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Abstract

We investigate two-stage parallel contests with a finite number of heterogeneous agents with various skills (cost of efforts) and reputations, as well as a finite number of heterogeneous contests. Each agent chooses which contest to enter in the first stage, and in the second stage, the agents who chose the same contest compete against one another for a single prize, the value of which is a combination of a nominal value and the reputation of the competing agents. We demonstrate that reputation in contests may lead to an unstable environment. We present a sufficient condition for resolving this instability, which results in the existence of a subgame perfect equilibrium with pure strategies. We then provide an algorithm for calculating it. We demonstrate that our equilibrium is far more likely to hold in an environment where the agents' skills and reputations are correlated

Keywords: Parallel contests; reputation; heterogenous agents, non-linear effort costs.

JEL classification: D44, D72, D82, J31

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1 Introduction

”Reputation, reputation, reputation! Oh, I have lost my reputation! I have lost the immortal part of myself, and what remains is bestial.” This scene from the play ”Othello” by William Shakespeare begins the book of Origgi (2019) to emphasize why reputation is so important in our lives, both personally and socially. In any situation, an agent’s reputation, which refers to his general quality or ability as perceived or assessed by other agents, primarily has a direct impact on his opponents. For example, in the well-known biblical tale of ”David and Goliath” (1 Samuel 17), Saul, the king of Israel, and his soldiers are battling the Philistines in the Valley of Elah. Goliath, the Philistine champion, challenges the Israelites to send out a champion of their own to decide the outcome in single combat twice daily for 40 days, but Saul and all of his soldiers are afraid to face Goliath because of his reputation as an unconquerable warrior. Finally, David accepts the challenge, defeats the opponent, and establishes his reputation as a valiant warrior before ascending to the throne of Israel. This story shows that, in any competition, an agent’s value of winning depends on the reputation of his opponents; the higher their reputation, the higher an agent’s value of winning.

In this paper, we assume that each agent has his own reputation and that an agent’s winning value in a contest is influenced by the reputations of all the agents in the contest. We study the issue of reputation in a two-stage model of parallel contests. In the first stage, agents simultaneously select the contest in which they want to compete, with the restriction that each agent may only compete in a single contest. The agents in each contest compete against one another in the second stage after choosing their contests, and the winner of each contest is awarded a prize based on a nominal value and the reputation of the contending agents.

Given that an agent’s expected payoff depends on the nominal prize as well as the endogenous allocation of agents in each contest and their reputations, choosing which contest to enter may not be an easy decision. An agent wants to compete against highly reputable opponents because it increases his winning value, which is dependent on his opponents’ reputations, but it also makes winning the prize more difficult because there is a correlation between agents’ reputations and their abilities. For instance, different universities have different reputations around the world, and the value of attending a particular university increases with its reputation. On the other hand, it is undoubtedly more challenging for young researchers to obtain tenure at universities with a strong reputation. Similarly, professional tennis players must decide which tournaments they want to enter throughout the year, some of which take place at the same time. A tournament with highly reputable players will increase the value of winning, but it will also be much more difficult to win.

We first consider $m \geq 2$ parallel contests, each of which has a generalized Tullock contest success function (Tullock 1980). We assume that the parallel contests include n heterogeneous agents with different reputation types and identical effort cost functions. Then, there are circumstances in which there is no pure strategy equilibrium, meaning that some agents always prefer to leave the contest they chose to enter since by doing so they can increase their expected payoffs. This is because the value of the payoff for the agents is endogenously defined by the reputations of the agents in the contest. We actually provide a simple illustration of how a highly reputable agent is constantly evading the less reputable agents, such that a pure strategy equilibrium does not exist. Then we present a sufficient condition that, if met, ensures a subgame perfect equilibrium with pure strategies. We use a variety of examples throughout the analysis to demonstrate the viability of the sufficient condition.

