1}
$$

where p_i are commodities prices, $p = (p_1, \ldots, p_n)$ (a row vector); x_i are quantities, $x = (x_1, \ldots, x_n)$ (a row vector); Y is money income; λ is marginal utility of income; and $U_i = \partial U / \partial x_i$. The optimality conditions (1) can then be solved to yield the resulting demand functions (assumed to be positive),

$$x_i = h_i(p_1, \ldots, p_n, Y) \qquad (i = 1, \ldots, n), \tag{2a}$$

together with the corresponding value of the Lagrange multiplier

$$\lambda = h_{n+1}(p_1, \ldots, p_n, Y). \tag{2b}$$

Inserting the solutions (2a) into the utility function U yields

$$U = U[h_1(p_1, \ldots, p_n, Y), \ldots, h_n(p_1, \ldots, p_n, Y)] \equiv V(p_1, \ldots, p_n, Y) \tag{3}$$

where V denotes the indirect utility function which is quasi-convex in the p_i. As previous authors have shown, this function is particularly convenient for the analysis of problems involving risk.

Since we are concerned with analyzing the welfare effects of price fluctuations, suppose that the prices of goods $1, \ldots, m$, where $1 \leqslant m \leqslant n$ are subject to infinitesimal changes. Assuming that money income remains fixed, and taking the differential of the budget constraint, one obtains

$$\sum_{i=1}^{m} x_i \, dp_i + \sum_{i=1}^{n} p_i \, dx_i = 0 \tag{4}$$

from which, together with the first order conditions (1), it follows that the resulting change in utility is

$$dV = dU = -\lambda \sum_{i=1}^{m} x_i \, dp_i \tag{5}$$

where $\lambda = \partial V/\partial Y \geq 0$ = marginal utility of income. Note that we have allowed for the number of commodities subject to price changes to vary between 1 and the total set n. The advantage of doing this is that it enables us to give a unified treatment of the various cases we shall want to consider below.

In order to determine the change in utility resulting from discrete changes in price it is necessary to integrate equation (5). If only one price is subject to change, this is simply a conventional one dimensional integral. However, if more than one price changes, the change in utility is given by a *line integral*, the value of which in general depends upon the path of integration. As discussed by Silberberg [**19**], and Burns [**3**], this line integral forms the basis for the various consumer's surplus measures. The well known differences among these measures are a reflection of this fundamental property of line integrals, namely that in general they depend upon the path of integration.

Hence an important question concerns the conditions under which the line integral is independent of the path of integration.[5] For the case $2 \leq m \leq n$ the necessary and sufficient conditions for path independence are[6]

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i} \quad \text{for all } i, j = 1, \ldots, m. \tag{6}$$

That is, the uncompensated cross price derivatives in the demand functions for those goods subject to price change (goods $1, \ldots, m$) are symmetric, from which it follows that the income elasticities of demand for those same goods are equal. To establish this latter proposition, consider the Slutsky equation

$$\frac{\partial x_i}{\partial p_j} = \left(\frac{\partial x_i}{\partial p_j}\right)_{\bar{U}} - x_j \frac{\partial x_i}{\partial Y} \qquad (i, j = 1, \ldots, n) \tag{7}$$

where $(\partial x_i/\partial p_j)_{\bar{U}}$ denotes the compensated price derivative. From the corresponding relationship for $\partial x_j/\partial p_i$, and using the symmetry of the compensated cross price derivatives together with (6), we immediately deduce

$$\frac{\partial x_i}{\partial Y}\frac{Y}{x_i} = \frac{\partial x_j}{\partial Y}\frac{Y}{x_j} = \eta \quad \text{say} \qquad (i, j = 1, \ldots, m). \tag{8}$$

Differentiating the consumer budget constraint yields the familiar weighted

[5] As discussed in the literature, consumer's surplus can still be a valid and accurate measure of compensating and equivalent variations in income, even when it does not accurately measure utility differences. Further, even when it is path dependent, it can be a valid, though approximate, welfare measure along the standard rectangular paths.

