Some Extensions and Analysis of Flux and Stress Theory

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Forces and Cauchy Stresses on Manifolds
Cauchy Stress Theory on Manifolds

Reminder:

- The classical Cauchy theory for the existence of stress uses the \textit{metric structure} of the Euclidean space.
- How would you generalize the notion of stress and Cauchy’s postulate so the theory can be formulated on a general manifold?
Added Benefit

- Such a stress object will *unify* the classical Cauchy stress and Piola-Kirchhoff stress.

- If you consider a material body as a manifold, all configurations of the body, in particular, the current configuration and any reference configuration, are *equivalent charts* in terms of the manifold structure of the body.

- The transformation from the Cauchy stress to the Piola-Kirchhoff stress will be just a transformation rule for two different representations of the *same stress object*. 
In Classical Continuum Mechanics

The force on a body \( \mathcal{B} \) in the material manifold \( \mathbb{R}^3 \) is given by

\[
F_{\mathcal{B}} = \int_{\mathcal{B}} b_{\mathcal{B}} \, dV + \int_{\partial \mathcal{B}} t_{\mathcal{B}} \, dA.
\]

- \( b_{\mathcal{B}} \) is the body force on \( \mathcal{B} \);
- \( t_{\mathcal{B}} \) is the surface force on \( \mathcal{B} \).

The force system \( \{(b_{\mathcal{B}}, t_{\mathcal{B}})\} \) is considered as a set function.
Cauchy’s Postulates for the dependence on \( B \).

- The \textit{body force} \( b_B \) does not depend on the body, i.e., \( b_B(x) = b(x) \).
- The \textit{surface force} at a point on the boundary of a control volume depends on the normal to the boundary at that point, i.e., \( t_B(x) = \Sigma_x(n(x)) \).
- \( \Sigma_x \) is assumed to be \textit{continuous}.
- There is a vector field \( s \) on the material manifold, the \textit{ambient force} or \textit{self force} (usually taken as zero), such that

\[
I_B = \int_B b_B \, dv + \int_{\partial B} t_B \, da = \int_B s \, dv.
\]

\textbf{Cauchy’s Theorem:} \( \Sigma_x \) is linear.
Obstacles to the Generalization to Manifolds:

- You cannot integrate vector fields on manifolds.
- You do not have a unit normal if you do not have a Riemannian metric.

Basic modifications:

- Use *integration of forms* on manifolds to integrate scalar fields.
- Write the force in terms of *power* expanded for various velocity fields so you integrate a scalar field.
- Use dependence on the *tangent space* instead of direction of the normal.
- Use *restriction* of forms for Cauchy’s formula.
Preliminaries for Continuum Mechanics on Manifolds

\( \mathcal{U} \) is the material manifold, \( \dim \mathcal{U} = m \);

\( \mathcal{B} \) a body is an \( m \)-dimensional submanifold on \( \mathcal{U} \).

\( \mathcal{M} \) is the physical space manifold, \( \dim \mathcal{M} = \mu \).

A configuration of a body \( \mathcal{B} \) is an embedding

\[ \kappa: \mathcal{B} \rightarrow \mathcal{M}. \]

A velocity is a mapping

\[ w: \mathcal{B} \rightarrow T\mathcal{M} \] such that, \( \tau_\mathcal{M} \circ w = \kappa \) is a configuration.

Alternatively, if

\[ \kappa^*(\tau_\mathcal{M}): W = \kappa^*(T\mathcal{M}) \rightarrow \mathcal{U} \]

is the pullback, a velocity at \( \kappa \) may be regarded as a section

\[ w: \mathcal{U} \rightarrow W. \]
Velocity Fields

\[ \mathcal{B} \]

\[ \kappa \]

\[ \mathcal{M} \]

\[ \omega \]
Bundles and Pullbacks

\[ W = \kappa^* E \]

\( E \) — a bundle

\( \pi \) — projection

\( W = \kappa^* E \)

A body

\( \mathcal{M} \) — space manifold

\( E \) — a bundle (e.g., \( T\mathcal{M} \))

\( \mathcal{U} \)

\( W = \kappa^* E \)
Sections of Bundles

$W = \kappa^*E$

$E$—a bundle

$E_x$—a section (e.g., $T\mathcal{M}$)

