Marginal Conditional Stochastic Dominance

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This paper introduces the concept of Marginal Conditional Stochastic Dominance (MCSD), which states the conditions under which all risk-averse individuals, when presented with a given portfolio, prefer to increase the share of one risky asset over that of another. MCSD rules also answer the question of whether all risk-averse individuals include a new asset in their portfolio when assets' returns are correlated. MCSD criteria are expressed in terms of the probability distributions of the assets and of the underlying portfolio. An empirical application of MCSD is provided using stocks traded on the New York Stock Exchange. MCSD rules are used to show that, in the long run, one cannot assert that the market portfolio is inefficient.
(*Stochastic Dominance; Gini; Market Efficiency; Concentration Curves; Portfolio Diversification*)

1. Introduction

Portfolio theory deals with the choice of investments that maximize an investor's expected utility. For this purpose, a complete utility function needs to be specified, and one must know the joint distribution of all investment alternatives. These requirements are an obstacle to any empirical application that derives an optimal portfolio.

There are two approaches to meeting such demanding conditions. The first is to restrict either the set of permissible distribution functions or the type of utility functions to a specific class, such as one based on the mean and the variance. Using this assumption, Tobin's (1958) *portfolio separation theorem* provides a solution that separates the portfolio decision process from the utility-maximization question. Necessary and sufficient conditions for separation were also established by Cass and Stiglitz (1970), who completely characterize the set of utilities allowing separation, and by Ross (1978), who investigated asset distributions and their interrelationships. These methods allow the determination of relationships between asset prices.

The alternative approach is not to restrict the utility function, but rather to derive weak conditions based on probability distributions. This is the basis of the theory of stochastic dominance. First-degree stochastic dominance establishes conditions based on preferences by all investors with increasing utility functions. Second-degree stochastic dominance (SSD) considers investors with concave and increasing utilities. SSD states the probabilistic conditions under which all risk-averse individuals prefer one risky asset over another.[1]

Bawa's (1982) and Levy's (1992) extensive research bibliographies indicate that SSD conditions perform well in applied economics and finance when the decision problem is preference for a single asset or policy. But in optimal portfolio selection, as Bawa (1975) shows, SSD performs rather poorly. Indeed, Fishburn (1978) and Bawa et al. (1985) demonstrate that a convex combination of securities can dominate a given portfolio in terms of SSD even if individual securities do not. Hence one has to search through all possible combinations of portfolios to find the efficient one.

To obtain an efficient portfolio using SSD, all the conditions necessary for dominance must be met. One of these conditions states that the expected return of a dominating portfolio must be at least equal to the expected return of a dominated portfolio (otherwise a risk-neutral investor would prefer the latter). Hence, if some assets in the portfolio have different expected returns, no efficient (stochastically dominating) portfolio can be

[1] Since these weak assumptions are widely accepted by economists as representing investor behavior, this paper considers only SSD.

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found. To see this, assume that portfolio A is in the SSD-efficient set. Then one can always find another portfolio, B, with an expected return higher than portfolio A, implying that A does not dominate B. This is one basic reason why identifying efficient portfolios by SSD is problematic. It is therefore necessary to provide a weaker SSD concept that will perform satisfactorily with portfolios.

Moreover, since economic circumstances change, optimal behavior requires that individuals respond by altering existing positions. In practice, however, most investors do not adjust their portfolio positions as frequently as economic situations change: once adopted, positions are not altered casually because of economies of scale, trade restrictions, or the investor's engouement for certain assets. Most investors partly revise their holdings without altering the core of the portfolio.

Marginal Conditional Stochastic Dominance (MCSD) states the probabilistic conditions under which all risk-averse individuals, given a portfolio of assets, prefer to increase the share of one risky asset over that of another. MCSD is a more confining concept than SSD because it considers only marginal changes in holding risky assets in a given portfolio.

MCSD rules can be interpreted as an extension of Arrow's theorem—"a risk-averter takes no part of an unfavourable or barely fair gamble; on the other hand *he always takes some part of a favourable gamble*"—(Arrow 1970, p. 100; italics in the original). MCSD, however, considers investors with random wealth (i.e., who hold a given portfolio with random return). We generalize Arrow's theorem and ask under what conditions will every risk averter who holds a given portfolio take some part of the gamble, where the gamble lies in increasing the holding of one asset at the expense of another. That is, in a random portfolio context, MCSD sorts out the favorable gambles.

In this paper we develop the conditions for MCSD and apply the rules to financial market data. We borrow the concept of concentration curves from the income-distribution literature and demonstrate their importance in constructing MCSD criteria. We also introduce the necessary conditions for MCSD using the mean and the systematic risk of individual investments. Using New York Stock Exchange data, we test the efficiency of the market portfolio. We first obtain the sets of securities that MCSD dominate others given the market portfolio for three different periods. These sets enable us to identify securities whose shares should be altered in order to construct alternative portfolios that dominate the market portfolio.

The fact that some securities dominate others in the MCSD sense indicates that the market portfolio is not efficient under second-degree stochastic dominance as well, implying that no expected-utility maximizer will hold the market portfolio. But pooling the three data sets and considering a longer time span negates the dominance, implying that in the long run the market cannot be shown to be inefficient.

The following section provides a brief introduction to second-degree stochastic dominance. Marginal conditional stochastic dominance criteria are developed in §3. In §4 we show the implications of MCSD with respect to portfolio diversification and the introduction of new assets. In §5 we adapt MCSD methodology to suit financial market data. Section 6 applies the MCSD criteria to New York Stock Exchange securities using CRSP daily returns data.

2. Stochastic Dominance

Second-degree Stochastic Dominance (SSD) expresses the conditions under which all risk-averse individuals prefer one risky asset to another. Necessary and sufficient conditions for SSD were developed independently by Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970). The literature on stochastic dominance has recently been reassessed by Levy (1992).

Let F and G be the cumulative distributions of two risky alternatives, X and Y, respectively, each with finite mean. By definition, X dominates Y by SSD if, for all nondecreasing concave utility functions U,[2]

$$E_F U(X) \geq E_G U(Y), \tag{1}$$

where $E_F U(X)$ and $E_G U(Y)$ are expected utilities using distributions F and G respectively.

[2] X and Y denote end-of-period wealth under the two alternatives.

THEOREM 1. *X dominates Y by SSD if and only if*

$$\int_{-\infty}^{z} [G(t) - F(t)]dt \geq 0 \quad \text{for all } z, \quad -\infty \leq z \leq \infty. \quad (2)$$

Let us review some of the problems associated with applications of Theorem 1 in a portfolio context.

(1) For portfolio-choice problems, the practical application of stochastic dominance is rather restricted because it involves infinite pairwise comparisons of alternative probability distributions. Unless the set of probability distributions is restricted, efficient portfolio selection and risk diversification using SSD rules present insuperable computational problems (Hadar and Russell 1974, Levy and Levy 1982). Applied research in portfolio choice therefore favors alternative parametric approaches that, under specific conditions, are consistent with expected utility (Ali 1975). Most models use the expected value and a measure of dispersion that accounts for risk. The most commonly used risk parameters are: variance, semi-variance (Porter 1973), mean absolute deviation, and Gini's mean difference (Yitzhaki 1982). These models provide general necessary conditions for SSD, with sufficient conditions satisfied only in specific cases.