[6] For an extensive discussion of this issue, see Burns [**2**, **3**].

income elasticities relationship[7]

$$(9) \qquad \sum_{i=1}^{n} s_i\eta_i = 1, \quad \sum_{i=1}^{n} s_i = 1,$$

where $s_i = p_i x_i / Y$ is the share of consumer budget allocated to commodity i, $\eta_i = (\partial x_i/\partial Y)(Y/x_i)$ is income elasticity of demand for commodity i; utilizing (8) these can be written as

$$(9') \qquad \eta \sum_{i=1}^{m} s_i + \sum_{i=m+1}^{n} s_i\eta_i = 1.$$

It therefore follows that if *all* commodity prices change (i.e. $m = n$), then $\eta = 1$. However, if only a *subset* change ($m < n$), only the weaker restriction (8) is imposed; η need *not* equal unity.

Further insight is obtained by considering the indirect utility function V. The fundamental duality properties of V enable the demand functions to be expressed in terms of the indirect utility function as follows:[8]

$$(10) \qquad x_i = -\frac{\partial V/\partial p_i}{\partial V/\partial Y} = -\frac{\partial V/\partial p_i}{\lambda} \qquad (i = 1, \ldots, n).$$

Differentiating (10) with respect to Y yields

$$(11) \qquad \frac{\partial^2 V}{\partial Y \partial p_i} = -\frac{\partial^2 V}{\partial Y^2} x_i - \frac{\partial V}{\partial Y}\frac{\partial x_i}{\partial Y} \qquad (i = 1, \ldots, n)$$

or, equivalently,

$$(11') \qquad \frac{\partial \lambda}{\partial p_i} = -x_i \frac{\partial \lambda}{\partial Y} - \lambda \frac{\partial x_i}{\partial Y} \qquad (i = 1, \ldots, n).$$

The conditions under which consumer's surplus is path independent and is an accurate measure of changes in utility, can also be analyzed in terms of V. If only the prices $p_1, \ldots, p_m$ ($1 \leq m \leq n$) are subject to discrete change, then in order for $\int \Sigma_{i=1}^{m} x_i \, dp_i$ to be proportional to ΔV, it is necessary and sufficient that λ be independent of $p_1, \ldots, p_m$, and of course this condition implies the path independence of $\int \Sigma_{i=1}^{m} x_i \, dp_i$. Introducing this requirement, we have[9]

$$(12) \qquad \lambda = \lambda(p_{m+1}, \ldots, p_n, Y).$$

Integrating (12), this implies an indirect utility function of the form

$$(13) \qquad V = \psi(p_1, \ldots, p_n) + \gamma(p_{m+1}, \ldots, p_n, Y)$$

which, applying the duality condition (10), yields the following demand functions

[7] See, e.g. Samuelson [**17**].

[8] This relationship is originally due to Roy [**15**]. For a more recent discussion of duality, see, e.g., Lau [**10**].

[9] It should be made clear that the independence of λ with respect to a set of prices is a cardinal property, which of course in a context of risk preference is perfectly appropriate.

for goods $1, \ldots, m$:

$$x_i = \frac{-\psi_i(p_1, \ldots, p_n)}{\lambda(p_{m+1}, \ldots, p_n, Y)} \qquad (i = 1, \ldots, m) \tag{14}$$

where $\psi_i \equiv \partial\psi/\partial p_i$. The demand functions in (14) can be characterized as having equal income elasticities which are independent of the first m prices.[10] Moreover, differentiating x_i with respect to p_j, $j = 1, \ldots, m$, yields

$$\frac{\partial x_i}{\partial p_j} = \frac{-\psi_{ij}(p_1, \ldots, p_n)}{\lambda(p_{m+1}, \ldots, p_n, Y)} = \frac{-\psi_{ji}(p_1, \ldots, p_n)}{\lambda(p_{m+1}, \ldots, p_n, Y)} = \frac{\partial x_j}{\partial p_i} \qquad (i, j = 1, \ldots, m), \tag{15}$$

which is consistent with the integrability conditions (6).

An immediate consequence of (12) is that

$$\frac{\partial \lambda}{\partial p_i} = 0, \qquad 1 \leqslant m \leqslant n \qquad (i = 1, \ldots, m).$$

Setting these partial derivatives to zero in (11′) yields

$$x_i \frac{\partial \lambda}{\partial Y} + \lambda \frac{\partial x_i}{\partial Y} = 0, \qquad 1 \leqslant m \leqslant n \qquad (i = 1, \ldots, m),$$

and hence

$$\frac{\partial x_i}{\partial Y}\frac{Y}{x_i} = \frac{-\partial^2 V/\partial Y^2 \cdot Y}{\partial V/\partial Y}, \qquad 1 \leqslant m \leqslant n \qquad (i = 1, \ldots, m). \tag{16}$$