$\kappa$—a body

$\mathcal{B}$—a body

$\mathcal{M}$—a space manifold

Reuven Segev: Geometric Methods, March 2001
Force Densities

\[ F_B(w) = \int_B b_B(w) + \int_{\partial B} t_B(w), \]

for linear

\[ b_B(x): W_x \to m \bigwedge T_x \mathcal{U}, \text{ and } t_B(y): W_y \to m-1 \bigwedge T_y \partial B. \]

- \( b_B \) is a section of

\[ L(W, m \bigwedge (T_B)) = m \bigwedge (T_B, W^*), \]

- \( t_B \) is a section of

\[ L(W, m-1 \bigwedge (T \partial B)) = m-1 \bigwedge (T \partial B, W^*), \]

— \( W^* \)-valued forms.
Vector Valued Forms

- $\gamma_x \in L(W_x, \bigwedge^k(T_xP))$, $P \subset \mathcal{U}$ a submanifold, $k \leq \dim(P)$.
- $\tilde{\gamma}_x : (T_xP)^n \to W_x^*$, alternating, multi-linear.

\[ \tilde{\gamma}_x \in \bigwedge^k(T_x\mathcal{U}, W_x^*), \text{ a (co-)vector valued form.} \]

- The requirement

\[ \tilde{\gamma}_x(v_1, \ldots, v_k)(u) = \gamma_x(u)(v_1, \ldots, v_k), \]

for any collection of $k$ vectors $v_1, \ldots, v_k$, and $u \in W_x$, generates an isomorphism

\[ L(W_x, \bigwedge^k(T_xP)) = \bigwedge^k(T_x\mathcal{U}, W_x^*). \]
What Will Cauchy’s Theorem and Formula Look Like?

For scalars, the flux form was an $(m - 1)$-form $J$ on an $m$-dimensional manifold. By restriction, the Cauchy formula, $\tau_B = \iota^*(J)$, induces an $(m - 1)$-form on $T_x\partial B$.

- For the case of force theory, $t_B(w)$ is an $(m - 1)$-form, the flux of power, where $t_B(x): W_x \to \bigwedge^{m-1} T^*x\partial B$.
- The natural generalization: at each point $x$ there is a linear mapping $\sigma_x: W_x \to \bigwedge^{m-1} T^*xU$, called the stress at $x$, such that $t_B(w) = \iota^*(\sigma(w))$. In other words,

$$t_B = \iota^* \circ \sigma,$$

is the required Cauchy formula.
The Cauchy Postulates: Notes.

The dependence of $t_B(x)$ on the subbody $B$ through the tangent space to $B$ is assumed to be continuous in the tangent space and point $x$. This aspect, that we neglected before, should be meaningful.

- The collection of hyperplanes, $G_{m-1}(TU)$—the Grassmann bundle, i.e., $(G_{m-1}(TU))_x$ is the manifold of $(m-1)$-dimensional subspaces of $T_xU$.

- The mapping that assigns the surface forces to hyperplanes will be referred to as the Cauchy section. At each point it is a mapping

$$
\Sigma_x : G_{m-1}(T_xU) \rightarrow L(W_x, \bigwedge^{m-1} (G_{m-1}(T_xU))^*)
$$
The Cauchy Postulates: The Cauchy Section

More precisely, consider the diagram

\[
\begin{array}{ccc}
\pi_G^*(W) & \xrightarrow{\pi_G^*(\pi)} & G_{m-1}(T\mathcal{U}) \\
\uparrow & & \downarrow \pi_G \\
W & \xrightarrow{\pi} & \mathcal{U}
\end{array}
\]

Then, the Cauchy section is a section

\[
\Sigma: G_{m-1}(T\mathcal{U}) \rightarrow L(\pi_G^*(W), \bigwedge^{m-1}(G_{m-1}(T\mathcal{U}))^*)
\]

- It is assumed that \( \Sigma \) is smooth.
The Cauchy Postulates: Boundedness

We need the analog of the boundedness assumption

$$\left| \int_B \beta + \int_{\partial B} \tau_B \right| \leq \int_B s,$$

where eventually we get $\tau_B = i^* (J)$ and $\int_{\partial B} \tau_B = \int_B dJ$.

