Some Extensions and Analysis of Flux and Stress Theory

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Generalized Bodies

The Material Structure Induced by an Extensive Property
Organisms

- *Material points, bodies and subbodies* are primitive concepts in continuum mechanics. These notions are somehow related to the conservation of mass.

- In growing bodies, material points are added and removed from the body.

- Examples: fingerprints, birthmarks are distinguishable.

- An *organism* has a body structure although mass is not preserved. Can formalize this idea?

- Assume we have an extensive property.
The Material Structure Induced by an Extensive Property

In the classical case we have the flux vector field $h$. It can be integrated to give us a material structure.

A material point is identified with an integral line (a flow line). This procedure may induce material structure associated with any extensive property, e.g., color and energy.

- $h$ will be the velocity field of the material points.

- Can we generalize the same idea for the general manifold case where the flow $(m-1)$-form replaces the vector field?
The Case where a Volume Element is Specified

It is not necessary to have a metric structure in order that the flux form $J$ be represented by a vector field.

Assume that you have a *volume element* $\theta$ ($m$-form) on $\mathcal{U}$. This may be thought of as the density of the property $p$ if it is positive or another positive property, e.g., mass.

- **Given $J$ and $\theta$, find a vector $v$** such that for every pair of tangent vectors, $u, w$,
  \[
  \theta(v, u, w) = J(u, w)
  \]
  written as
  \[
  J = v \cdot \theta.
  \]

- For a given $\theta$ there is a unique such vector $v$—the *kinematic flux*—a generalization of the velocity field.
- The vector field $v$ depends linearly on the flux $J$. 

The Flux Bundle

Let us examine how the kinematic flux $v$ varies as we vary the volume element.

Since the space of $m$-forms at $x$ is 1-dimensional, as we vary the volume element the resulting vectors $v$ remain on a line (1-D subspace of the tangent space).

- Another characterization: *If a surface element (say the one defined by the vectors $u$, $w$) contains the line, the flux through it vanishes.*
- This is analogous to the situation with the velocity field.
- A collection of subspaces is referred to as a *distribution*. This distribution is the *flux bundle*. 

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Generalized Body Points

Integral manifolds of the distribution, the 1-dimensional flux bundle in this case, are submanifolds whose tangent space at a point is the corresponding line of the flux bundle at that point.

In general such integral manifolds need not exist (higher dimensions), however they always exist for 1-dimensional bundles as is the case here.

- Each integral line manifold is identified with a body point.
- Actual formulation is done on space-time manifold to allow time dependent fluxes. There $\beta$ is included in $\tau$ and $dJ = s$. 

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Frames in Space-Time

an event e

Space-Time U

Time Axis

(t, x)

Cartesian Product

Space

a frame

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Flux and Stress Theories

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Property-Induced Fibration and Frame

No volume element: Fibration
—no real valued time is assigned to events

A volume element: Integrable vector field
—real valued time is assigned to events
Space Formulation VS. Space-Time Formulation

Space Formulation
\[ \dim \mathcal{U} = 3 \]
\[ \dim \mathcal{B} = 3 \]
Balance
\[ \int_{\mathcal{B}} \beta + \int_{\partial \mathcal{B}} \tau = \int_{\mathcal{B}} s \]
surface term
2-form on a 3-D manifold
source term
3-form on a 3-D manifold
flux form
\( J \)—3 components
variables
—time dependent
field equation
\( \beta + dJ = s \)

Space-Time Formulation
\[ \dim \mathcal{E} = 4 \]
\[ \dim \mathcal{R} = 4 \]
Balance
\[ \int_{\partial \mathcal{R}} t = \int_{\mathcal{R}} s \]
surface term
3-form on a 4-D manifold
source term
4-form on a 4-D manifold
flux form
\( \widehat{J} \)—4 components
variables
—fixed values at events
field equation
\( d\widehat{J} = s \)
Flow Potentials

- Although we do not have vector velocity fields, we have material points.
- In addition, we have analogs for the flow potentials.
- In the case $s = 0$ we obtain (say the 4-D case) $dJ = 0$.
- Assume that $A$ is any $(m - 2)$-form on $\mathcal{U}$. Then, $J = dA$ satisfies the differential balance equation—$A$ is a flow potential. Since in general,

$$
\int_{\partial M} i^* \omega = \int_{M} d\omega,
$$

for every control region $\mathcal{B}$

$$
\int_{\mathcal{B}} dJ = \int_{\partial \mathcal{B}} i^*(J) = \int_{\partial \mathcal{B}} i^*(dA) = \int_{\partial(\partial \mathcal{B}) = \emptyset} i^*(i^*(A)) = 0.
$$
Summary: The Structure on Space-Time manifold Associated with an Extensive Property