Defining $\rho = (-\partial^2 V/\partial Y^2 \cdot Y)/(\partial V/\partial Y)$, the income elasticity of the marginal utility of income, which in turn is the Arrow-Pratt measure of relative risk aversion, we see that for the consumer's surplus measure to be an accurate measure of changes in utility we require that the income elasticities of those goods subject to price variation all equal the coefficient of relative risk aversion.[11] It is important to emphasize that (16) holds over the entire range of m, including the important case where only one commodity is subject to price variations. At the other extreme where $m = n$, so that all prices change, all income elasticities, and hence the coefficient of relative risk aversion ρ, must equal unity; see (9′). In this case the indirect utility function is of the form

$$V = \psi(p_1, \ldots, p_n) + k \ln Y, \quad k \text{ constant} > 0, \tag{17}$$

implying corresponding demand functions

$$x_i = -\psi_i(p_1, \ldots, p_n)Y/k \qquad (i = 1, \ldots, n). \tag{18}$$

Demand functions of the form (18) are obtained if and only if the underlying direct utility function is homothetic. The restrictiveness of this latter assumption has

[10] This interpretation was pointed out to us by a referee. It is an immediate consequence of applying the definition of income elasticity to (14).

[11] See Arrow [**1**], Pratt [**14**].

been widely discussed throughout the literature, and makes explicit the special nature of surplus measures.

3. THE EFFECTS OF PRICE STABILIZATION

As in the original Waugh [24] analysis, it is assumed that prices are exogenous to the consumer and that through buffer stock activities they can be stabilized at their respective arithmetic means. With these assumptions and using consumer's surplus measures, Waugh demonstrated that consumers prefer price instability.

In view of the restrictive assumptions underlying the use of consumer's surplus, we shall evaluate the benefits from stabilization in terms of the indirect utility function $V(p_1, \ldots, p_n, Y)$. As long as some prices fluctuate randomly, V is also a random variable. We suppose that a subset of prices $p_1, \ldots, p_m$, $1 \leq m \leq n$, are stabilized at their arithmetic means $\bar{p}_1, \ldots, \bar{p}_m$, and we evaluate the benefits from stabilization by comparing the expected utility under variable prices with the utility obtained when the stabilized prices are at their means. That is, the evaluation involves comparing

$$E\{V[p_1, \ldots, p_m, p_{m+1}, \ldots, p_n]\} \quad \text{with}$$

$$V[\bar{p}_1, \ldots, \bar{p}_m, p_{m+1}, \ldots, p_n],$$

where the expectation E is over the subset of stabilized prices $p_1, \ldots, p_m$. Assuming that the prices have finite variances and are not subject to any perfect linear relationship between them, this comparison clearly hinges on the convexity/concavity properties of V in terms of the prices being stabilized.[12] As before we are letting m extend over the range $(1, n)$ in order to allow for the possibility that only a subset of commodity prices are stabilized. We begin by considering the case $m = 1$.

Stabilization of a Single Commodity Price

The case $m = 1$ corresponds to much of the partial equilibrium analysis in the more applied literature and for that reason alone merits separate consideration. It is also the case for which the sharpest results can be obtained. Assuming that the price being stabilized is p_1, we see that consumers lose (gain) from stabilization according as

$$\frac{\partial^2 V}{\partial p_1^2} > (<) 0. \tag{19}$$

To evaluate $\partial^2 V/\partial p_1^2$ differentiate both sides of (10) for $i = 1$, with respect to p_1, yielding

$$\frac{\partial^2 V}{\partial p_1^2} = \frac{-\partial^2 V}{\partial Y \partial p_1} x_1 - \frac{\partial V}{\partial Y}\frac{\partial x_1}{\partial p_1}. \tag{20}$$

[12] This involves the use of a multivariate extension of Jensen's inequality. For a precise statement of the necessary theorem, expressed in a form convenient for the present context, see Turnovsky [22].