(2) Portfolio theory implies a continuous search for efficiency. Once the investor is faced with new alternatives, the entire optimization process must be rerun, because the additional investment options affect the optimal decision. Furthermore, in some instances the initial investment position cannot be altered as, for example, investors' future wages and income, Social Security benefits, or assets with low liquidity. Thus, investors must decide upon new alternatives without necessarily trading their entire portfolio.

(3) We can view the search for efficiency as involving a continuous process of marginal decisions. If the entire portfolio is not eligible for change, we look instead at individual assets. The criterion of conditional stochastic dominance lets us answer the question: Given a portfolio, under what conditions will all risk-averse individuals prefer one particular option over another, provided they have to hold the rest of the portfolio? Conditions satisfying this problem were established by Jewitt (1987).

Our aim is to reformulate the portfolio-optimization problem, generalizing the concept of stochastic dominance so that it can be applied more readily to financial markets. We define Marginal Conditional Stochastic Dominance (MCSD) as follows: Given a portfolio of risky assets, under what conditions do all risk-averse investors prefer *marginally* increasing the share of one asset over another? MCSD is not an alternative to SSD; it is an instrument used to reach SSD.

3. Marginal Conditional Stochastic Dominance Criteria

Assume you are analyzing the portfolio of an expected utility-maximizing individual who holds a portfolio of risky assets and has the opportunity to invest in a new asset. Under what conditions would you recommend that the individual should include the new asset in his portfolio by reducing, at the margin, the share of an existing asset in the portfolio? Stated another way, assume that the share of one asset is marginally increased at the expense of an alternative asset, keeping the initial wealth constant. Under what conditions will all risk-averse individuals prefer the change?

Consider an investor with a concave utility function, $U(\cdot)$, who holds a portfolio of n assets. Let W_0 be initial wealth, W final wealth, and r_i the rate of return on asset i. The portfolio $\{\alpha\}$ is defined by the shares α_i such that:

$$\sum_{i=1}^{n} \alpha_i = 1, \quad (3)$$

while final wealth is defined by:

$$W = W_0\left(1 + \sum_{i=1}^{n} \alpha_i r_i\right). \quad (4)$$

Given portfolio $\{\alpha\}$, is there an asset k that, if increased by reducing asset j, will lead to a change that will be preferred by all risk-averse individuals? The answer to this question determines the MCSD criterion.

Let $d\alpha_k$ be the marginal change in holding asset k. From (3):

$$d\alpha_k + d\alpha_j = 0. \quad (5)$$

Hence the marginal change in expected utility is

$$dEU(W) = EU'(W)dW = E_r U'(W)\cdot W_0 \cdot (r_k d\alpha_k + r_j d\alpha_j), \quad (6)$$

where E_r is the expectation with respect to all assets' returns. Inserting (5), and assuming that $d\alpha_k$ is positive, yields

$$\frac{dEU(W)}{d\alpha_k} = EU'(W)\cdot W_0 \cdot (r_k - r_j). \quad (7)$$

Asset k is said to dominate asset j, given portfolio $\{\alpha\}$, if Equation (7) is positive for all risk-averse individuals.[3] Equation (7) is similar in spirit to Arrow's result, but refers to random wealth.

While Arrow shows how absolute risk aversion leads to an increase in the risky asset as wealth grows, our MCSD theorem presents the necessary and sufficient conditions for dominance in terms of Absolute Concentration Curves (ACCs). Widely used in income inequality studies, these functions are less common in the field of finance. We therefore first explain and define the ACC concept and then present the theorem.

Let P be the portfolio's rate of return:

$$P = \sum_{i=1}^{n} \alpha_i r_i. \quad (8)$$

We define $\mu_i(p)$ as the conditional expected rate of return on asset i when the portfolio's return equals p:

$$\mu_i(p) = E(r_i | P = p).$$

The conditional expected return on asset i shows the contribution of asset i to the portfolio, given a return p. This is a basic element that is also used to compute the asset's expected return and its systematic risk (beta) with respect to the portfolio.[4] In a sample of discrete observations, one estimates the conditional expected return $\mu_i(p)$ by following these steps: (1) One finds the set of assets' returns that yield a return p on the given portfolio. (2) One averages all the realizations of asset i in that set. The ACC of asset i with respect to portfolio $\{\alpha\}$ is defined as the cumulative conditional expected return on asset i as a function of the cumulative distribution of the portfolio.[5]

Formally:

$$ACC_i^{\alpha}(\xi) = \int_{-\infty}^{p} \mu_i(t) f_\alpha(t)dt \quad \text{for } \infty \geq p \geq -\infty, \quad (9)$$

where p is *implicitly* defined by the cumulative distribution:

$$\xi = \int_{-\infty}^{p} f_\alpha(t)dt. \quad (10)$$

To compute the ACC of an asset one first determines the values for ξ and then uses Equation (10) to calculate p. For a given probability, ξ, the ACC is the cumulative return on asset i, given that the portfolio's return is less than p. In the special case where the portfolio's rates of return[6] are used instead of $\mu_i(p)$ [in Equation (9)], the resultant absolute concentration curve is labeled the Absolute Lorenz Curve (ALC) for portfolio $\{\alpha\}$:[7]

$$ALC_\alpha(\xi) = \int_{-\infty}^{p} t f_\alpha(t)dt \quad \text{for } \infty \geq p \geq -\infty, \quad (11)$$

where p is implicitly determined by (10).

The basic properties of ACCs are:

(1) $ACC_i^{\alpha}(0) = 0$,

(2) $ACC_i^{\alpha}(1) = \mu_i$,

(3) $\partial ACC_i^{\alpha}(\xi)/\partial\xi = \mu_i(p)$.

Properties (1) and (2) define the lower and upper levels of ACC_i^{α} on the cumulative probability range. Property (3) states that the slope of ACC_i^{α} equals the conditional expected return on asset i, given the portfolio's rate of return p.

[3] The term "dominate" will henceforth refer to MCSD.

[4] The systematic risk (beta) of an asset is defined as $\mathrm{cov}(r_i, P)/\sigma_P^2$, where the denominator is the variance of the portfolio return. But $\mathrm{cov}(r_i, P)$ can be written as

$$\mathrm{cov}(r_i, P) = E[(r_i - \mu_i)(P - \mu_P)] = E[r_i(P - \mu_P)] = \int_{-\infty}^{\infty} \mu_i(p)(p - \mu_P) f_\alpha(p)dp,$$

where $f_\alpha(p)$ is the density function of the portfolio.

[5] "Concentration" is the term used by Italian statisticians to describe the compactness of the income distribution. In portfolio analysis we should interpret the term as the variability of a random variable.

[6] So that $\mu_\alpha(p) = p$.

[7] Shorrocks (1983) uses ALCs (referred to as Generalized Lorenz Curves) to formulate necessary and sufficient conditions for SSD rules.

We can now state the major result of the paper:

THEOREM 2 (MCSD). *Given portfolio* $\{\alpha\}$, *asset k dominates asset j for all concave U on W if and only if*:

$$\mathrm{ACC}_k^\alpha(\xi) \geq \mathrm{ACC}_j^\alpha(\xi) \quad \text{for } 1 \geq \xi \geq 0,$$

with at least one strong inequality.

A formal proof of the Theorem follows the next numerical example, which is designed to illustrate the concepts of ACC and MCSD. Theorem 2 is analogous to Theorem 1, but dominance is conditional on the portfolio level. Indeed, SSD uses cumulative distributions to establish the conditions under which one portfolio is preferred to another, while MCSD uses concentration curves to assess the preference for one asset over another, when the portfolio is given. Theorem 2 distinguishes between the wealth level of the portfolio and the rates of return on single assets, as utility is defined in terms of total wealth, and portfolio diversification is based on individual rates of return. This is spelled out in Equation (7), where MCSD expresses the interrelation between marginal utility and rates of return.