Later, we assume that there are n heterogeneous agents with two possible types, each with a different non-linear cost of effort and a different reputation. Furthermore, the costs of effort are non-linear and not explicitly given. All we know about these costs of effort is that they are increasing and weakly convex in effort. We also assume some ranking, including the agents' costs of effort, that clarifies the relative strength of the agents of different types, and we assume the same ranking for reputation, meaning that agents with low costs of effort are more reputable. In that case, it is impossible to characterize the subgame perfect equilibrium since the costs of effort are not explicitly given, so the agents' expected payoffs over the parallel contests cannot be calculated. To overcome this obstacle, we use revealed preference techniques to demonstrate the existence of a subgame perfect equilibrium with pure strategies and provide an algorithm that enables us to calculate this equilibrium, namely, the allocation of agents across the contests. The algorithm is based on finite sequential steps, each of which includes an instruction that enables some types of agents to move across the contests. Our parallel contests have a subgame perfect equilibrium with pure strategies when the algorithm reaches a point at which no agent wants to change his position. At this point, the algorithm is said to have reached an equilibrium.

To summarize, we demonstrate that reputation in contests may lead to an unstable environment. Our goal is to determine when the reputation effect on contests leads to a stable outcome, that is, the conditions under which a subgame perfect equilibrium exists. We establish sufficient conditions that, in some cases, become necessary for the existence of this equilibrium, and we demonstrate that achieving these conditions is significantly more likely when agents' skills (costs of effort) and reputations are closely aligned. On the other hand, when agents have reputations that are unrelated to their skill sets (a low rank for an agent's cost of effort and a high rank for his reputation), the entire model becomes unstable.

1.1 Related literature

There are several papers dealing with parallel contests. These include the works of Konrad and Kovenock (2012) and Juang et al. (2020) who examine models of parallel all-pay contests with homogeneous players, and Azmat and Möller (2009), who study two parallel Tullock contests with homogeneous players. Azmat and Möller (2018) investigate parallel all-pay contests with a continuum of heterogeneous agents, and Morgan et al. (2018) examine two parallel contests involving a continuum of heterogeneous agents in which each agent's performance is deterministic but noisy. In each of the aforementioned models, the authors presumptively assume that agents are homogenous and have a linear cost of effort. Those who take into account heterogeneous agents also presumptively assume a continuum of agents. Cohen et al. (2023), in contrast, deal with a finite number of heterogeneous agents whose effort costs are non-linear. The current model deviates from Cohen et al. (2023) by assuming that the prizes are not constant and depend on the types (reputations) of the participating agents. This modification is crucial because, in contrast to all the papers mentioned above, an additional contestant does not necessarily disadvantage the other contestants.

In the second stage of our model of parallel contests, each contest takes place using a Tullock CSF (see, for example, Skaperdas 1996, Baye and Hoppe 2003, Ewerhart 2015, and Sela 2020). Based on the uniqueness of the equilibrium in one-stage generalized Tullock contests either in complete or incomplete information (see Szidarovszky and Okuguchi 1997, Yamazaki 2008, Einy et al. 2015, and Ewerhart and Quartieri 2020), our algorithm converges to the equilibrium in the case of non-linear costs of effort. The uniqueness of the equilibrium is important because it implies that there will always be a single equilibrium in every contest in the second stage for any change in the allocation of agents determined by the algorithm in the first stage.

The theoretical economic literature (e.g., Wilson 1985; Mailath and Samuelson 2015) primarily focuses on reputation effects in repeated games where each player builds his own reputation based on prior actions. Some of these models, for example, assume that a long-run player playing against a series of short-lived opponents can develop a reputation for playing in a certain way and thus benefit from commitment power (see, for instance, Kreps and Wilson, 1982; Kreps et al., 1982; Milgrom and Roberts, 1982; Cripps et al., 2004; Ely et al., 2008). However, in our two-stage model, an agent's reputation is fixed and unaffected by his actions. The common theme in our paper and the related literature is that interactions with highly reputable agents are beneficial to an agent.

The model by Damiano et al. (2010) considers a continuum of agents with heterogeneous types (ability or reputation) who must choose between two organizations of fixed sizes. Their interpretation of reputation is similar to that in our model. They assume that an agent's payoff increases with the average type of organization they join, but this payoff also increases with their ranking within the organization. Damiano et al. (2012) expand on their previous model by incorporating a competition between the two organizations, with each organization having the power to decide how to award prizes. Each organization benefits from the average type, and the agent's payoff increases with the nominal prize as well as the average type in the organization he chooses. In both of these models, agents simultaneously select the organization they wish to join. However, unlike our model, in which agents compete against one another in each contest (organization) and the outcome is determined by their actions, in their models, competitions among agents within these organizations are solely based on their types; that is, the agents are not active during that stage of the competition.