Substituting for $\partial^2 V/\partial Y \partial p_1$ from (11) we obtain

$$\frac{\partial^2 V}{\partial p_1^2} = x_1^2 \frac{\partial^2 V}{\partial Y^2} + \frac{\partial V}{\partial Y}\left(x_1 \frac{\partial x_1}{\partial Y} - \frac{\partial x_1}{\partial p_1}\right) \tag{21}$$
$$= x_1^2 \frac{\partial^2 V}{\partial Y^2} + \frac{\partial V}{\partial Y}\left(2x_1 \frac{\partial x_1}{\partial Y} - \left(\frac{\partial x_1}{\partial p_1}\right)_{\bar{U}}\right).$$

Now using the definitions introduced above for ρ, η_i, and defining

$$e_i = \frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i}$$

as the own uncompensated price elasticity of demand for commodity i, $i = 1, \ldots, n$, (21) can be written as

$$\frac{\partial^2 V}{\partial p_1^2} = \frac{x_1 \partial V/\partial Y}{p_1}(s_1(\eta_1 - \rho) - e_1). \tag{21'}$$

Assuming $x_1 > 0$, then, provided the consumer is unsatiated so that $\partial V/\partial Y > 0$, the criterion determining the gains or losses from price stabilization can be expressed in terms of

$$\operatorname{sgn}\left(\frac{\partial^2 V}{\partial p_1^2}\right) = \operatorname{sgn}\,(s_1(\eta_1 - \rho) - e_1). \tag{22}$$

From (22) it is seen that whether or not consumers benefit from the stabilization of p_1 depends upon: (i) the own price elasticity of demand for good one; (ii) the coefficient of relative risk aversion; (iii) the share of the consumer's budget allocated to good one; (iv) the income elasticity of demand for good one. More specifically, it follows from (22) that the desirability of price stability decreases with the magnitude of the price and income elasticities but increases with the coefficient of relative risk aversion. The direction of the impact of an increase in the budget share depends upon $(\eta_1 - \rho)$. While it is certainly possible for $\partial^2 V/\partial p_1^2 < 0$ (in which case the consumer gains from price stabilization, contrary to the Waugh proposition), empirical evidence would tend to suggest that $\partial^2 V/\partial p_1^2 > 0$, in which case the Waugh result remains true. For example, taking $e = -0.2$, $\eta = 0.6$, $\rho = 1$, $s_1 = 0.3$ as typical estimates for a composite commodity such as food, (22) is certainly found to be positive.

Using (10), equation (21') can be written in the following illuminating form:

$$\frac{-\partial^2 V/\partial p_1^2 \cdot p_1}{\partial V/\partial p_1} = s_1(\eta_1 - \rho) - e_1. \tag{23}$$

The term on the left hand side is the elasticity of the marginal indirect utility with respect to price and therefore measures the coefficient of relative risk aversion with respect to price risk; i.e., σ is defined analogously to ρ, the coefficient with respect to income risk.[13] Thus equation (23) provides a simple explanation of the degree of relative risk aversion with respect to price risk, in terms of the coefficient of relative risk aversion with respect to income risk, together with the parameters s_1, η_1, e_1, which describe properties of the riskfree demand function.[14] The significance of (23) is that it provides a *quantitative* (as well as qualitative) measure of assessing the effects of price instability. For example, taking the parameter values, given in the paragraph above, (23) implies a coefficient of relative risk aversion with respect to price risk of -0.32.

Since our main concern is with deriving qualitative propositions, we revert to the qualitative version of the criterion (22). From the Slutsky equation (7), this can be expressed in the equivalent form[15]

$$\text{sgn}\left(\frac{\partial^2 V}{\partial p_1^2}\right) = \text{sgn}\,(s_1(2\eta_1 - \rho) - e_1^c) \tag{24}$$

where

$$e_i^c = \left(\frac{\partial x_i}{\partial p_i}\right)_{\bar{U}} \cdot \frac{p_i}{x_i}$$

is compensated own price elasticity of demand for commodity i, $i = 1, \ldots, n$. Using the fact that $e_i^c < 0$, enables us to state the following formal propositions:

PROPOSITION 1: *A sufficient condition for consumers to lose from the stabilization of the price p_1 is*

$$2\eta_1 - \rho \geq 0, \tag{25}$$

i.e., the income elasticity of demand for good one must not be less than half the coefficient of relative risk aversion.

PROPOSITION 2: *A necessary condition for consumers to prefer p_1 to be stabilized is*

$$2\eta_1 - \rho < 0,$$

i.e., the coefficient of relative risk aversion exceed twice the income elasticity of demand.

