Extensions of Flux Theory

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Objects of Interest

- **Fluxes and stresses**: as fundamental objects of continuum mechanics.
- **Geometric aspects**: Formulations that do not use the traditional geometric and kinematic assumptions. For example, Euclidean structure of the physical space, mass conservation. Materials with micro-structure (sub-structure), growing bodies.
- **Analytic aspects**: Irregular bodies and flux fields. Fractal bodies.
Flux Theory?

Derive the existence of the flux vector field \( j \), e.g., the heat flux vector field or the electric current density, and its properties from global balance laws, e.g., balance of energy or conservation of charge.

Relevant Operations:

- **Total Flux (Flow) Calculation:**
  \[
  \int_{A} j \cdot n \, dA.
  \]

- **Gauss-Green Theorem:**
  \[
  \int_{\partial B} j \cdot n \, dA = \int_{B} \text{div} \, j \, dV.
  \]
Questions Regarding the Operations

- **Total Flux Calculation:**
  \[ \int_A j \cdot n \, dA. \]
  - How irregular can \( A \) be?

- **Gauss-Green Theorem:**
  \[ \int_{\partial B} j \cdot n \, dA = \int_B \text{div} \, j \, dV. \]
  - How irregular can \( B \) be?
  - How irregular can \( j \) be?
Examples:
Balanced Extensive Properties

In terms of scalar extensive property $p$ with density $\rho$ in space, one assumes for every “control region” $B \subset \mathcal{U} \cong \mathbb{R}^3$:

- Consider $\beta$, interpreted as the \textit{time derivative} of the density $\rho$ of the property, so for any control region $B$ in space, $\int_B \beta \, dV$ is the rate of change of the total content of the property inside $B$.

- For each control region $B$ there is a \textit{flux density} $\tau_B$ such that $\int_{\partial B} \tau_B \, dA$ is the \textit{total flux (flow)} of the property out of $B$.

- There is a function $s$ on $\mathcal{U}$ such that for each region $B$

$$\int_{B} \beta \, dV + \int_{\partial B} \tau_B \, dA = \int_{B} s \, dV.$$

Here, $s$ is interpreted as the \textit{source density} of the property $p$ (e.g., $s = 0$ for mass and electric charge).
Fluxes: Traditional Cauchy Postulate and Theorem

Cauchy’s postulate and theorem are concerned with the dependence of $\tau_B$ on $B$.

- It uses the metric properties of space.
- $\tau_B(x)$ is assumed to depend on $B$ only through the unit normal to the boundary at $x$.
- The resulting Cauchy theorem asserts the existence of the flux vector $j$ such that $\tau_B(x) = j \cdot n$. 
Assumptions Again:

In terms of a scalar extensive property with density $\rho$ in space, one assumes that there are operators $T(\partial B)$, the \textit{total flux operator}, and $S(B)$ the \textit{total content} operator, such that for every “control region” $B \subset \mathcal{U} \cong \mathbb{R}^3$ (we take $s = 0$):

- **Balance:** $T(\partial B) + S(B) = 0$
- **Regularity:** $S(B) = \int_B \beta_B \, dV$, and $T(\partial B) = \int_{\partial B} \tau_B \, dA$
- **Locality (pointwise):** $\beta_B(x) = \beta(x)$, and $\tau_B(x) = \tau(x, \mathbf{n})$
- **Continuity:** $\tau(\cdot, \mathbf{n})$ is continuous.

\textbf{Note:} It follows from the balance and regularity assumptions that

- $|\partial B| \to 0$ implies $T(\partial B) \to 0$,
- $|B| \to 0$ implies $T(\partial B) \to 0$

$| \cdot |$ being either the area or volume depending on the context.
The Results:

**Cauchy’s Theorem**

asserts that \( \tau(x, n) \) depends linearly on \( n \). There is a vector field \( j \) such that

\[
\tau = j \cdot n.
\]

Considering smooth regions and flux vector fields such that Gauss-Green theorem may be applied, the balance may be written in the form of a differential equation as

\[
\text{div } j + \beta = s.
\]
Traditional Proof:

- Consider the infinitesimal tetrahedron. Since the area is in an order of magnitude larger than the volume, the volume terms are negligible.
- Thus, $\sum_i A_i \tau(n_i) = 0$.
- Also, $\sum_i A_i n_i = 0$.
- Hence,

$$\tau \left( \frac{A_1}{A_4} n_1 + \frac{A_2}{A_4} n_2 + \frac{A_3}{A_4} n_3 \right) = \frac{A_1}{A_4} \tau(n_1) + \frac{A_2}{A_4} \tau(n_2) + \frac{A_3}{A_4} \tau(n_3)$$
Contributions in Continuum Mechanics

- Gurtin & Williams: 1967,
- Gurin & Martins: 1975,
- Gurtin, Williams & Ziemer: 1986,
- Noll & Virga: 1988,
- Degiovanni, Marzocchi & Musesti: 1999, . . .
The Proposed Formulation


- Building blocks: $r$-dimensional oriented cells in $E^n$.
- Formal vector space of $r$-cells: polyhedral $r$-chains.
- Complete w.r.t a norm: Banach space of $r$-chains.
- Elements of the dual space: $r$-cochains.

Relevance to Flux Theory

- The total flux operator on regions is modelled mathematically by a cochain.
- Cauchy’s flux theorem is implied by a representation theorem for cochains by forms.
Features of the Proposed Formulation

- It offers a common point of view for the analysis of the following aspects: class of domains, integration, Stokes’ Theorem, and fluxes.
- Allows irregular domains and flux fields.
- The co-dimension not limited to 1. Allows membranes, strings, etc. Not only the boundary is irregular, but so is the domain itself.
- Compatible with the formulation on general manifolds where no particular metric is given.
Outline

- Cells and polyhedral chains
- Algebraic cochains
- Norms and the complete spaces of chains
- The representation of cochains by forms:
  - Multivectors and forms
  - Integration
  - Representation
  - Coboundaries and differentiable balance equations
Cells and Polyhedral Chains
A cell, $\sigma$, is a non empty bounded subset of $E^n$ expressed as an intersection of a finite collection of half spaces.

The **plane of** $\sigma$ is the smallest affine subspace containing $\sigma$.

The **dimension** $r$ of $\sigma$ is the dimension of its plane. Terminology: an $r$-cell.

The boundary $\partial \sigma$ of an $r$-cell $\sigma$ contains a number of $(r-1)$-cells.
Recall: An orientation of a vector space is determined by a choice of a basis. Any other basis will give the same orientation if the determinant of the transformation is positive. A vector space can have 2 orientations.

An oriented $r$-cell is an $r$-cell with a choice of one of the two orientations of the vector space associated with its plane.

The orientation of $\sigma' \in \partial \sigma$ is determined by the orientation of $\sigma$:

- Choose independent $(v_2, \ldots, v_r)$ in $\sigma'$.
- Order them such that given $v_1$ in the plane of $\sigma$ which points out of $\sigma'$, $(v_1, \ldots, v_r)$ are positively oriented relative to $\sigma$. 

![Diagram of oriented cells](https://via.placeholder.com/150)
Polyhedral Chains: Algebra into Geometry

- A **polyhedral r-chain** in $E^n$ is a formal linear combination of $r$-cells
  \[ A = \sum a_i \sigma_i. \]

- The following operations are defined for polyhedral chains:
  - The polyhedral chain $1\sigma$ is identified with the cell $\sigma$.
  - We associate multiplication of a cell by $-1$ with the operation of inversion of orientation, i.e., $-1\sigma = -\sigma$.
  - If $\sigma$ is cut into $\sigma_1, \ldots, \sigma_m$, then $\sigma$ and $\sigma_1 + \ldots + \sigma_m$ are identified.
  - Addition and multiplication by numbers in a natural way.

- The space of polyhedral $r$-chains in $E^n$ is now an **infinite-dimensional vector space** denoted by $A_r(E^n)$.

- The **boundary of a polyhedral r-chain** $A = \sum a_i \sigma_i$ is $\partial A = \sum a_i \partial \sigma_i$. Note that $\partial$ is a linear operator $A_r(E^n) \longrightarrow A_{r-1}(E^n)$. 