- Balance laws are formulated in terms of forms.
- The flux vector field is replaced by a flux \((m - 1)\)-form in the \(m\)-dimensional space.
- Flow lines still make sense using the flux bundle.
- Generalized body points may be associated with an arbitrary extensive property—organisms.
- A particularly compact formulation in space-time.
- A positive extensive property induces a material frame.
Stresses for Generalized Bodies
Forces for Generalized Bodies

- Force densities are linear mappings on the values of the generalized velocities.
- In the case where a material structure is induce by an extensive property and a volume element is given, the induced generalized velocity $w$ depends linearly on the flux form $J$.
- It would be a natural generalization to replace generalized velocities by flux forms as fields on which forces operate to produce power.
- The physical dimension of forces will not be power per unit velocity but power per unit flux of the property $p$.
- For the spacetime formulation $F_\mathcal{B}(J) = \int_{\partial \mathcal{B}} t_\mathcal{B}(J)$, $\mathcal{B} \subset \mathcal{E}$.
- $t_\mathcal{B}(e): \wedge^{m-1} T^* e \mathcal{E} \rightarrow \wedge^{m-1} T^* e \partial \mathcal{B}$. 

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Flux and Stress Theories

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Consider the energy extensive property. It has a flux density term \( \int_{\partial B} \tau^{(e)} \) and a corresponding flux form \( J^{(e)} \) such that \( \tau^{(e)} = i^* \circ J^{(e)} \).

On the other hand the flux density of energy may be written in terms of the boundary force as \( t_B(J) \).

Cauchy’s theorem implies that \( t_B = i^* \circ \sigma \) so the energy flux density is \( \tau^{(e)} = i^* \circ J^{(e)} = i^* \circ \sigma(J) \). Hence,

\[
J^{(e)} = \sigma(J).
\]

The Cauchy stress is the linear mapping that transforms the flux of the property \( p \) into the flux of energy.

\[ \sigma_e : \bigwedge^{m-1} T^*_e \mathcal{E} \to \bigwedge^{m-1} T^*_e \mathcal{E}. \] The stress at a point (event) is a linear transformation on the space of \( (m-1) \)-forms.

May be applied to “resources” other then energy?
Local Representation of Stress-Tensors

- Denote by \( \{ \hat{e}^i \} \) the basis of the \( m \)-dimensional space of \( (m - 1) \)-forms. Denote its dual basis by \( \{ \hat{e}_j \} \).

- Since the stress at a point is a linear transformation on the space of \( (m - 1) \)-forms it may be represented in the form \( \hat{\sigma}^j_\hat{i} \hat{e}_j \otimes \hat{e}^i \).

- If we had a volume element \( \theta \) we would have an isomorphism \( \bigwedge^{m-1}(T^*U) \leftrightarrow TU \) of \( (m - 1) \)-forms and vectors, such that \( J \leftrightarrow v \) are given by \( \theta(v, u, w) = J(u, w) \).

- Thus, with a volume element and due to the following structure,

\[
\begin{array}{c}
\bigwedge^{m-1}(T^*U) \\
i_\theta^{-1}
\end{array} \xrightarrow{\sigma} \begin{array}{c}
\bigwedge^{m-1}(T^*U) \\
i_\theta
\end{array}
\begin{array}{c}
\bigwedge^{m-1}(T^*U) \\
\theta
\end{array} \xrightarrow{\tilde{\sigma}} \begin{array}{c}
TU \\
\end{array}
\begin{array}{c}
\end{array} \xrightarrow{\tilde{\sigma}} \begin{array}{c}
TU
\end{array},
\]

one may represent a stress \( \sigma \) by a linear transformation \( \tilde{\sigma} \) on \( TU \).

- **Surprisingly, \( \tilde{\sigma} \) is independent of the volume element \( \theta \). In fact, you can construct a natural isomorphism \( \sigma \leftrightarrow \tilde{\sigma} \) without a volume element.**
Maxwell Stress-Energy Tensor without a Metric

- Maxwell 2-form: \( g \), a flow potential for \( J \), i.e., \( J = dg \).
- Faraday 2-form: \( f \) such that \( df = 0 \).
- Assume a volume element and set \( w = i_\theta(J) \) to be the vector field representing the flux form.
- Define the stress-energy tensor as the section \( \sigma \) of 
  \[ L(\wedge^{m-1}(T^*U), \wedge^{m-1}(T^*U)) \] 
  by 
  \[ \sigma(J) = (w \downarrow g) \wedge f - (w \downarrow f) \wedge g. \]

- The power is 
  \[ d\sigma(J) = (w \downarrow f) \wedge J + (L_w g) \wedge f - (L_w f) \wedge g. \]

—a generalization of the Lorentz force \((w \downarrow f) \wedge J\). (\(L\) is the Lie derivative.) The two additional terms cancel in the traditional situation.