[13] While the elasticity measure appearing on the left-hand side of (23) has natural appeal, it must nevertheless be interpreted with some caution. For example, since $\partial V/\partial p < 0$, the sign of this measure is reversed from that of the corresponding measure with respect to income risk. It is also possible, if income is endogenous, for $\partial V/\partial p$ to become positive, changing the sign convention, or for $\partial V/\partial p = 0$, in which case the expression is not defined.

[14] We are grateful to a referee for suggesting this alternative interpretation of (21′) to us.

[15] Using the fact that

$$(\partial x_i/\partial p_i)_{\bar{U}} = \partial^2 e(p_1, \ldots, \bar{U})/\partial p_i^2$$

where $e(p_1, \ldots, p_n; \bar{U})$ is the minimum expenditure function, it is possible to interpret (24) in terms of the concavity of e with respect to prices.

PROPOSITION 3: *If the underlying direct utility function is homogeneous of degree one in quantities, then $\partial^2 V/\partial Y^2 = 0$, in which case consumers are risk neutral and will prefer price instability.*

Note as a further point that Proposition 1 includes the consumer's surplus measure as a special case. As we have shown above (see (16)) consumer's surplus is an accurate measure of changes in utility if $\eta_1 = \rho > 0$, when the inequality (24) is obviously met. In this case the uncompensated demand curve is downward sloping and it follows from (22) that

$$(22') \qquad \operatorname{sgn}\left(\frac{\partial^2 V}{\partial p_1^2}\right) = \operatorname{sgn}(-e_1) > 0.$$

The result that consumers lose from price stabilization stems solely from the demand curve being downward sloping, precisely as in the original Waugh analysis.

We should emphasize that Propositions 1–3 are based on the (conventional) assumption that income Y is exogenous. In a situation where income is price-dependent, such as in a trade or production model, the basic criterion (21) has to be modified to allow for the *net* trade position. Clearly these propositions would have to be revised accordingly. In some cases the restrictions can be relaxed; in others they will need to be strengthened.

With these comments in mind, our single price stabilization results can be summarized as follows. The original Waugh proposition based on consumer's surplus measures tends to be supported by the present more general utility evaluation. For plausible parameter values one would expect consumers to prefer price instability, particularly if s_1, the share of the budget allocated to the commodity is small, and the degree of risk aversion is low. However, for a sufficiently high degree of risk aversion, price stabilization can definitely prove to be beneficial.

Stabilization of an Arbitrary Number of Commodity Prices

We now consider the general case where m commodity prices $2 \leqslant m \leqslant n$ are subject to stabilization. The convexity/concavity properties of V in terms of $p_1, \ldots, p_m$ are summarized by the matrix

$$A = \left(\frac{\partial^2 V}{\partial p_i \partial p_j}\right), \quad 2 \leqslant m \leqslant n \qquad (i, j = 1, \ldots, m).$$

Consumers will prefer unstable to stable prices if A is positive definite; they will gain from stabilization if A is negative definite; if A is neither positive nor negative definite, nothing can be inferred about the benefits from stabilization without additional information on the relevant probability distributions.

The elements of A are obtained as before by differentiating (10) for commodity i, with respect to p_j, $j = 1, \ldots, m$. While the resulting expressions can be obtained in various ways, the following form proves to be most convenient

$$(27)\qquad A=\begin{pmatrix} \frac{\lambda x_1^2}{Y}(2\eta_1-\rho)-\lambda k_{11} & \cdots & \frac{\lambda x_1 x_m}{Y}(\eta_1+\eta_m-\rho)-\lambda k_{1m} \\ \vdots & & \vdots \\ \frac{\lambda x_m x_1}{Y}(\eta_1+\eta_m-\rho)-\lambda k_{m1} & \cdots & \frac{\lambda x_m^2}{Y}(2\eta_m-\rho)-\lambda k_{mm} \end{pmatrix}$$

where

$$k_{ij}=\left(\frac{\partial x_i}{\partial p_j}\right)_{\bar{U}}$$

are the price effects of the compensated demand curves. The substitution matrix $K=(k_{ij})$ $i, j=1,\ldots,n$, is negative definite, and the same property applies to any submatrix consisting of the first m commodity prices.[16]

The diagonal elements of (27) are of course identical to the terms appearing in (24). Indeed the factors determining whether or not consumers benefit from stabilization are direct generalizations of those for the $m=1$ case, including now in particular cross price effects. By examining the matrix A it is seen that the conditions for consumers to lose from stabilization are now more stringent than for the single commodity price stabilization case. Whereas the condition

$$(28)\qquad 2\eta_i-\rho\geqslant 0 \qquad (i=1,\ldots,m)$$

is sufficient for losses from stabilization if the price of commodity i alone is stabilized, this is no longer sufficient for a loss in welfare if the prices of other commodities are simultaneously stabilized. This can be immediately verified by considering the case where $m=2$.