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Polyhedral Chains: Illustration

\[ A = A_1 + A_2 \]
\[ \partial A = \partial A_1 + \partial A_2 \]
\[ \partial A \]

\[ \partial : \mathcal{A}_r \rightarrow \mathcal{A}_{r-1} \]
A Polyhedral Chain as a Function

\[ A = \sum a_i \sigma_i \]

\[ \partial A = \sum a_i \partial \sigma_i \]
Total Fluxes as Cochains

Basic Idea:

Regard the flux through a 2-dimensional chain as the action of a linear operator—a co-chain—on that chain.

A cochain: Linear $T: \mathcal{A}_r \rightarrow \mathbb{R}$. We write $T(B) = T \cdot B$.

Algebraic implications:

- additivity,
- interaction antisymmetry.

\[
T \cdot (-\sigma) = -T \cdot \sigma, \quad T \cdot (\sigma_1 + \sigma_2) = T \cdot \sigma_1 + T \cdot \sigma_2
\]
Norms and the Complete Space of Chains: 
Analysis into Geometry
The Norm Induced by Boundedness

**Boundedness:** $|T_{\partial B}| \leq N_2 |\partial B|$, $|T_{\partial B}| \leq N_1 |B|$. Setting $A = \partial B$, ...

**As a cochain:** $|T \cdot A| \leq N_2 |A|$, $|T \cdot \partial D| \leq N_1 |D|$, $A \in \mathcal{A}_r$, $D \in \mathcal{A}_{r+1}$.

Thus, for any $D \in \mathcal{A}_{r+1}$, and $A \in \mathcal{A}_r$:

$$|T \cdot A| = |T \cdot A - T \cdot \partial D + T \cdot \partial D|$$

$$\leq |T \cdot A - T \cdot \partial D| + |T \cdot \partial D|$$

$$\leq N_2 |A - \partial D| + N_1 |D|$$

$$\leq C_T (|A - \partial D| + |D|),$$

---

**Basic Idea (revised)**

Regard the flux as a *continuous linear functional* on the space of chains w.r.t. a norm

$$|T \cdot A| \leq C_T \|A\|,$$

where the *flat norm* (smallest) is given as

$$\|A\| = |A|^b = \inf_D \{|A - \partial D| + |D|\}.$$
Flat Chains

- The *mass* of a polyhedral $r$-chain $A = \sum a_i \sigma_i$ is $|A| = \sum |a_i| |\sigma_i|$.
- The *flat norm*, $|A|^\flat$, of a polyhedral $r$-chain:

$$|A|^\flat = \inf\{|A - \partial D| + |D|\},$$

using all polyhedral $(r + 1)$-chains $D$.

- Taking $D = 0$, $|A|^\flat \leq |A|$.
- If $A = \partial B$, taking $D = B$ gives $|A|^\flat \leq |B|$. Hence, $|\partial B|^\flat \leq |B|$.

- Completing $\mathcal{A}_r(E^n)$ w.r.t. the flat norm gives a Banach space denoted by $\mathcal{A}_r^\flat(E^n)$, whose elements are *flat* $r$-chains in $E^n$.
- Flat chains may be used to represent continuous and smooth submanifolds of $E^n$ and even irregular surfaces.
- The *boundary of a flat $(r + 1)$-chain* $A = \lim^\flat A_i$, is the a flat $r$-chain $\partial A = \lim \partial A_i$. The boundary operator is continuous and linear.
Flat Chains, an Example (Illustration - I):

\[ |A_i| = 2L, \]
\[ |A_i|^b \leq (L + 2)d_i \rightarrow 0. \]
\[ |A_i|^b \leq 2d_i + d_i^2 \rightarrow 0. \]
Example: The Staircase

The dashed lines are for reference only.

\[ |A_i|^b \leq 2^{i-1}2^{-2i} = 2^{-i}/2 \quad \implies \quad (B_i) \text{ a convergent series.} \]