A portfolio is composed of three independently distributed risky assets whose rates of return are:

Asset *A* yields—10% and 15% each with probability 0.5;

Asset *B* yields—5% and 10% each with probability 0.5;

Asset C yields—15% and 25% each with probability 0.5.

Assume that the initial portfolio is made up of 25% of *A*, 50% of *B*, and 25% of *C*. Table 1 presents the properties of the portfolio. The first column in the table shows the returns on the portfolio, *P*, ranked from the lowest to the highest return. Column (2) presents the probabilities associated with these returns, while column (3) shows the cumulative distribution of the portfolio. In columns (4), (5) and (6) we present the expected return on each asset $\{\mu_i(p),\ i = A, B, C\}$ *conditional* on the portfolio returns given in column (1). For example, the 6.25% return on P [$=0.25(-10\%) + 0.50(5\%) + 0.25(25\%)$] has a $\frac{1}{8}$ probability to occur, the cumulative distribution is equal to $\frac{5}{8}$ and the expected returns on assets *A*, *B*, *C*, conditional on the return of the portfolio being 6.25% are −10%, 5%, and 25% respectively.

Table 1 **An Example; Portfolio Returns and Assets Conditional Expected Returns**[a]

Portfolio Returns (1)	Probability (2)	Cumulative Distribution F_α (3)	Conditional Expected Returns $\mu_A(p)$ (4)	$\mu_B(p)$ (5)	$\mu_C(p)$ (6)
−3.75	1/8	1/8	−10.0	5.0	−15.0
−1.25	1/8	2/8	−10.0	10.0	−15.0
2.5	1/8	3/8	15.0	5.0	−15.0
5.0	1/8	4/8	15.0	10.0	−15.0
6.25	1/8	5/8	−10.0	5.0	25.0
8.75	1/8	6/8	−10.0	10.0	25.0
12.5	1/8	7/8	15.0	5.0	25.0
15.0	1/8	1	15.0	10.0	25.0

[a] Portfolio *P* is made up of 25% of asset *A*, 50% of asset *B*, and 25% of asset *C*. *A* yields −10% and 15% each with probability 0.5; *B* yields 5% and 10% each with probability 0.5; *C* yields −15% and 25% each with probability 0.5. The first column presents the returns on *P*; the second column provides the probabilities of returns: the third column shows the cumulative probability. Columns 4, 5, and 6, present the expected return on each asset *conditional* on portfolio returns *P*.

With the returns *P* ranked as in Table 1 and the cumulative distribution of the portfolio calculated, the ACCs for assets *A*, *B*, and *C* are obtained by adding, for each asset, the conditional expected returns $\mu_i(p)$ *multiplied* by the probability of occurrence of *p* as given in column (2). The results appear in Table 2, with graphical depiction in Figure 1. For example, the ACC of asset *B*, given the $\frac{3}{8}$ cumulative probability, is 2.5% and is obtained by adding $\frac{1}{8}(5\%) + \frac{1}{8}(10\%) + \frac{1}{8}(5\%)$. The cumulative probability distribution is always uniformly distributed and the horizontal axis in Figure 1 represents an ordering of all possible outcomes of the portfolio, from worst to best, when these outcomes are weighted by the probability of their occurrence. Two sections of equal length will be equally probable. When portfolio returns are ranked in ascending order, the vertical axis measures the *cumulative* expected return from asset *i*. Hence, ACC expresses the expected return of asset *i*, given that the portfolio return is lower than *p*. The ACC of asset *A* begins at (0, 0), where its contribution to an empty portfolio is zero, and ends at

Table 2 **An Example; Assets ACCs and Portfolio Absolute Lorenz Curve**

Cumulative Probability	Assets ACCs A	B	C	Portfolio Absolute Lorenz Curve
0	0.0	0.0	0.0	0.0
1/8	−1.25	0.625	−1.875	−0.468
2/8	−2.5	1.875	−3.75	−0.625
3/8	−0.625	2.5	−5.625	−0.312
4/8	1.25	3.75	−7.5	0.3125
5/8	0.0	4.375	−4.375	1.0937
6/8	−1.25	5.625	−1.25	2.1875
7/8	0.625	6.25	1.875	3.75
1	2.5	7.5	5.0	5.625

$(\mu_A, 1)$, where all possible portfolio outcomes are accounted for, and the expected contribution of asset A to P is its expected return.

The Absolute Lorenz Curve (ALC) of a portfolio is constructed in the same manner, with the portfolio's cumulative expected returns ranked according to the portfolio distribution. ALC is the weighted sum of the various ACCs (the weights being the assets' shares) and, from Property (3) above, is always convex.

From Figure 1 it is evident that asset B's ACC lies above those of assets A and C for the entire range of the probability distribution. Hence, by Theorem 2, åsset B dominates assets A and C. Indeed, a marginal increase in the share of B at the expense of asset A or C benefits any risk-averse individual holding the portfolio.

To support this claim, consider, for example, a logarithmic utility function of the type $U_1(W) = \ln(W)$, where $W = (1 + P)$. To verify that expected utility increases as a result of the change, increase B on account of A. Given the original portfolio, expected utility is $EU_1(W) = 0.053091$. Increasing the share of asset B to 51% at the expense of asset C raises expected utility to 0.053414. (To validate Theorem 2 numerically it must be demonstrated that expected utility does not decrease for *all* concave utility functions. This would be an impossible task.)

Let us now verify the converse of Theorem 2, when ACCs intersect (as shown in Figure 1 with assets A and C). In this case, one cannot assert that one asset dominates the other. For example, increasing the share of asset A to 26% on account of asset C decreases expected utility, given $U_1(W)$, to 0.05206. However, given an alternative concave utility function, $U_2(W)$, expected utility increases, implying that increasing A at the margin is desirable for U_2 but not for U_1.[8]

We now present the proof of the MCSD theorem.[9]

PROOF OF THEOREM 2.[10]

(a) Sufficiency. Assume that $\mathrm{ACC}_k^\alpha(\xi) \geq \mathrm{ACC}_j^\alpha(\xi)$ for all ξ. Then:

$$\int_{-\infty}^{z} \mu_k(t) f_\alpha(t)dt \geq \int_{-\infty}^{z} \mu_j(t) f_\alpha(t)dt \quad \text{for all } z. \tag{12}$$

Alternatively:

$$\int_{-\infty}^{z} [\mu_k(t) - \mu_j(t)] f_\alpha(t)dt \geq 0 \quad \text{for all } z. \tag{13}$$

Define $\beta(t) = [\mu_k(t) - \mu_j(t)] f_\alpha(t)$; hence, by Lemma 1 (presented at the end of the proof),

$$\int_{-\infty}^{z} U'(t)\beta(t)dt \geq 0 \quad \text{for all } z. \tag{14}$$

Because (14) also holds for $z = \infty$, one obtains:

$$\mathrm{EU}'(W)\cdot[r_k - r_j] \geq 0. \tag{15}$$

(b) Necessity. Assume that $\mathrm{EU}'(W)\cdot[r_k - r_j] > 0$ for all concave U. As

$$\mathrm{EU}'(W)\cdot[r_k - r_j] = \mathrm{EU}'(W)\cdot[E(r_k|W) - E(r_j|W)],$$

one obtains:

$$\int_{-\infty}^{\infty} U'(t)\cdot[\mu_k(t) - \mu_j(t)] f_\alpha(t)dt > 0. \tag{16}$$