- We write the scalar boundedness assumption for the power, so $\beta = b(w)$ and $\tau_B = t_B(w)$.
- We anticipate that $t_B = i^* \circ \sigma$. Hence, the bounded expression is

$$\left| \int_B b(w) + \int_{\partial B} t_B(w) \right| = \left| \int_B b(w) + \int_{\partial B} i^* (\sigma(w)) \right| = \left| \int_B b(w) + \int_{\partial B} d(\sigma(w)) \right|.$$

Thus, the expression should be bounded by the values of both $w$ and its derivative—the first jet $j^1(w)$. 
Consequences of the (Generalized) Cauchy Theorem

Since \( t_\mathcal{B}(w) = i^* (\sigma(w)) \), the total power is given as

\[
F_\mathcal{B}(w) = \int_\mathcal{B} b(w) + \int_{\partial \mathcal{B}} t_\mathcal{B}(w) = \int_\mathcal{B} b(w) + \int_\mathcal{B} d(\sigma(w)).
\]

- The density of \( F_\mathcal{B}(w) \) depends linearly on the values of \( w \) and its derivative.
- For manifolds, there is no way to separate the value of the derivative of a section from the value of the section. Hence \( j^1(w) \)—the first jet of \( w \) is a single invariant quantity that contains both the value and the value of the derivative.

Thus, the expression should be bounded by the values of both \( w \) and its derivative—the first jet \( j^1(w) \).
Variational Stresses
**Jets**

A jet of a section at $x$ is an invariant quantity containing the values of both the section and its derivative.

$J^1(W)_x$ — the collection of all possible values of jets at $x$ — the jet space.

$J^1(W)$ — the collection of jet spaces, the jet bundle.
Variational Stresses

We obtained

\[ F_\mathcal{B}(w) = \int_{\mathcal{B}} \left( b(w) + d(\sigma(w)) \right). \]

- The value of the power density at a point is linear in the jet of \( w \).
- Hence, there is a section \( S \), such that

\[ S_x : J^1(W)_x \to \bigwedge^m T^*_x \mathcal{U} \text{ such that } S_x(j^1(w)_x) = b(w) + d(\sigma(w)). \]

- We will refer to such a section \( S \) of \( L(J^1(W), \bigwedge^m (T^*_x \mathcal{U})) \) as a variational stress density. It produces power from the jets (gradients) of the velocity fields.
- Thus,

\[ F_\mathcal{B}(w) = \int_{\mathcal{B}} \left( b(w) + d(\sigma(w)) \right) = \int_{\mathcal{B}} S(j^1(w)). \]

Conclusion:

A Cauchy stress \( \sigma \) and a body force \( b \) induce a variational stress density \( S \).
Variational Stress Densities:

- Variational stress densities are sections of the vector bundle $L(J^1(W), \bigwedge^m (T^*U))$, i.e., at each point, is assigns an $m$-covector to a jet at that point, linearly.

- If $S$ is a variational stress density, then the power of the force $F$ it represents over the body $\mathcal{B}$, while the generalized velocity is $w$, is given by

$$F_{\mathcal{B}}(w) = \int_{\mathcal{B}} S(j^1(w)).$$

This expression makes sense as $S(j^1(w))$, is an $m$-form whose value at a point $x \in \mathcal{B}$ is $S(x)(j^1(w)(x))$.

- The local representation of $S$ is through the arrays $S_\alpha$ and $S^i_\beta$. The single component of the $m$-form $S(j^1(w))$ in this representation is

$$S_\alpha w^\alpha + S^j_\beta w^\beta_{ij}.$$
Linear Connections

\[ \Gamma \] — the connection mapping

Vertical component

Horizontal component

\[ W_x \quad W_y \]

\[ \pi \]

\[ \pi \]

\[ w \]

\[ T \pi \]

No connection

\[ \Gamma \] — the connection mapping

Flux and Stress Theories

Pisa, Oct. 2007
The Case where a Connection is Given:

- If a connection is given on the vector bundle $W$, the jet bundle is isomorphic with the Whitney sum $W \oplus B L(T B, W)$ by $j^1(w) \mapsto (w, \nabla w)$, where $\nabla$ denotes covariant derivative.
- A variational stress may be represented by sections $(S_0, S_1)$ of
  \[ L(W, \bigwedge (T^∗U)) \oplus B L(L(TU, W), \bigwedge (T^∗B)) \]
  so the power is given by (see Segev (1986))
  \[ F_B(w) = \int_B S_0(w) + \int_B S_1(\nabla w). \]