Sufficient conditions for A to be positive definite and therefore for consumers to lose from price stabilization are given by

$$(29a)\qquad 2\eta_i-\rho\geqslant 0 \qquad (i=1,\ldots,m),$$

$$(29b)\qquad \eta_i=\eta \qquad (i=1,\ldots,m).$$

Note that provided $m<n$ the common value of the income elasticities in (29b) need *not* equal ρ, the value for which consumer's surplus is path independent; it can be any value consistent with the budget constraint (9′).

The proof of this assertion is readily established by invoking (29b) and writing A in the form

$$(30)\qquad A=\frac{\lambda}{Y}(2\eta-\rho)x'x-\lambda K$$

[16] See Samuelson [**17**].

where the prime denotes vector transpose. Consider now the quadratic form $u'Au$ where $u' = (u_1, \ldots, u_m)$ is an arbitrary row vector. Thus we find

$$u'Au = \frac{\lambda}{Y}(2\eta - \rho)(xu)'(xu) - \lambda u'Ku.$$

The term $u'Ku > 0$, by virtue of the negative definiteness of K, while the term $(xu)'(xu)$ being an exact square is nonnegative.[17] Hence provided $\lambda > 0$, condition (29) implies $u'Au > 0$, so that A is positive definite. We may summarize this result with the following proposition:[18]

PROPOSITION 4: *Sufficient conditions for consumers to lose from the stabilization of a subgroup of prices* ($m < n$) *is that commodities in the subgroup all have equal income elasticities of demand and that this common elasticity not be less than half the coefficient of relative risk aversion.*

One important case in which the conditions of Proposition 4 will be met is if the direct utility function U is homothetically separable in the subgroup of commodities whose prices are being stabilized. That is, U is of the form

$$U = F[U^1(x_1, \ldots, x_m), U^2(x_{m+1}, \ldots, x_n)] \tag{31}$$

where U^1, U^2 are homogeneous of degree 1 and F is positive, finite, continuous, and strictly monotonically increasing in its two arguments with $F(0,0) = 0$. Calculating the income elasticities of demand from equation (10) and using results of Lau [**10**] (in particular Theorem 4, Corollary 1, and Lemma 1), we can show that the utility function (31) implies $\eta_i = \rho$, $i = 1, \ldots, m$. The notion of homothetic separability is rather intuitively appealing and thus strengthens the applicability of this proposition.

Note also that in the special case for which consumer's surplus is both path independent and measures changes in utility, we again have from (16) that $\eta_i = \eta = \rho$, so that again both (29a), (29b) are met. This enables us to state:

PROPOSITION 5: *Imposing the restrictions under which consumer's surplus is path independent and measures changes in utility, consumers will lose from having a subgroup* ($m < n$) *of commodities stabilized, thereby generalizing the single commodity result.*

Finally, we turn to the case where $m = n$ so that *all* commodity prices are to be stabilized. In this case condition (16) together with (9′) implies that the common value of η in (29b) is unity, so that (29a) becomes

$$2 - \rho \geqslant 0. \tag{29a′}$$

[17] With x defined to be a row vector, and u as a column vector, (xu) is obviously a scalar.

[18] In connection with Propositions 4 and 5, it is clear that if the subset m prices move proportionately, related by $p_i = \bar{p}_i\varepsilon$, $i = 1, \ldots, m$, $m < n$ (and are therefore perfectly correlated), the corresponding commodities can be viewed as a single commodity, in which case Propositions 1–3 are applicable. The case where all prices move proportionately is given in Proposition 7.

Moreover, as discussed in Section 2, the conditions for consumer's surplus to measure changes in utility in this case requires $\eta_i = \rho = 1$, in which case (29a′) will also be met. We may thus state:

PROPOSITION 6: *A sufficient condition for consumers to lose from the stabilization of all prices is that all income elasticities be unity (i.e. the utility function be homothetic) and that the coefficient of relative risk aversion ρ be less than 2. For consumer's surplus to be an exact measure of utility change in this case requires $\rho = 1$, in which case this measure will also (correctly) indicate a loss from stabilization. Proposition 6 is essentially the same as Theorem 3 (ii) of Hanoch* [**9**].