Using the definition of $\beta(t)$, (16) can be rewritten as:

$$\int_{-\infty}^{z} U'(t)\beta(t)dt + \int_{z}^{\infty} U'(t)\beta(t)dt > 0. \tag{17}$$

[8] Consider the piecewise linear function:

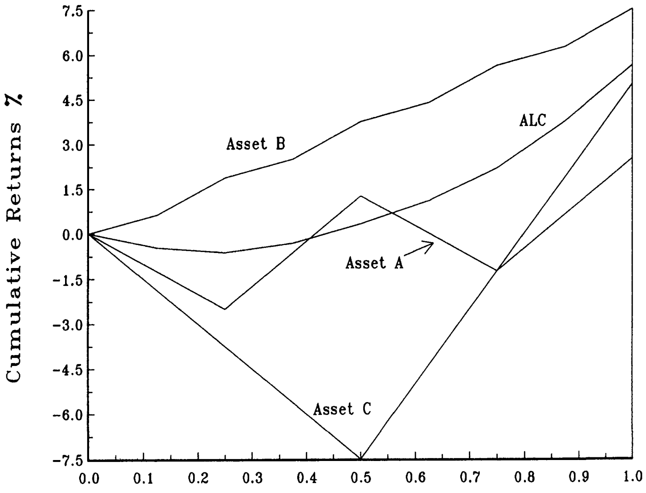
$$U_2(W) = \begin{cases} W & \text{if } W \leq 1, \\ 1 + 0.3(W - 1) & \text{if } W > 1. \end{cases}$$

Given the original portfolio, expected utility is $\mathrm{EU}_2(W) = 1.0125$. Increasing the share of B to 51% at the expense of C raises $\mathrm{EU}_2(W)$ to 1.012968.

[9] The theorem has been proved in a different context by Yitzhaki and Olkin (1988). The proof is presented here for the sake of completeness.

[10] As W_0 is a positive constant, it will be ignored in the proof.

Figure 1 Assets ACCs and Portfolio ALC



Because the inequality holds for all concave U, it must also hold for the specific function:

$$U'(t) = \begin{cases} c & \text{for } t \leq z, \\ 0 & \text{for } t > z. \end{cases} \tag{18}$$

Hence:

$$c \int_{-\infty}^{z} \beta(t)dt > 0 \quad \text{for all } z. \tag{19}$$

By varying z, and because

$$\beta(t) = [\mu_k(t) - \mu_j(t)] f_\alpha(t)$$

and F_α is a monotone function of z, one obtains:

$$\text{ACC}_k^\alpha(\xi) > \text{ACC}_j^\alpha(\xi) \quad \text{for all } \xi. \quad \text{Q.E.D.}$$

LEMMA 1. *Let $U'(t)$ be a nonnegative and nonincreasing function, and let $\beta(t)$ be a function such that:*

$$\int_{-\infty}^{z} \beta(t)dt \geq 0 \quad \textit{for all } z.$$

Then

$$\int_{-\infty}^{z} U'(t)\beta(t)dt \geq 0 \quad \textit{for all } z. \tag{20}$$

For proof of the lemma see Yitzhaki (1982).

4. Theoretical Implications and Extensions

The first implication of Theorem 2 relates to whether portfolio diversification pays as well as to the value of additional investment opportunities. Consider an asset, one that is not originally held in portfolio $\{\alpha\}$, being offered to the investor. Under what conditions will all risk-averse individuals include some share of the new asset in their portfolio?

This problem was considered by Samuelson (1967) and McEntire (1984), who examined independently distributed assets. For that class of distributions, a necessary and sufficient condition for asset k to be included in the portfolio by *all* risk-averse individuals is that $\mu_k \geq \mu_i$, where i is any asset in $\{\alpha\}$. Theorem 2 includes this result as a special case, and therefore it can be described as an extension of Samuelson and McEntire for any type of probability distribution.

To show that Samuelson's result is a particular case of Theorem 2, consider a portfolio composed of two independent assets, whose rates of return are r_1 and r_2. An additional investment opportunity with rate of return r_k presents itself. Following Theorem 2, asset k will be included in the portfolio by all risk-averse investors if

$$\text{ACC}_k^\alpha(\xi) \geq \text{ACC}_1^\alpha(\xi) \quad \text{or} \quad \text{ACC}_k^\alpha(\xi) \geq \text{ACC}_2^\alpha(\xi), \quad \text{for all } 1 \geq \xi \geq 0.$$

As these assets are independently distributed, they produce special ACCs. First, because asset k is not included in the portfolio and its distribution is independent of the distribution of the portfolio, its ACC is a straight line with slope μ_k connecting the origin to $(\mu_k, 1)$. Second, as shown in the Appendix, $\text{ACC}_i^\alpha(\xi)$ (for $i = 1, 2$) always lies below the straight line connecting $(0, 0)$ and $(\mu_i, 1)$. Therefore, if $\text{ACC}_k^\alpha(1) \geq \text{ACC}_i^\alpha(1)$ for $i = 1, 2$, it also holds for all ξ. Hence, given that assets are independent, $\mu_k \geq \mu_i$ is a sufficient condition for security k to be included in the portfolio.

The second implication of Theorem 2 involves necessary conditions for MCSD. The ACC approach to MCSD requires pairwise comparisons of alternatives; this is a lengthy process that can be made more efficient by applying the following necessary conditions.

THEOREM 3. *If asset k dominates asset j given portfolio* $\{\alpha\}$, *then*:

$$(1) \quad \mu_k \geq \mu_j, \tag{21}$$

$$(2) \quad \mu_k - 2\,\mathrm{cov}[r_k, F(P)] \geq \mu_j - 2\,\mathrm{cov}\,[r_j, F(P)], \tag{22}$$

where $\mathrm{cov}[r_j, F(P)]$ *is the covariance of the return on asset j and the cumulative probability distribution of portfolio* $\{\alpha\}$.

The first condition is derived from the basic property (2) of ACCs. Because $\mathrm{ACC}_k^\alpha(1) \geq \mathrm{ACC}_j^\alpha(1)$ is a necessary condition for security k to dominate security j, the mean of k must be higher or equal to the mean of j. This result, which is obtained independently from the choice of probability distributions, implies that assets whose portfolio shares are increased at the expense of others have higher expected returns regardless of their risk. (Otherwise, the requirement for a favorable gamble will be violated and a risk neutral individual will not increase the shares of favorable assets.)

The proof of the second necessary condition is as follows: From Theorem 2, because $\mathrm{ACC}_k^\alpha(\xi) \geq \mathrm{ACC}_j^\alpha(\xi)$ for all $1 \geq \xi \geq 0$, is a necessary condition for MCSD, it also holds that:

$$\int_0^1 \mathrm{ACC}_k^\alpha(F)dF \geq \int_0^1 \mathrm{ACC}_j^\alpha(F)dF$$

if k dominates asset j.