Note, \[ |B_i - B_j| = \left| \sum_{k=j+1}^{i} A_k \right| \leq \sum_{k=j+1}^{i} |A_k| \leq \sum_{k=j+1}^{\infty} |A_k| \leq \sum_{k=j+1}^{\infty} 2^{-k}/2, \quad \forall \quad i > j. \]
Example: the Van Koch Snowflake

$A_i$ contains $4^i$ triangles of side length $3^{-i}$. Each time the length increases by $2 \cdot 3^{-i} \cdot 4^i = 2 \left( \frac{4}{3} \right)^i$. Hence, $|B_i| \to \infty$.

\[
|A_i|^b \leq 4^i \frac{\sqrt{3}}{2} 3^{-i} 3^{-i} = \frac{\sqrt{3}}{2} \left( \frac{2}{3} \right)^i
\]
The Representation of Cochains by Forms

Objectives:

- Create an algebraic language to treat chains and cochains,
- A representation theorem for cochains in terms of fields and integration.
Multivectors

- A **simple r-vector** in $V$ is an expression of the form $v_1 \wedge \cdots \wedge v_r$, where $v_i \in V$.

- An **r-vector** in $V$ is a formal linear combination of simple $r$-vectors, together with:
  
  1. $v_1 \wedge \cdots \wedge (v_i + v'_i) \wedge \cdots \wedge v_r$
     
     $= v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r + v_1 \wedge \cdots \wedge v'_i \wedge \cdots \wedge v_r$;
  
  2. $v_1 \wedge \cdots \wedge (av_i) \wedge \cdots \wedge v_r = a(v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r)$;
  
  3. $v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_r$
     
     $= -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r$.

- The r-vector vanishes if the vectors are linearly dependent.

- The collection, $V_r$, of $r$-vectors is a vector space and \( \text{dim } V_r = \frac{n!}{(n-r)!r!} \).

- Given a basis $\{e_i\}$ of $V$, the r-vectors $\{e_{\lambda_1 \ldots \lambda_r} = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_r}\}$, such that $1 \leq \lambda_1 < \cdots < \lambda_r \leq n$, form a basis of $V_r$. 
The Representation of Polyhedral Chains by Multivectors

- Given an oriented $r$-simplex $\sigma$ in $E^n$, with vertices $\{p_0 \ldots p_r\}$, the $r$-vector of $\sigma$, $\{\sigma\}$, is $\{\sigma\} = v_1 \wedge \cdots \wedge v_r / r!$, where the $v_i$ are defined by $v_i = p_i - p_0$ and are ordered such that they belong to $\sigma$’s orientation.

$\{\sigma\}$ represents the plane, orientation and size of $\sigma$—the relevant aspects.

- The $r$-vector of a polyhedral $r$-chain $\sum a_i \sigma_i$, is

$$\{\sum a_i \sigma_i\} = \sum a_i \{\sigma_i\}.$$
Why an \( r \)-covector?

For the 3-dimensional example, we want to measure the flux through any infinitesimal cell \( \sigma \), \( \{ \sigma \} = v \wedge u \).

- Denote by \( T(\sigma) \) the flux through that infinitesimal element.
- As \( T(\sigma) \) depends only the plane, orientation and area, we expect
  \[
  T(\sigma) = \hat{T}(\{ \sigma \}).
  \]

- Balance: \( \hat{T} \) is linear
  \[
  \hat{T}(\sigma) = \tau \cdot \{ \sigma \},
  \]
  where \( \tau \) is a linear mapping of multi-vectors to real numbers—an \( r \)-covector.
Rough Proof

Consider the infinitesimal tetrahedron $X, A, B, C$ generated by the three vectors $u, v, w$.

— Use right-handed orientation.

— Balance implies:

$$T(v, u) + T(v, w) + T(u, v + w) - T(u + v, w) = 0.$$ 

— Same for $X, B, C, E$ and $X, C, D, E$

$$T(w, u) + T(u + v, w) + T(v, u) - T(v, w + u) = 0$$

$$T(w, u) - T(v + w, u) - T(v, w) + T(v, w + u) = 0.$$ 

— Add up to obtain: $T(u, v + w) = T(u, v) + T(u, w)$. 