We will refer to the section $S_1$ of $L(L(TU, W), \bigwedge^m(T^∗B))$ as the \textit{variational stress tensor}.
- With an appropriate definition of the divergence, a force may be written in terms of a body force and a surface force.
Problem: Relation Between Variational and Cauchy Stresses

- Can we extract the generalized Cauchy stress $\sigma$ from the variational stress $S$ invariantly?
- There is a linear $p_\sigma : L(J^1(W), \wedge^m(T^*\mathcal{B})) \to L(W, \wedge^{m-1}(T^*\mathcal{B}))$ that gives a Cauchy stress $\sigma = p_\sigma(S)$ to any given variational stress $S$.
- Locally, if $\sigma$ is represented by $\sigma_{\beta\hat{i}}$ such that $\sigma_{\beta\hat{i}}w^\beta$ is the $i$-th component of the $(m-1)$-from $\sigma(w)$, locally $p_\sigma$ is given by

$$
(x^i, S_\alpha, S^j_\beta) \mapsto (x^i, \sigma_{\beta\hat{i}})
$$

where,

$$
\sigma_{\beta\hat{i}} = (-1)^{i-1}S^{+i}_\beta, \quad (\text{no sum over } i).
$$

- Can you write a generalized definition of the divergence that applies even without a connection? ✔ Locally, the divergence $\text{Div } S$ is given by

$$
(S^{i}_{\alpha,i} - S_\alpha).
$$
The Vertical Subbundle of the Jet Bundle:

- Let $\pi_0^1: J^1(W) \rightarrow W$ be the natural projection on the jet bundle that assign to any 1-jet at $x \in B$ the value of the corresponding 0-jet, i.e., the value of the section at $x$.
- We define $VJ^1(W)$, the vertical sub-bundle of $J^1(W)$, to be the vector bundle over $B$ such that
  \[ VJ^1(W) = (\pi_0^1)^{-1}(0), \]
where $0$ is the zero section of $W$.
- There is a natural isomorphism
  \[ I^+: VJ^1(W) \rightarrow L(TU, W). \]
The Vertical Subbundle $VJ^1(W)$:
The Vertical Component of a Variational Stress:

- Let $\iota_V: V^1(W) \rightarrow J^1(W)$ be the inclusion mapping of the sub-bundle.
- Consider the linear injection, $\iota_n = \iota_V \circ (I^+)^{-1}: L(T\mathcal{U}, W) \rightarrow J^1(W)$.
- Thus we have a linear surjection
  \[
  \iota^*_n: L(J^1(W), \bigwedge (T^*\mathcal{B})) \rightarrow L(L(T\mathcal{U}, W), \bigwedge (T^*\mathcal{B}))
  \]
  given by $\iota^*_n(S) = S \circ \iota_n$.
- For a variational stress $S$, we will refer to
  \[
  S^+ = \iota^*_n(S) \in L(L(T\mathcal{U}, W), \bigwedge (T^*\mathcal{B}))
  \]
  as the vertical component of $S$. (The symbol of the variational stress).
- If the variational stress is represented locally by $(S_\alpha, S^j_\beta)$, then, $S^+$ is represented locally by $S^+_{\alpha} = S^i_\alpha$.
- Clearly, one cannot define invariantly (without a connection) a "horizontal" component to the stress.
Variational Fluxes:

- Since the jet of a real valued function $\varphi$ on $\mathcal{B}$ can be identified with a pair $(\varphi, d\varphi)$ in the trivial case where $W = \mathcal{B} \times \mathbb{R}$, the jet bundle can be identified with the Whitney sum $W \oplus_{\mathcal{B}} T^*\mathcal{U}$.

- $VJ^1(W)$ can be identified with $T^*\mathcal{U}$ and the vertical component of the variational stress is valued in $L(T^*\mathcal{U}, \Lambda^m(T^*\mathcal{B}))$. We will refer to sections of $L(T^*\mathcal{U}, \Lambda^m(T^*\mathcal{B}))$ as variational fluxes.