By definition:

$$\int_0^1 \mathrm{ACC}_k^\alpha(F)dF = \int_{-\infty}^{\infty}\int_{-\infty}^{z} \mu_k(t)f_\alpha(t)dt f_\alpha(z)dz. \tag{23}$$

Integrating (23) by parts yields

$$= -[1 - F_\alpha(z)]\int_{-\infty}^{z} \mu_k(t)f_\alpha(t)dt\,|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}[1 - F_\alpha(t)]\mu_k(t)f_\alpha(t)dt. \tag{24}$$

The first term in (24) is zero, provided the mean is finite. Hence the integral (23) becomes:

$$= \mu_k - \int_{-\infty}^{\infty} F_\alpha(t)\mu_k(t)f_\alpha(t)dt. \tag{25}$$

Since $E[F(P)] = \frac{1}{2}$, adding and subtracting $(\frac{1}{2})\mu_k$ from (25) yields

$$= \frac{1}{2}\mu_k - \int_{-\infty}^{\infty}\left[F(t) - \frac{1}{2}\right]\mu_k(t)f_\alpha(t)dt. \tag{26}$$

By the definition of the covariance,

$$\int_0^1 \mathrm{ACC}_k^\alpha(F)dF = \frac{1}{2}\mu_k - \mathrm{cov}[r_k, F(P)], \tag{27}$$

whence the result of Theorem 3.

The implications of Theorem 3 are powerful in that they allow the ranking of risky assets in terms of their excess expected return over their systematic risk. Indeed, as shown in the mean-Gini capital asset pricing model (MG-CAPM) developed by Shalit and Yitzhaki (1984),[11] systematic risk is expressed as:

$$\beta_k = \frac{2\,\mathrm{cov}[r_k, F(P)]}{\Gamma_P}, \tag{28}$$

where $\Gamma_P = 2\,\mathrm{cov}[P, F(P)]$ is the Gini of the market portfolio.

The mean-Gini (MG) model is similar to the mean-variance (MV) paradigm, except that the Gini index replaces the variance as a measure of risk. It is a well-known fact that, unless assets' returns are normally distributed, or the utility function is quadratic, MV fails to represent expected-utility maximizers. On the other hand, as shown by Yitzhaki (1982), MG provides necessary conditions for SSD ordering of any type of distribution and, also, necessary and sufficient conditions for SSD when cumulative distributions intersect at most once. The equilibrium conditions under MG are similar to the CAPM conditions under MV except that the Gini index replaces the variance. In a market with risk-averse investors facing identical investment opportunities, MG equilibrium becomes:

$$\mu_k = r_F + \beta_k(\mu_P - r_F), \tag{29}$$

where r_F is the risk-free rate, μ_P is the market portfolio return, and β_k is given by Equation (28).

Therefore, going back to Theorem 3, a necessary condition for asset k to dominate asset j is:

$$\mu_k - \beta_k\Gamma_P \geq \mu_j - \beta_j\Gamma_P, \quad \text{or} \tag{30}$$

[11] For an evaluation of the Gini as a measure of portfolio risk and its comparison with the variance, see Bey and Howe (1984) and Okunev (1988).

$$\frac{(\mu_k - \mu_j)}{\Gamma_P} \geq \beta_k - \beta_j, \tag{31}$$

i.e., the difference between two securities' expected returns must be greater than the difference in their systematic risks defined in terms of MG-CAPM.[12]

Equation (31) establishes the second necessary condition for obtaining a set of securities such that no other security dominates it. The first prerequisite was attained by ranking the securities in descending order of expected returns. Pairwise comparison between these expected returns determines the set of securities that meet condition (21). This set is further reduced by checking whether the difference between the expected returns per unit of portfolio risk is greater than the difference in MG systematic risks.[13]

This powerful result, using risk and mean return, allows for a complete ordering of investment alternatives, while MCSD criteria establish only a partial ordering. This, of course, is an advantage when no dominance can be determined using ACCs, but where a decision maker nevertheless wants to rank investment alternatives. In that case, the mean-Gini necessary conditions for MCSD will provide an answer that, as we know, does not necessarily satisfy the sufficient conditions.

5. Market Efficiency and the MCSD Evidence

MCSD rules enable us to evaluate whether the market portfolio can be viewed as an efficient portfolio generated from expected-utility maximization. Consider an expected-utility-maximizing investor who holds the market portfolio.[14] If that portfolio is efficient, changing the proportions of some securities will not raise expected utility. If expected utility does increase, the market portfolio can be viewed as not efficient, and MCSD rules can be used to determine which securities should have their shares adjusted.

In §3 we developed MCSD criteria for a given level of initial wealth. With financial data, the value of wealth changes with each observation. Hence the level of initial wealth must be treated as a random variable, and MCSD rules must be changed accordingly.

Equation (7) shows that, given initial wealth W_0, asset k dominates asset j if:

$$E_r[U'(W) \cdot W_0 \cdot (r_k - r_j) | W_0] \geq 0, \tag{32}$$

where E_r is the expectation taken with respect to all asset returns. If initial wealth is considered a random variable, Condition (7) is modeled as:

$$E_{W_0} E_r[U'(W) \cdot W_0 \cdot (r_k - r_j) | W_0] \geq 0. \tag{33}$$

This condition can be written as:

$$E_r E_{W_0}[U'(W) \cdot W_0 \cdot (R_k - R_j)] \geq 0, \tag{34}$$

where $R_k = (1 + r_k)$.

The term $W_0 \cdot R_k$ is viewed as the value of the portfolio if the individual's entire wealth is invested in asset k. Therefore, MCSD rules using value levels (rather than rates of return) must be specified. When initial wealth is viewed as a random variable, the necessary and sufficient condition for MCSD becomes:

$$E_r E_{W_0}[U'(W) \cdot W_0 \cdot (R_k - R_j)] \geq 0$$

if and only if

$$\mathrm{ACC}_k^W(\xi) \geq \mathrm{ACC}_j^W(\xi) \quad \text{for all } 1 \geq \xi \geq 0,$$

where

$$\mathrm{ACC}_i^W(\xi) = \int_{-\infty}^{W^*} E(W_0 \cdot R_i | W = w) f_W(w) dw$$

$$\text{for } i = k, j \quad \text{and} \quad \infty \geq W^* \geq -\infty,$$

and where W^* is implicitly defined by

$$\xi = \int_{-\infty}^{W^*} f_W(w) dw.$$

In a dynamic context these conditions allow for the rebalancing of the portfolio following the MCSD results. For each period, the proportion of dominating securities is increased at the expense of the dominated securities. For the next period, MCSD rules are reevaluated ac-

[12] Mean-Gini systematic risks can also account for degrees of risk aversion if the extended Gini parameter is used as a measure of dispersion (Shalit and Yitzhaki 1989). In that respect, additional necessary conditions based on the extended Gini parameter can be formulated.

[13] Bawa and Lindenberg (1977) use the Lower Partial Moment to derive necessary conditions for SSD in a fashion analogous to the mean-Gini conditions to SSD.

[14] If the intertemporal returns are independent, then maximizing one period's return will necessarily yield a maximum return for any number of periods.

cording to the rebalanced portfolio, conditional on the wealth carried forward from the previous period. In an efficient market situation, rebalancing the portfolio will be a continuous process conditional on all available information. In such a market, no security will ever dominate another.

The following empirical examination of MCSD assumes that individuals hold their wealth in a portfolio of assets represented by the stock market. As we use only the ranking of wealth in the analysis, any monotone transformation of wealth can be used as a proxy.

6. Market Efficiency and MCSD Evidence on the NYSE

To illustrate the MCSD rules we present an application that assesses the set of dominant stocks on the New York Stock Exchange (NYSE). We use daily returns data from the Center for Research in Securities Prices (CRSP) for three samples of the 102 most-traded securities on the Exchange.[15] MCSD tests were developed and performed on a personal computer, although the FORTRAN and C algorithms can be adapted to a larger machine, and can accommodate the entire CRSP data set with nonmissing returns.