The literature on status in contests is very similar to that on reputation in contests (see, for example, Hopkins and Kornienko 2004, Kosfeld and Neckermann 2011, Bhattacharya and Dugar 2014, and Charness et al. 2014). Both situations are similar in that an agent prefers to be surrounded by high-type agents. However, in our model of contests with reputation, an agent wants to be in a group with high-type (highly reputable) agents, and the distribution of the agents in the other groups does not affect his payoff, while in status contests, an agent wants to be in the highest status class with the high-type agents, but the distribution of the agents in the other groups have a significant impact on his payoff. For example, Moldovanu et al. (2007), Dubey and Geanakoplos (2010), and Drugov and Ryvkin (2020), assume that each agent cares about the number of agents in the categories above and below him. In particular, agents receive a positive utility that is proportional to the number of agents in lower status categories and a negative utility that is proportional to the number of agents in higher status categories.

The rest of the paper is organized as follows: We describe the model of two-stage parallel contests with reputation in Section 2. We analyze the second stage in Section 3. We analyze the first stage of our model in Section 4, where agents have asymmetric reputations but similar cost functions. In Section 5, we analyze the first stage of our model, where agents have asymmetric cost functions as well as asymmetric reputations. Section 6 discusses the robustness and strength of our results, while Section 7 concludes. The appendix contains the proofs.

2 The model

Consider a set of n asymmetric agents, $N = \{1, 2, \ldots, n\}$, and a set of m asymmetric contests, $M = \{1, 2, \ldots, m\}$, where $m < n$. Each agent $i \in N$ has a reputation type α_i where $\alpha_i \geq \alpha_{i+1}$ for $i = 1, 2, ..., n-1$, and each contest $k \in M$ has a nominal prize v_k. In the first stage, each agent chooses a contest $k \in M$ to compete in where the set of agents who choose to compete in contest k is denoted by N_k, and n_k is the number of agents who choose to compete in contest k. In the second stage, after all of the agents have selected a contest to compete in, agents who have chosen the same contest compete against one another to win the contest. The agents in the contest k simultaneously exert their efforts. Let x_i denote the effort of agent $i \in N_k$. The probability that agent i will win contest k is $p_i = \frac{g_i(x_i)}{\sum_{j \in N_k} g_j(x_j)}$ where $g_i(x_i)$, which is called the production function, is assumed to be zero at $x_i = 0$, continuous, strictly increasing, and weakly concave in x_i for all $x_i \in \mathbb{R}_+$. Agent i who exerts an effort of x_i bears the cost of his effort $\hat{c}_i(x_i)$ regardless of whether he wins or loses, where $\hat{c}_i(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is the effort cost function, which is strictly increasing and weakly convex in the effort x_i. In addition, the winning value for the winner of contest k is a function $f_k(v_k, \bar{\alpha}_k)$ where v_k is the nominal prize of contest k and $\bar{\alpha}_k = (\alpha_i)_{i \in N_k}$ is the vector of all reputation types of the agents who have chosen contest k. The value function $f_k(v_k, \bar{\alpha}_k)$ is increasing in both the nominal prize v_k and the reputation types of the agents in contest k, $(\alpha_i)_{i\in N_k}$.The assumption that the value function is based on all contestants' reputation types is not critical and is made to simplify the calculations because, under this assumption, all contestants in the same contest have the same value, whereas if a contestant's value function is based on all contestants' reputations in the contest, excluding his own reputation, the contestants in the same contest have different value functions.[1]