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Or Using Multi-Vectors

- Consider the infinitesimal tetrahedron $D$ generated by the three vectors $u, v, w$ and let $A = \partial D$.
- $|A|^b \leq |A - \partial D| + |D| \to 0$, as the volume of the tetrahedron decreases.
- Thus, $\lim T(\{A\}) = 0$.

--- Use right-handed orientation.

Thus: $T(u \wedge v) + T(v \wedge w) + T(w \wedge u) + T((w - v) \wedge (v - u)) = 0$.

Using: $(w - v) \wedge (v - u) = w \wedge v - w \wedge u + v \wedge u = -u \wedge v - v \wedge w - w \wedge u$,

we conclude: $T(u \wedge v + v \wedge w + w \wedge u) = T(u \wedge v) + T(v \wedge w) + T(w \wedge u)$. 
Reminder: Dual Spaces of Vector Spaces

- For a vector space $\mathcal{W}$, $\mathcal{W}^*$—the *dual space*—is the collection of all linear mappings, $T : \mathcal{W} \rightarrow \mathbb{R}$ (also *linear functionals, covectors*).

- In our case, flat chains are in $\mathcal{A}_r^\flat (E^n)$, and the total fluxes, being continuous linear functionals of chains, are $T \in \mathcal{A}_r^\flat (E^n)^*$.

- For an infinite dimensional vector space on which a norm $\|w\|$ is defined, one also requires that $T$ is continuous. The condition for continuity (assuming linearity) is

$$|T(w)| \leq C_T \|w\|.$$ 

- This provides a procedure for generating new mathematical objects. Define a vector space and a norm and consider its dual space.

- *Representation Theorems*: represent the action of the linear functionals on vectors by known mathematical operations (inner products, integration).
Multi-Covectors

- An $r$-covector is an element of $V^r$—the dual space of $V_r$.
- $r$-covectors can be expressed using covectors:

$$V^r = (V^*)_r$$

$(V^*)_r$ is the space of multi-covectors, i.e., constructed as $V_r$ using elements of the dual space $V^*$:

$$\tau = f_{\lambda_1 \ldots \lambda_r} e^{\lambda_1} \wedge \cdots \wedge e^{\lambda_r}, \quad \lambda_i < \lambda_{i+1}.$$  

- $r$-covectors may be identified with alternating multilinear mappings:

$$V^r = L_A^r(V, \mathbb{R}), \quad \text{by} \quad \tau(v_1 \wedge v_2 \wedge \cdots \wedge v_r) = \bar{\tau}(v_1, \ldots, v_r).$$

- This is a simple example of a representation theorem for functionals.
Riemann Integration of Forms Over Polyhedral Chains

- An *r-form* in $Q \subset E^n$ is an $r$-covector valued mapping in $Q$.
- An $r$-form is continuous if its components are continuous functions.
- The *Riemann integral* of a continuous $r$-form $\tau$ over an $r$-simplex $\sigma$ is defined as
  \[
  \int_{\sigma} \tau = \lim_{k \to \infty} \sum_{\sigma_i \in S_k \sigma} \tau(p_i) \cdot \{\sigma_i\},
  \]
  where $S_i \sigma$ is a sequence of simplicial subdivisions of $\sigma$ with mesh $\to 0$, and each $p_i$ is a point in $\sigma_i$.
- The Riemann integral of a continuous $r$-form over a polyhedral $r$-chain $A = \sum a_i \sigma_i$, is defined by
  \[
  \int_A \tau = \sum a_i \int_{\sigma_i} \tau.
  \]
Lebesgue Integral of Forms over Polyhedral Chains

- An $r$-form in $E^n$ is *bounded and measurable* if all its components are bounded and measurable.

- The **Lebesgue integral** of an $r$-form $\tau$ over an $r$-cell $\sigma$ is defined by

$$\int_{\sigma} \tau = \int_{\sigma} \tau(p) \cdot \left\{ \sigma \right\} \frac{dp}{|\sigma|},$$

where the integral on the right is a Lebesgue integral of a real function.