The use of ex-post financial data to assess investment efficiency might be subject to criticism. We nevertheless use these data, as they are the data most available to the analyst. In this case we assume that realized returns are related to anticipated returns. If this were not the case, investors interested in maximizing realized returns would always invest in assets with lower anticipated returns.

Each sample consists of 201 daily returns on 102 securities *plus* the market daily return as expressed by the CRSP value-weighted average return with dividends. The first sample includes data from January 2, 1985, through October 17, 1985; the second sample is comprised of daily returns from February 5, 1986, through November 19, 1986; and the third sample is from March 18, 1987, through December 31, 1987.

Brown and Warner (1985) note several problems with using daily data: although they lessen the temporal aggregation bias, daily returns cause some difficulties with the market index because they include infrequently traded assets (see, e.g., Gibbons and Ferson 1985). The MCSD method, which requires only the ranking of the market proxy, obviates these problems. We also avoid lower trading frequency problems by choosing the 102 securities with the highest volume on the NYSE.

A measure of wealth by which alternatives are ranked is needed to establish dominance. Because the utility functions do not have to be specified, any monotone transformation of individual wealth will be appropriate. Therefore we use the market index as a proxy of daily changes in wealth. For each sample, the index, which starts with $W_{t=0} = 1$, is computed as follows:

$$W_t = \prod_{i=0}^{t} (1 + r_{P_i}) \quad \text{for } t = 1, \ldots, 201,$$

where r_{P_i} is the CRSP-value-weighted average market return with dividends for day i.

The data are sorted by the market index, and the market Absolute Lorenz Curve is computed. Stock returns are multiplied by the lagged market index to obtain the value levels that would have been earned had individuals invested all their initial wealth in the particular securities. Cumulative values are computed to establish the ACCs for the securities.

Dominance is obtained by determining the set of dominating securities. First, the algorithm calculates for each firm its mean return and Gini beta to check whether the necessary conditions of Theorem 3 hold. Using these necessary conditions drastically reduces the number of securities for which concentration curves are calculated for the MCSD test. Next, dominance is assessed following Theorem 2.

The results for the first sample appear in Table 3. For each dominating firm there is a variable list of dominated firms. For example, in the first period, given the realization of the market portfolio, *General Motors* stock dominated the performance of 49 other stocks. Marginally increasing the proportion of *General Motors* at the expense of one dominated firm would have improved portfolio performance.

The fact that Table 3 can be produced indicates that the market was inefficient in that period. Market efficiency requires that no one security dominate any other security. This is true only for firms that do not appear

[15] The list of securities is available from the authors upon request.

Table 3 Marginal Conditional Stochastic Dominant Firms Traded on the NYSE for the Period JAN 2, 1985 to OCT 17, 1985

DOMINATING FIRM	Dominated Firms				
ABBOT LABORATORIES	Detroit Edison	Digital Equipment	Gillette	Schlumberger	
A T & T	Eastman Kodak				
ANHEUSER BUSCH	Gillette				
ARCHER DANIELS MIDLAND	Eastman Kodak				
BETHLEHEM STEEL	Texas Instruments				
DOW CHEMICAL	Detroit Edison	Eastman Kodak			
EXXON	Merck				
FEDERATED DEPT. STORES	Gillette				
GENERAL ELECTRIC	Eastman Kodak				
GENERAL MOTORS	Alcan Aluminium	Anheuser Busch	American Elc. Pwr.	Avon Products	American Express
	BellSouth	Chevron	Chrysler	Central & S.W.	Consolidated Edison
	Citicorp	CSX	Detroit Edison	Digital Equipment	Du Pont De Nemours
	Eastman Kodak	Exxon	Ford Motors	GTE	General Electric
	Gillette	Goodrich, BF	Hewlett Packard	Illinois Power	K Mart
	Kroger	Merck	Middle South Utils.	Mobil	Morgan, J. P.
	Motorola	Niagara Mohawk	Occidental Petrol.	Pacific Gas & Elec.	Philadelphia Elec.
	Pinnacle West Cptl.	Polaroid	Public Svc. Enterpr.	RJR Nabisco	Schlumberger
	Sears Roebuck	Southern	Tenneco	Texaco	Texas Instruments
	Texas Utils.	USX	USF & G	Unisys	MARKET PORTFOLIO
GOODYEAR	Eastman Kodak				
HOUSTON INDUSTRIES	Detroit Edison				
JOHNSON & JOHNSON	Detroit Edison	Gillette	Kroger	Procter & Gamble	Public Svc. Enterpr.
	Southwestern Bell	U S West			
LIMITED	Bellsouth	Gillette	Illionois Power	Philadelphia Elec.	Texaco
MARION LABORATORIES	Gillette				
MC DONALDS	Alcan Aluminium	CSX	Digital Equipment	Eastman Kodak	Ford Motor
	GTE	IBM	Minnesota Mg. & Mf.	Philadelphia Elec.	Procter & Gamble
	RJR Nabisco	Texas Instruments			
MORGAN JP	Gillette				
NORTHEAST UTILITIES	Eastman Kodak				
PUBLIC SVC. ENTERPR.	Detroit Edison				
SALOMON	Digital Equipment	RJR Nabisco	Schlumberger		
SMITHKLINE BECKMAN	American Elec. Pwr.	Detroit Edison	Illinois Power	IBM	Gillette
	Merck	Minnesota Mg. & Mf.	Pacific Telesis	Philadelphia Elec.	Primerica
	RJR Nabisco	Schlumberger	Texas Instruments	USX	U S West
	MARKET PORTFOLIO				
SYNTEX	Avon Products	Bellsouth	Central & S.W.	Consolidated Edison	Detroit Edison
	Eastman Kodak	Gillette	Pacific Telesis	RJR Nabisco	Southwestern Bell
UPJOHN	Gillette				
WAL MART STORES	Alcan Aluminium	American Elec. Pwr.	Avon Products	Bellsouth	CSX
	Central & S.W.	Chrysler	Coca Cola	Consolidated Edison	Detroit Edison
	Du Pont Nemours	Eastman Kodak	Ford Motors	GTE	General Electric
	Gillette	Houston Inds.	IBM	Illinois Power	Kroger
	Merck	Minnesota Mg. & Mf.	Motorola	Northeast Utilities	Ohio Edison
	Pacific Gas & Elec.	Pacific Telesis	Philadelphia Elec.	Pinnacle West Cptl.	Primerica
	Public Svc. Enterpr.	RJR Nabisco	Schlumberger	Sears Roebuck	Southern
	Southwestern Bell	Texaco	Texas Instruments	Texas Utilities	USX
	Unisys	U S West	MARKET PORTFOLIO		
WASTE MANAGEMENT	Detroit Edison				
XEROX	Schlumberger				
ZAYRE	American Express	Bellsouth	Chrysler	Digital Equipment	Gillette
	Illinois Power	K Mart	Occidental Petrol	Pacific Gas & Elec.	Philadelphia Elec.
	Polaroid	RJR Nabisco	Sears Roebuck	Southern	Texaco
	Texas Instruments	USX			

in Table 3, i.e., firms in the sample that neither dominate nor are dominated.