We assume that adding an agent to a contest has a decreasing marginal effect on the value function f_k as the number of agents increases. Formally, denote by $f_k(v_k, \overline{\alpha}_k^m)$ the value function in contest k with m agents of reputation types $\overline{\alpha}_k^m = (\alpha_i)_{i=1}^m$ and by $f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s)$ the value function in contest k with m agents of the reputation types $\overline{\alpha}_k^m = (\alpha_i)_{i=1}^m$ plus $s > 0$ agents of the reputation types $\overline{\alpha}_s = (\alpha)_{j=1}^s$, namely, the additional s agents have the same reputation type α. Denote $R_{k,m,s} = |f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s) - f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_{s-1})|$ where $\overline{\alpha}_s$ is a vector of length of s and $\overline{\alpha}_{s-1}$ is a vector of length of $s-1$. Then we assume that $R_{k,m,s} > R_{k,m,s+1}$ for any given reputation vector $\overline{\alpha}_k^m$, $m = 2, 3, ...$ and for every reputation vectors $\overline{\alpha}_s$ and $\overline{\alpha}_{s-1}$. In particular, we assume that $\lim_{m\to\infty} R_{k,m,s} = 0$. In other words, the number of agents in the contest reduces the impact of an additional agent on the value function.

The expected payoff of agent $i \in N_k$ is:

$$f_k(v_k, \bar{\alpha}_k)\frac{g_i(x_i)}{\sum_{j\in N_k} g_j(x_j)} - \hat{c}_i(x_i). \tag{1}$$

Let $y_j = g_j(x_j)$ for all $j \in N_k$. Then, the expected payoff given by (1) is equivalent to

$$f_k(v_k, \bar{\alpha}_k)\frac{y_i}{\sum_{j\in N_k} y_j} - c_i(y_i), \tag{2}$$

where $c_i = g_i^{-1} \circ \hat{c}_i$.[2] The assumptions on g_i and $\hat{c}_i$ imply that c_i is a well-defined function with $c_i(0) = 0$, and that c_i is strictly increasing and weakly convex with respect to y_i. Thus, we will consider the maximization function given by (2) and assume that the agents' costs of effort can be ranked such that for all $x \geq 0$, $c_i(x) \leq c_{i+1}(x)$, $i = 1, ..., n-1$. Then, we can say that an agent with a lower cost function has a higher ability than other agents with higher cost functions. This competition will be known as the model of two-stage parallel contests with reputation.[3]

[1]It is worth noting that our main findings in this paper about the existence of equilibrium are valid even if we assume that an agent's value function is solely determined by the reputation of the other agents in the contest.

[2]This transformation was proposed by Szidarovszky and Okuguchi (1997), and Yamazaki (2008).

[3]We can assume a much more general contest success function here than the one given by (1). However, we also require the

To study the dynamics of the competition, we first analyze the second stage, in which each agent has already chosen the specific contest in which he competes, and now each agent i in each contest k chooses the effort level to compete with the other agents who have chosen the same contest. Following that, we analyze the competition in the first stage, in which the agents choose the contest, based on the analysis of the second stage.

3 The second stage

There are m contests taking place at the same time in the second stage. At this stage of the competition, analyzing contest $k \in M$ is sufficient because the m contests are independent of one another. The maximization problem of agent i in contest k is

$$\max_{x_i} f_k(v_k, \bar{\alpha_k}) \frac{x_i}{\sum_{j \in N_k} x_j} - c_i(x_i). \tag{3}$$

The FOC (first-order condition) of this maximization problem is

$$f_k(v_k, \bar{\alpha}_k) \frac{\sum_{j \in N_k} x_j - x_i}{\left(\sum_{j \in N_k} x_j\right)^2} \leq c_i'(x_i).$$

Without loss of generality, it is assumed that all the agents are active in contest k. If there are inactive agents, only the active agents will be considered. Following that, solving the maximization problem (3) yields the well-known result:

Proposition 1 *In the second stage of the model of two-stage parallel contests with reputation, if $i \in N_k$, then agent i's equilibrium effort is*

$$x_i = p_i \sum_{j \in N_k} x_j,$$

where p_i is agent i's winning probability and is given by

$$p_i = 1 - \frac{(n_k - 1)c_i'(x_i)}{\sum_{j \in N_k} c_j'(x_j)}, \tag{4}$$

and agent i's expected payoff is given by

$$u_i = f_k(v_k, \bar{\alpha}_k)(p_i)^2. \tag{5}$$

uniqueness of the equilibrium in the second stage of each contest, so we consider the contest success function given by (1).