Hanoch also establishes some further propositions, which we restate for completeness, using present notation, and reinterpret in our present context.

PROPOSITION 7: *If all prices move proportionately (i.e. in effect there is a single composite commodity) and $\rho < 2$, consumers will lose from price stabilization irrespective of the homotheticity of the underlying utility function.*

PROPOSITION 8: *A cannot be negative definite unless $m = n = 1$ and $\rho > 2$. Hence V cannot be strictly concave in any subset of prices containing more than one element.*

Proposition 7 can be immediately established by writing $p_i = \bar{p}_i\varepsilon$, where ε is a random variable, and by considering the convexity of V in ε. Proposition 8 follows from the fact that V is quasi-convex in p and therefore in general cannot be strictly concave. In the present context this means that there are *no* conditions (independent of the underlying probability distributions for p_i) which will ensure that the simultaneous stabilization of any subset of commodity prices will positively benefit consumers. Provided V is not convex (when we know they will lose), whether consumers gain or not will depend in part upon the parameters of the relevant probability distributions.

Strengthening the condition on the underlying direct utility function U to concavity (from quasi-concavity) is equivalent to the assumption of risk aversion in the Arrow-Pratt sense; that is $\partial^2 V/\partial Y^2 < 0$ if and only if U is concave in x. Hanoch summarizes this relationship by interpreting it to mean that risk aversion with respect to commodity bundles is equivalent to risk aversion with respect to income. Risk neutrality in the Arrow-Pratt sense implies $\partial^2 V/\partial Y^2 = 0$, in which case the indirect utility function is

$$V = \psi(p_1, \ldots, p_n) + \gamma(p_1, \ldots, p_n)Y$$

implying demand functions of the form

$$x_i = \frac{-[\psi_i(p_1, \ldots, p_n) + \gamma_i(p_1, \ldots, p_n)Y]}{\gamma(p_1, \ldots, p_n)}. \quad (32)$$

In the case where only one commodity price is stabilized risk neutrality was shown to ensure that consumers lose from price stabilization. However, for $m>1$, this proposition cannot be established conclusively as can be seen by setting $\rho=0$ in (27) for the case $m=2$. A necessary and sufficient condition for the demand function (32) to meet the requirements of Proposition 6 is

$$\psi_i=0 \qquad (i=1,\ldots,n)$$

in which case (32) reduces to (18).

4. CHOICE OF NUMERAIRE AND CERTAINTY PRICE

A basic property of the indirect utility function is that it is homogeneous of degree zero in prices and income. This means that its value is unchanged if one divides through by some numeraire commodity and works in terms of normalized prices. While we have not introduced an explicit numeraire commodity, it is clear that implicitly the numeraire is money income, so that we are essentially dealing with real prices of the form $q_i=p_i/Y$.

As Flemming, Turnovsky, and Kemp [**6**] have recently discussed, the concavity/convexity properties of functions such as V, which in effect use the arithmetic mean of the normalized price as the basis for the certainty comparison, may often depend critically upon the choice of numeraire. This of course is unsatisfactory, since the choice of numeraire is presumably an arbitrary one. Flemming, Turnovsky, and Kemp have shown how with a function homogeneous of degree zero, this numeraire dependence can be avoided if instead of using the arithmetic mean, the *geometric mean* of the relevant random variable(s) is used as the appropriate certainty price.

The issue can be illustrated most clearly in the case of Proposition 7, which assumes in effect only one (composite) commodity, and only one price, satisfying the budget constraint

$$px=Y$$

where p, x are now scalars. With no optimization possible, the indirect utility function is simply

$$V(Y/p)\equiv V(1/q). \tag{33}$$

Treating Y as given, Proposition 7 (originally due to Hanoch) involves determining the sign of $\partial^2 V/\partial q^2$. Differentiating (33) twice with respect to q, and using the definition of ρ, this is given by

$$\operatorname{sgn}\left(\frac{\partial^2 V}{\partial q^2}\right)=\operatorname{sgn}(2-\rho) \tag{34}$$

and inspection of (34) immediately yields the proposition. On the other hand, because V is homogeneous of degree zero in p and Y, one would expect fluctuations in p and Y to have the same qualitative effects on consumer's welfare.