One interesting feature is the comparison of the behavior of the market portfolio with that of individual assets. We can see whether the market portfolio dominates or is dominated by individual securities by comparing the ALC of the market portfolio with the ACCs of the individual securities. If the market portfolio dominates one security, increasing the share of *all* securities in the portfolio and reducing the proportion of the dominated security improves the portfolio for all risk-averse investors.

For example, in the first sample, increasing the share of *General Motors* at the expense of all other stocks would improve the investor's position. As can be seen, the transitivity property of the MCSD binary relation holds; for example, *General Motors* dominates *Exxon*, which in turn dominates *Merck*; thus *General Motors* also dominates *Merck*.

Necessary conditions for MCSD are given in Table 4. Firms are ranked in decreasing order of their expected return minus their Gini systematic risk ($\mu - \beta\Gamma$). The expectation is that the dominating firms in Table 3 will appear at the head of the list in Table 4, and that the dominated firms will be at the bottom. However, one must also account for the possibility of the expected return of a dominating firm exceeding that of a dominated one. Furthermore, the rankings in Table 4 indicate only *necessary* conditions for MCSD. For example, *Coca Cola*, placed 56th on the list, has an excess expected return over its Gini systematic risk of 1.09578, that is, greater than the $\mu - \beta\Gamma$ of all the firms below it. But *Coca Cola* does not dominate even one firm; on the contrary, it is dominated by *Wal Mart Stores*, which has a greater expected return and a higher $\mu - \beta\Gamma$.

For the second sample period (February 5, 1986, to November 19, 1986),[16] we again show that the market was not efficient because we find a set of dominated and dominating firms. The dominated firms are reduced to a score of the worst performers (such as *Tenneco*, *Niagara Mohawk Power*, and *Bellsouth*, to cite only a few). The *market portfolio* dominates one firm, *Tenneco*, implying that reducing the share of that company's stock in the portfolio and increasing the shares of all other firms would improve portfolio performance. According to the mean-Gini necessary conditions for MCSD, it is shown that the firms with the highest $\mu - \beta\Gamma$ will dominate the firms with a lower $\mu - \beta\Gamma$.

Dominance was also obtained for the third sample, which includes the October 1987 crash. Because of the high volatility in stock prices in that period, adjustments leading to an efficient market portfolio were imperfect, producing a large set of dominating firms.

The three samples produce different sets of dominating and dominated firms. Not surprisingly, dominating firms in the first sample become dominated in the second and third samples, implying that the market acts on good investment opportunities. The question is: does dominance exist in the long run (i.e., is the market portfolio inefficient)?

To answer this question we pool the three samples to form one data set of 603 daily returns. Because the samples are disjoint, the wealth index (measured by the accumulated market index) is compiled to include the missing days. Hence, the market index constructed for 758 days, starts with 1 on December 31, 1984, and ends with 1.598 on December 31, 1987. The data are ranked with respect to the increasing order of the market index, ACCs for all the securities were calculated, and the MCSD test was performed.

For the pooled sample, no security dominates any other. Given this result, we cannot assert that the market portfolio was inefficient and that investors could benefit from changing their positions in this portfolio. In the long run, therefore, the market portfolio can be considered efficient.

This statement, however, must be qualified: first, one weakness of the data analysis is that sample ACCs are treated as if they were population concentration curves and that there was therefore no need for statistical tests of the robustness of the results, and no confidence intervals or significance levels were computed. Second, sampling variability may affect the likelihood of finding a dominance relationship; indeed, the smaller the sample, the less stable the ACC, and one is bound to find dominance even if there is no dominance in the population. Special attention is, of course, needed when comparing periods with different sizes of observations without the proper adjustment. We are not aware of a

[16] The tables for the second and third sample periods are available from the authors upon request.

Table 4 Firms Ranked with Respect to Mean-Beta*Γ for the Period JAN 2, 1985 to OCT 17, 1985

Firm	Mean	Beta	Rank	Firm	Mean	Beta	Rank
1 GENERAL MOTORS	1.124203	0.955292	1.099447	53 CHRYSLER	1.122229	1.016575	1.095885
2 LIMITED	1.124486	0.974390	1.099235	54 USX	1.122044	1.011825	1.095822
3 MARION LABORATORIES	1.123897	0.972829	1.098686	55 **MARKET PORTFOLIO**	1.121715	1.000000	1.095800
4 ZAYRE	1.123851	0.972167	1.098658	56 COCA COLA	1.121888	1.007461	1.095780
5 UPJOHN	1.124220	1.010737	1.098027	57 BRISTOL MYERS	1.121644	0.998588	1.095766
6 STORAGE TECHNOLOGY	1.121186	0.905267	1.097726	58 SEARS ROEBUCK	1.121501	0.993073	1.095766
7 WAL MART STORES	1.122908	0.974480	1.097655	59 PRIMERICA	1.121905	1.008876	1.095760
8 MORGAN, J. P.	1.122656	0.974799	1.097394	60 COMMONWEALTH ED.	1.121427	0.991943	1.095721
9 PEPSICO	1.123257	0.998163	1.097389	61 U S WEST	1.121642	1.001981	1.095675
10 SYNTEX	1.122860	0.987164	1.097277	62 TEXAS UTILS.	1.121384	0.992199	1.095672
11 USF & G	1.122912	0.992153	1.097200	63 AMERICAN ELEC. PWR.	1.121354	0.991123	1.095669
12 ABBOTT LABS.	1.122797	0.989898	1.097144	64 PACIFIC TELESIS	1.121458	0.995417	1.095662
13 ANHEUSER BUSCH	1.123268	1.012153	1.097039	65 AMR	1.121609	1.001280	1.095661
14 SMITHKLINE BECKMAN	1.122433	0.980830	1.097015	66 GILLETTE	1.121986	1.015923	1.095658
15 JOHNSON & JOHNSON	1.122485	0.986680	1.096916	67 UNITED TECHNOLOGIES	1.121777	1.009380	1.095619
16 WASTE MANAGEMENT	1.122881	1.003775	1.096868	68 TENNECO	1.121235	0.989783	1.095585
17 XEROX	1.122658	0.995929	1.096848	69 PHILADELPHIA ELEC.	1.121677	1.008261	1.095548
18 PILLSBURY	1.122999	1.012095	1.096771	70 GOODYEAR	1.121209	0.990632	1.095536
19 SALOMON	1.121868	0.968997	1.096757	71 BANKAMERICA	1.119762	0.936683	1.095487
20 FEDERATED DEPT. STORES	1.122267	0.986218	1.096709	72 AVON PRODUCTS	1.122698	1.056021	1.095331
21 MC DONALDS	1.122274	0.986739	1.096702	73 BOEING	1.122227	1.039255	1.095295
22 KRAFT NEW	1.122887	1.010424	1.096702	74 LONG ISLAND LIGHT	1.121714	1.020872	1.095258
23 CHEVRON	1.122503	0.996721	1.096673	75 K MART	1.120813	0.987481	1.095222
24 HOUSTON INDUSTRIES	1.122306	0.991577	1.096610	76 GENERAL ELECTRIC	1.121442	1.012531	1.095202
25 UNION CARBIDE	1.123892	1.054023	1.096578	77 DETROIT EDISON	1.121107	1.000275	1.095185
26 CITICORP	1.121774	0.972554	1.096570	78 CSX	1.121655	1.025588	1.095077
27 EXXON	1.122212	0.989582	1.096567	79 RJR NABISCO	1.120584	0.985983	1.095032
28 DOW CHEMICAL	1.122497	1.000646	1.096565	80 IBM	1.121320	1.014595	1.095027
29 GOODRICH, B. F.	1.122002	0.982046	1.096552	81 GTE	1.121009	1.005411	1.094954
30 MOBIL	1.121940	0.979831	1.096548	82 PHILIP MORRIS	1.120775	0.997007	1.094938
31 OCCIDENTAL PETROL.	1.122464	1.001376	1.096514	83 MIDDLE SOUTH UTILS.	1.119688	0.960793	1.094789
32 POLAROID	1.122742	1.012864	1.096494	84 FORD MOTORS	1.121385	1.026559	1.094782
33 DU PONT DE NEMOURS	1.122192	0.991862	1.096488	85 MINNESOTA MNG. & MF.	1.121138	1.017582	1.094768
34 AMERICAN EXPRESS	1.121866	0.983779	1.096371	86 INCO	1.121022	1.014174	1.094740
35 PINNACLE WEST CPTL.	1.122014	0.990141	1.096355	87 BETHLEHEM STEEL	1.120882	1.009614	1.094718
36 FEDERAL NATL. MTG.	1.122538	1.011878	1.096315	88 PROCTER & GAMBLE	1.121424	1.032390	1.094670
37 ARCHER DANIEL MIDLAND	1.122096	0.995116	1.096307	89 NAVISTAR INTL.	1.120469	0.996209	1.094653
38 PACIFIC GAS & ELEC.	1.121824	0.987077	1.096244	90 HOSPITAL CORP. AMER.	1.120420	0.996496	1.094596
39 BELLSOUTH	1.121962	0.994750	1.096183	91 SCHLUMBERGER	1.120451	0.998326	1.094580
40 OHIO EDISON	1.121956	0.994680	1.096179	92 UNISYS	1.121008	1.028116	1.094365
41 NORTHEAST UTILS.	1.122020	0.997444	1.096171	93 MOTOROLA	1.121069	1.035648	1.094231
42 CONSOLIDATED EDISON	1.121861	0.992023	1.096153	94 PHILLIPS PETROLEUM	1.121046	1.035625	1.094208
43 A T & T	1.121561	0.982026	1.096112	95 NATIONAL SEMICOND.	1.120934	1.034325	1.094130
44 TEXACO	1.121926	0.997164	1.096084	96 INTERNATIONAL PAPER	1.120284	1.012282	1.094051
45 CENTRAL & S.W.	1.121889	0.996504	1.096065	97 DIGITAL EQUIPMENT	1.120962	1.038579	1.094048
46 KROGER	1.121755	0.993086	1.096020	98 EASTMAN KODAK	1.120713	1.029645	1.094030
47 ILLINOIS POWER	1.121377	0.978811	1.096011	99 HEWLETT PACKARD	1.120390	1.018224	1.094003
48 MERCK	1.122165	1.010972	1.095966	100 ADVANCED MICRO. DVCS.	1.120253	1.016575	1.093909
49 PUBLIC SVC. ENTERPR.	1.121578	0.988513	1.095960	101 VARITY	1.121535	1.077886	1.093601
50 SOUTHWESTERN BELL	1.121811	0.998551	1.095933	102 ALCAN ALUMINIUM	1.120309	1.045119	1.093225
51 SOUTHERN	1.121594	0.991027	1.095912	103 TEXAS INSTRUMENTS	1.119850	1.031907	1.093108
52 NIAGARA MOHAWK	1.121633	0.993389	1.095890				