So far, we have analyzed the second stage of the model of two-stage parallel contests with reputation. In the following section, we will analyze the first stage of this competition, focusing on the effect of reputation on the agents' strategies in the first stage. The first stage analysis will be divided into two cases. In case A, we will assume that all the agents have the same cost functions, whereas in case B, we will assume that agents have different cost functions that can be ranked as described above.

4 The first stage - case A

Given the analysis of the second stage, in which the agents choose pure strategies (efforts) according to Proposition 1, we will say that our model of two-stage parallel contests with reputation has a subgame perfect equilibrium in pure strategies if, in the first stage, the agents choose their contests using pure strategies such that, given the choices of the other agents, no agent wishes to switch to a different contest.

It is unclear whether an agent would prefer to compete against high-type or low-type agents. This is because highly reputable agents increase the winning value of the contest, whereas low-cost (high-ability) agents are difficult to compete with. These two effects must be considered by each agent when determining his best strategy. In the following section, we will assume that all the agents have the same cost function, $c_i(x) = c_j(x), \forall i, j \in N$, and analyze their dynamics solely based on their reputation, whereas in the following section, we will look at a more general situation in which cost functions differ between agents. Even when agents have the same cost function, the two effects exist; however, in terms of competitiveness, there is no difference between competing against a highly reputable agent and competing against a less-reputable agent. Thus, agents have an incentive to follow highly reputable agents because the winning value is higher with these agents, and the chances of winning are independent of agent type. As such, we will analyze the competition dynamics solely based on the reputations of the agents, and we will specifically find subgame perfect equilibrium with pure strategies for these competitions.

Even if all the agents' cost functions are symmetric, the analysis of the model of parallel contests with reputation is complex. Furthermore, it is clear in some cases that there is no subgame perfect equilibrium in pure strategies. This is especially true when there is no clear relationship between the agents' reputations and abilities (cost of effort), as in the case under consideration. The reason for this is that a highly reputable agent can significantly increase the value of the prize and, in particular, the expected payoffs of the other agents in the contest in which he competes, providing an incentive for other agents to follow him to every contest he chooses. Such a situation is depicted in the following example.

Example 1 *Consider a set of three agents $N = \{h, l, l\}$ with one highly-reputable agent (h-type) and two less-reputable agents (l-type), and a set of two contests $M = \{1, 2\}$ with nominal prizes of $v_1 = v_2 = 10$. The h-type agent has a reputation level of $\alpha_h = 30$, while the l-type agent has a reputation level of $\alpha_l = 1$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha_k}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

For both types of agents, the cost function is $c_i(x_i) = x_i$. To understand the contest dynamics, we look at every possible agent allocation across the two contests and calculate the agents' expected payoffs in the second stage. Following that, we will be able to identify subgame perfect equilibria in pure strategies, specifically, agents' allocations across the two contests such that no agent wishes to move to another contest. There are six possible permutations for the agents' allocation across the contests; however, due to the symmetry of the contests, only three are relevant. Because of the symmetry of the cost functions, the effort, payoff, and realized prize (the level of the value function given the allocation of agents) are the same for all the agents in the same contest. The table below depicts all possible allocations and the corresponding expected payoffs for the agents.

	Contest 1	*Contest 2*	*Contest* 1			*Contest* 2		
			Realized Prize	*Effort*	*Payoff*	*Realized Prize*	*Effort*	*Payoff*
1	$\{h, l, l\}$	$\emptyset$	*106.66*	*23.70*	*11.85*	-	-	-
2	$\{h, l\}$	$\{l\}$	*155*	*38.75*	*38.75*	*10*	-	*10*
3	$\{h\}$	$\{l, l\}$	*300*	-	*300*	*10*	*2.5*	*2.5*

Table 1: The expected payoff, effort and realized prize for the agents.