This however, is not so. Calculating $\partial^2 V/\partial Y^2$, we see

$$\text{(35)} \qquad \operatorname{sgn}\left(\frac{\partial^2 V}{\partial Y^2}\right) = \operatorname{sgn}(\rho)$$

which depends solely upon the sign of risk aversion. Consumers lose from fluctuations in income if and only if they are risk averse in the Arrow-Pratt sense (which of course is the basis for the definition!). Comparing (34) and (35), it is seen that the welfare implications of stability depend upon the choice of numeraire.

The procedure proposed by Flemming, Turnovsky, and Kemp is to consider the geometric mean of the random variable as the appropriate certainty price. In this case

$$\text{(36)} \qquad V(Y/p) \equiv V[e^{\ln Y - \ln p}] \equiv V[e^{z-r}]$$

where $z = \ln Y$, $r = \ln p$. The desirability of price stabilization depends upon the convexity/concavity of V in $r = \ln p$; i.e. upon sgn $\partial^2 V/\partial r^2$. Differentiating (36) twice with respect to r, this is determined by

$$\text{(37)} \qquad \operatorname{sgn}\left(\frac{\partial^2 V}{\partial r^2}\right) = \operatorname{sgn}(1-\rho).$$

Hence whether or not consumers benefit from having p stabilized at its *geometric mean* depends upon whether or not the coefficient of relative risk aversion exceeds unity. Moreover, precisely the same criterion is obtained if the fluctuations are in income rather than price. In this case the effects of stabilization depend upon $\partial^2 V/\partial z^2$, which by differentiation is given by

$$\text{(38)} \qquad \operatorname{sgn}\left(\frac{\partial^2 V}{\partial z^2}\right) = \operatorname{sgn}(1-\rho),$$

illustrating the numeraire-invariance property.

The traditional measures of risk aversion are based on the presumption that the appropriate certainty measure is the arithmetic mean and that the increase in risk consists of an arithmetic mean preserving spread (AMPS). As a further point, Flemming, Turnovsky, and Kemp argue that where the certainty price is the geometric mean and the increase in risk is defined by a geometric mean preserving spread (GMPS), the appropriate definition of risk aversion should be defined in terms of utility of the logarithm $H(\ln X)$ say, where X denotes the random variable. That is, a person shows risk aversion (risk preference) with respect to a GMPS according as $H''(\ln X) \lesseqgtr 0$. Also analogous to the usual arithmetic definitions they define the coefficient of absolute risk aversion with respect to a GMPS as

$$a = \frac{-H''(\ln X)}{H'(\ln X)}$$

with the coefficient of relative risk aversion defined analogously to ρ above.

Defining

(39) $G(X) \equiv H(\ln X)$

and differentiating (39) twice with respect to X immediately yields

$$\frac{-H''(\ln X)}{H'(\ln X)}+1=\frac{-G''(X)X}{G'(X)}$$

and hence

(40) $a+1=\rho,$

indicating a very simple relationship between relative risk aversion with respect to an AMPS and absolute risk aversion with respect to GMPS. Using (37) and (40), corresponding to Proposition 7 we can state:

PROPOSITION 7′: *If all prices move proportionately (i.e. there is a single composite commodity), consumers will gain or lose from having price stabilized at its geometric mean according to whether they are risk averse or risk loving with respect to a GMPS.*

The same kinds of issues are raised with respect to the other propositions discussed above. While we do not propose to discuss these modifications in detail, it is worth reconsidering the case where p_1 is to be stabilized at its geometric rather than its arithmetic mean. In this case the indirect utility function is

$$V(p_1, p_2, \ldots, p_n, Y) \equiv V(e^{r_1}, p_2, \ldots, p_n, Y)$$

from which it can be shown that the desirability of stabilization depends upon

(25′) $$\operatorname{sgn}\left(\frac{\partial^2 V}{\partial r_1^2}\right)=\operatorname{sgn}\,(s_1(2\eta_1-\rho)-e_1^c-1).$$

Thus the critical criterion is modified to include an additional term -1, the consequence of which is that the simple sufficiency condition (26) is no longer applicable. Indeed, the extra term tends to make price stabilization much more desirable. This is hardly surprising since the geometric mean is lower than the arithmetic mean and clearly stabilization at a lower price is relatively more attractive to consumers.

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