- This is extended by linearity to domains that are polyhedral chains by

$$\int_{A} \tau = \sum a_i \int_{\sigma_i} \tau,$$

for $A = \sum_i a_i \sigma_i$. 

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The Cauchy Mapping

- The Cauchy mapping, $D_T$, for the cochain $T$, gives $D_T(p, \alpha)$, at the point $p$ in the direction $\alpha$ defined by the cell $\sigma$, defined as:

$$D_T(p, \alpha) = \lim_{i \to \infty} T \cdot \frac{\sigma_i}{|\sigma_i|}, \quad \alpha = \frac{\sigma_i}{|\sigma_i|}$$

where all $\sigma_i$ contain $p$, have $r$-direction $\alpha$ and $\lim_{i \to \infty} \text{diam}(\sigma_i) = 0$.

- The Cauchy mapping for a given cochain $T$, of $r$-directions is analogous to the dependence of the flux density on the unit normal.
**The Representation Theorem**

**Whitney:**

- **The analog to Cauchy’s flux theorem.** For each flat $r$-cochain $T$ there is an $r$-form $\tau = \tau_T$ that represents $T$ by

$$T \cdot A = \int_A \tau_T,$$

for every flat $r$-chain $A.$
Coboundaries and Balance Equations

- The *coboundary* $dT$ of an $r$-cochain $T$ is the $(r+1)$-cochain defined by
  \[ dT \cdot A = T \cdot \partial A. \]

  A very general form of “Stokes’ theorem”.

- Thus, $d$ is the *dual of the boundary operator*:
  \[
  \mathcal{A}^b_{r+1}(E^n) \xrightarrow{\partial} \mathcal{A}^b_r(E^n)
  \]
  \[
  \mathcal{A}^b_{r+1}(E^n)^* \leftarrow d=\partial^* \mathcal{A}^b_r(E^n)^*.
  \]

- The coboundaries of flat cochains are flat, as the boundary operator is continuous.

- Hence, there is a flat cochain $S$ satisfying the global balance equation:
  \[ S \cdot A + T \cdot \partial A = 0, \quad \forall A, \quad \implies \quad dT + S = 0. \]

  A very general form of the balance equation.
The Local Balance Equation

If \( \tau_T \) is a form that represents the total flux operator \( T \), then, by the representation theorem applied to \( dT \), there is a form representing \( dT \)

\[
d_0\tau = \tau_{dT}.
\]

Thus,

\[
dT \cdot B = T \cdot \partial B \quad \text{is represented by} \quad \int_B d_0\tau = \int_{\partial B} \tau_T.
\]

Let \( \beta \) be the \( r \)-form representing the rate of content operator \( S \) so

\[
T(\partial B) + S(B) = 0 \quad \text{is represented by} \quad \int_{\partial B} \tau_T + \int_B \beta = 0.
\]

One obtains the local expression

\[
d_0\tau + \beta = 0.
\]
Stokes’ Theorem for Differentiable Forms on Polyhedral Chains

- The exterior derivative of a differentiable \( r \)-form \( \tau \) is an \( (r + 1) \)-form \( d\tau \) defined by

\[
d\tau(p) \cdot (v_1 \wedge \cdots \wedge v_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i-1} \nabla_{v_i} \tau(p) \cdot (v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{r+1})
\]

where \( \hat{v}_i \) denotes a vector that has been omitted, and \( \nabla_{v_i} \) is a directional derivative operator.

- Stokes’ theorem for polyhedral chains, based on the fundamental theorem of differential calculus, states that

\[
\int_A d\tau = \int_{\partial A} \tau
\]

for every differentiable \( r \)-form \( \tau \) and an \( (r + 1) \)-polyhedral chain \( A \).
The Local Balance Equation for Differentiable Cochains

- Reminder:
  - If $\tau_T$ is a form that represents the total flux operator $T$, then, by the representation theorem applied to $dT$, there is a form representing $dT$
    \[ d_0 \tau = \tau_{dT}. \]
  - One obtains the local expression
    \[ d_0 \tau + \beta = 0. \]

- If $\tau_T$ is differentiable, then, $d_0 \tau = d\tau$, the exterior derivative.
Thanks