- There is a natural isomorphism

$$i\wedge : \Lambda^{m-1}(T^*\mathcal{B}) \to L(T^*\mathcal{U}, \Lambda^m(T^*\mathcal{B}))$$

given by $i\wedge(\omega)(\varphi) = \varphi \wedge \omega$. 
The Cauchy Stress Induced by a Variational Stress:

- Consider the *contraction* natural vector bundle morphism

\[ c: L(L(TU, W), \bigwedge^m (T^* B)) \oplus_B W \to L(T^* U, \bigwedge^m (T^* B)) \]

given by

\[ c(B, w)(\phi) = B(w \otimes \phi), \]

for \( B \in L(L(TU, W), \bigwedge^m (T^* B)) \), \( w \in W \), and \( \phi \in T^* U \), where

\((w \otimes \phi)(v) = \phi(v)w\). We also write \( w \downarrow B \) for \( c(B, w) \).

- For a section \( S^+ \) of \( L(L(TU, W), \bigwedge^m (T^* B)) \) and a vector field \( w \), \( w \downarrow S^+ \) is a *variational flux*.

- Consider the mapping

\[ i_{\sigma}: L(L(TU, W), \bigwedge^m (T^* B)) \to L(W, \bigwedge^{m-1} (T^* B)) \]

such that \( i_{\sigma} \circ S^+(w) = i_{\wedge}^{-1}(w \downarrow S^+) \). It is linear and injective.
Cauchy Stresses and Variational Stresses (Contd.)

- \( p_\sigma = i_\sigma \circ i^* : L(J^1(W), \Lambda^m(T^* B)) \rightarrow L(W, \Lambda^{m-1}(T^* B)) \) is a linear mapping (no longer injective) that gives a Cauchy stress to any given variational stress.

- Locally, \( \sigma \) is represented by \( \sigma_{\beta \hat{i}} \) such that \( \sigma_{\beta \hat{i}} w^\beta \) is the \( i \)-th component of the \( (m - 1) \)-from \( \sigma(w) \).

- Locally \( p_\sigma \) is given by

\[
(x^i, S_\alpha, S^j_\beta) \mapsto (x^i, \sigma_{\beta \hat{i}})
\]

where,

\[
\sigma_{\beta \hat{i}} = (-1)^{i-1} S^i_\beta, \quad (no \ sum \ over \ i).
\]
The Divergence of a Variational Stress:

- For a given variational stress $S$ and a generalized velocity $w$, consider the difference, an $m$-form,

$$d(p_\sigma(S)(w)) - S(j^1(w)).$$

- Locally, the difference is represented by

$$\left( S^i_{\alpha,i} - S_\alpha \right) w^\alpha$$

- This shows that the difference depends only on the values of $w$ and not its derivative.

- Define the generalized divergence of the variational stress $S$ to be the section $\text{Div}(S)$ of the vector bundle $L(W, \wedge^m (T^*B))$ satisfying

$$\text{Div}(S)(w) = d(p_\sigma(S)(w)) - S(j^1(w)) = d\sigma(w) - S(j^1(w)),$$

$\sigma = p_\sigma(S)$, for every generalized velocity field $w$. 
The Principle of Virtual Power:

- Given a variational stress $S$, the expression for the power is
  \[ F_B(w) = \int_B S(j^1(w)). \]

- Using the previous constructions and Stokes’ theorem we have
  \[ F_B(w) = \int_{\partial B} i^*_B(\sigma(w)) - \int_B \text{Div}(S)(w), \]
  where, $\sigma = p_\sigma(S)$ is the the Cauchy stress induced by the variational stress $S$, and $i^*_B$ is the restriction of $(m-1)$-forms on $B$ to $\partial B$.

- Thus we have for $t_B(w) = i^*_B(\sigma(w)) = i^*_B(p_\sigma(S)(w))$ and $\text{Div} S + b_B = 0$, a force for each subbody $B$ of the form
  \[ F_B(w) = \int_{\partial B} b_B(w) + \int_B t_B(w). \]
Conclusions:

- The mapping relating the values of variational stress fields and Cauchy stresses

\[ p_\sigma : L(J^1(W), \wedge(T^* U)) \rightarrow L(W, \wedge (T^* U)), \]

is linear, surjective, but not injective.

- However, the mapping between the fields

\[ p : S \mapsto (\sigma, b), \quad \sigma = p_\sigma \circ S, \quad b = -\text{Div} \, S, \]

is injective.

- The inverse, \( p^{-1} : (\sigma, b) \mapsto S \), is given by

\[ S(x)(A) = b_x(w_x) + d\sigma(w)_x, \]

for any vector field \( w \) whose jet at \( x \) is \( A \).