Beginning with allocation 1, in which all the agents chose the same contest, the h-type agent with a payoff of 11.85 will increase his payoff by switching to the other contest, bringing us to allocation 3. However, in *allocation 3, one l-type from contest 2, who now receives a payoff of 2.5, would prefer to move to contest 1, which brings us to allocation 2, where he will receive a payoff of 38.75. Now, in allocation 2, the l-type agent who competes in contest 2 and has a payoff of* 10 *would choose to move to contest 1, bringing us to allocation 1 in which he receives a payoff of 11.85. As a result, there is no allocation from which no agent wishes to move on, and thus no subgame perfect equilibrium in pure strategies exists in this case.*

The example 1 shows that a subgame perfect equilibrium in pure strategies does not always exist in our model of two-stage parallel contests with reputation. This occurs when there is a movement loop, as in the preceding example, specifically when a highly-reputable agent moves from contest i to contest j, which is followed by the movement of less-reputable agents and a chain reaction that returns us to an infinite loop. Identifying the conditions under which these loops do not exist will provide us the sufficient conditions for the existence of equilibrium. First, we order the agents' incentives to shift from place to place based on their level of reputation.

Lemma 1 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function $c_i(x) = c(x)$ for all $i \in N$ and for all $x \geq 0$. Let $i, j \in N_k$ and $\alpha_i \geq \alpha_j$. Then if agent i does not want to switch from contest k to contest s, neither does agent j.*

Lemma 1 demonstrates that if highly-reputable agents do not want to move from one contest to another, less-reputable agents in the same contest do not want to move either. In particular, if the agent with the highest reputation in a contest does not want to move to a different contest, then all the agents in the same contest will not want to move either. This enables us to analyze the corner equilibrium points as follows:

Proposition 2 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function, $c_i(x) = c(x)$ for all $i \in N$ and for all $x \geq 0$. Let $\bar{\alpha}$ be the vector of all the agents reputation types in N and α_1 be the highest reputation type. Then, if for all $s \in M, s \neq k$:*

$$\frac{\sqrt{f_k(v_k, \bar{\alpha})}}{\sqrt{f_s(v_s, \alpha_1)}} \geq n,$$

there is a subgame perfect equilibrium in which all the agents choose to compete in contest k.

According to proposition 2, when all the agents are allocated to contest k and the other contests remain empty, if the highest-reputable agent does not want to move to another contest, then there is a subgame perfect equilibrium in which, in the first stage, all the agents choose to compete in contest k. It is important to note that a corner equilibrium can exist when all the agents prefer to compete in contest k rather than contest s even if $v_k < v_s$. Using the same approach, we can identify not only corner equilibria but also interior equilibria. In particular, we will have such an equilibrium when the most reputable agent in every contest does not want to move.

Proposition 3 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function, $c_i(x) = c(x)$ for all $i \in N$ and for all $x \geq 0$. Let $\alpha_1 \geq \alpha_j$ for all $j \in N$. If there are a sufficient number of agents with the highest reputation type of α_1, then there is a subgame perfect equilibrium with pure strategies.*

Proposition 3 demonstrates that if there are a sufficient number of highly reputable agents in the competition, a pure strategy equilibrium must exist. Of course, the critical number of highly reputable agents is determined by the shape of the value function. The following example illustrates this phenomenon by examining a scenario with two types of agents, one with a low reputation and one with a high reputation. This example particularly illustrates that the number of highly reputable agents does not need to be very large for an equilibrium to exist.

Example 2 *Consider the set of agents $N = \{1, 2, 3, ..., n\}$ with $h > 1$ highly-reputable (h-type) and $l > 1$ less-reputable (l-type) agents, that is, $\alpha_h > \alpha_l$. There are two parallel contests, $M = \{1, 2\}$ with nominal prizes of $v_1 > v_2$. All the agents regardless of their reputation have the same cost function $c(x_i) = x_i$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