statistical testing procedure of stochastic dominance as all empirical work using SSD does not provide statistical testing of the results.

Furthermore, we did not check whether a convex set of securities dominates the market portfolio. For example, Fishburn (1978) and Bawa et al. (1985) show that a convex combination of securities can dominate (SSD) a given portfolio even if the individual securities do not. Had we used a less stringent approach, we might have found a set of dominating securities. Therefore one can say the test for efficiency is a joint test of efficiency and stringency.

7. Concluding Remarks

Marginal Conditional Stochastic Dominance establishes the conditions whereby risk-averse individuals, given a particular level of wealth, prefer risky assets. The concept has practical applications to financial market data. The criteria, which use absolute concentration curves, are based on the *ranking* of the market portfolio and not on the market index itself. Therefore ACCs are exempt from criticisms that use of a market portfolio proxy might occasion.

MCSD criteria can also be used to determine an optimal portfolio. By continuously modifying proportions of the dominating and dominated assets in the portfolio one obtains an efficient allocation in which no single asset dominates another. This procedure requires that portfolio rankings be updated for each iteration. Although this can be a lengthy process, it does lead to a solution. In the meantime, MCSD has useful practical applications in determining the set of dominating and dominated securities. Further research is needed into statistical tests for MCSD rules.[17]

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Appendix A

PROPOSITION. *Let α be a portfolio composed of two independently distributed assets,* 1 *and* 2. *Then* $\text{ACC}_i^\alpha(\xi)$ *(for* $i = 1, 2$*) always lies below the line connecting* (0, 0) *and* $(\mu_i, 1)$.

Let $p = R_1 + R_2$ be the return on α, where R_i is the return multiplied by the share of i in α. Because 1 and 2 are two independent assets, the density function of the portfolio is the density function of the sum of two independent random variables (DeGroot 1986, p. 167):

$$f_\alpha(P) = \int_{-\infty}^{\infty} f_1(s) f_2(P - s)\, ds, \tag{A.1}$$

where f_1 and f_2 are the density functions of assets 1 and 2. The cumulative probability distribution of the portfolio is

$$F_\alpha(P) = \int_{-\infty}^{\infty} f_1(s) F_2(P - s)\, ds. \tag{A.2}$$

The conditional expected value of R_1 given P is

$$\mu_1(P) = \frac{\int_{-\infty}^{\infty} s f_1(s) f_2(P - s)\, ds}{\int_{-\infty}^{\infty} f_1(s) f_2(P - s)\, ds}. \tag{A.3}$$

The ACC of asset 1 is defined by

$$\text{ACC}_1^\alpha(\xi) = \int_{-\infty}^{z} \mu_1(P) f_\alpha(P)\, dP, \tag{A.4}$$

where $z = F_\alpha^{-1}(\xi)$.
Substituting (A.3) into (A.4) yields:

$$\text{ACC}_1^\alpha(\xi) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} s f_1(s) f_2(p - s)\, ds\, dp. \tag{A.5}$$

Changing the order of integration,

$$\text{ACC}_1^\alpha(\xi) = \int_{-\infty}^{\infty} s f_1(s) F_2(z - s)\, ds. \tag{A.6}$$

The straight line connecting (0, 0) with $(\mu_1, 1)$ is labeled the Line of Independence (LOI), which is given by:

$$\text{LOI}_1^\alpha(\xi) = \mu_1 \xi. \tag{A.7}$$

Using (A.2), the vertical difference between LOI and ACC is:

$$\text{LOI}_1^\alpha(\xi) - \text{ACC}_1^\alpha(\xi) = \int_{-\infty}^{\infty} (\mu_1 - s) f_1(s) F_2(z - s)\, ds. \tag{A.8}$$

To complete the proof, one has to show that Equation (A.8) is nonnegative for all ξ. To demonstrate that, consider, first, $\beta(s) = (\mu_1 - s) f_1(s)$, which has the property:

$$\int_{-\infty}^{X} \beta(s)\, ds \geq 0 \quad \text{for all } X,$$

and second, $F_2(z - s)$ is nonnegative and declining with s. Hence, by using Lemma 1, the proof is completed.

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