In the first stage, both h-type and l-type agents decide which contest to enter. The number of l-type and h-type agents in contests $i, i = 1, 2$, is denoted as l_i and h_i respectively. According to Proposition 1, the expected payoff of the agents who chose to compete in contest i is

$$v_i \frac{h_i \alpha_h + l_i \alpha_l}{(h_i + l_i)^3}.$$

Assume, without loss of generality, that h-type agents from contest 2 want to switch to contest 1. In this case, we let h-type agents move from contest 2 to contest 1 one by one. If there is at least one h-type agent *left in contest 2 who has no incentive to move to contest 1 then by Lemma 1, all the l-type agents in contest 2 do want to move either to contest 1 and we actually have an equilibrium allocation of the agents between the two contests. Accordingly, assume that $h-1$ h-type agents compete in contest 1, meaning that there is exactly one h-type agent left in contest 2. This agent will have no incentive to move to contest 1 if his expected payoff in contest 2 is greater than his expected payoff in contest 1 when he joins this contest. This occurs when*

$$v_1 \frac{h\alpha_h + l_1\alpha_l}{(h+l_1)^3} \leq v_2 \frac{\alpha_h + l_1\alpha_l}{(1+l_1)^3}, \tag{6}$$

which yields

$$\frac{(h+l_1)^3}{(l_2+1)^3} \geq \frac{v_1(h\alpha_h + l_1\alpha_l)}{v_2(\alpha_h + l_2\alpha_l)}.$$

Now, since

$$\frac{v_1(h\alpha_h + l_1\alpha_l)}{v_2(\alpha_h + l_2\alpha_l)} < \frac{v_1\alpha_h(h+l_1)}{v_2\alpha_l(1+l_2)},$$

a sufficient condition for the last h-type in contest 2 to stay there is

$$\frac{(h+l_1)^3}{(l_2+1)^3} > \frac{v_1\alpha_h(h+l_1)}{v_2\alpha_l(1+l_2)},$$

or, alternatively,

$$\left(\frac{h+l_1}{l_2+1}\right)^2 > \frac{v_1\alpha_h}{v_2\alpha_l}.$$

This holds if

$$h > \sqrt{\frac{v_1\alpha_h}{v_2\alpha_l}}(l_2+1) - l_1. \tag{7}$$

We can see from (7), that as the number of h-type agents increases, an equilibrium with pure strategies will be reached.

When a new agent enters a contest, it has two simultaneous effects on the value of the other agents in that contest. The reputation and competitiveness effects. The reputation effect can increase or decrease the value of the prize depending on the new agent's reputation in comparison to the reputations of the other agents in the contest. However, because there is one more agent to compete against for winning, the competitiveness effect reduces the expected payoff of the agents in the contest. According to Proposition 3, if there are a sufficient number of highly-reputable agents, the competitiveness effect will have a greater absolute value than the reputation effect, and no agent will be motivated to leave one contest and move to another. In that case, we say that an additional agent in a contest has a negative cumulative (competitiveness plus reputation) effect. The cumulative effect is defined formally as follows:

Definition 1 *We say that the introduction of a new agent into a contest has a positive (or negative) cumulative effect if, as a result of the new agent's entry into the contest, the expected payoff increases (decreases) for all other agents in that contest. Formally, if all the agents have the same cost function $c_i(x) = c_j(x), \forall i, j \in N$, a β-type agent, with reputation type of α_β has a negative cumulative effect on contest k if, given the the value function f_k of the nominal prize v_k, and the reputation vector $\bar{\alpha}_k$ of the n_k agents, there exists*

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2} > \frac{f_k(v_k, \bar{\alpha}_k, \alpha_\beta)}{(n_k+1)^2}. \tag{8}$$

When a new agent enters a competition, the cumulative effect, which takes reputation and competitiveness effects into account, can be either positive or negative. When the cumulative effect is negative, either the agent's reputation is low relative to the reputations of the agents already competing, or the agent's reputation is high, but the competitiveness effect has a greater impact, resulting in a lower expected payoff for the agents who already compete. On the other hand, when the cumulative effect is positive, it shows that the reputation effect outweighs the competitiveness effect, which is always negative. In our model of two-stage parallel contests with reputation, the negative cumulative effect of all agents is critical to the existence of equilibrium. This means that even if the most reputable agent enters a competition, the expected payoff of the agents currently competing in that competition will decrease. Then, we will show that a pure-strategy equilibrium will exist.

According to Proposition 3, if there are a sufficient number of highly reputable agents dispersed among the contests, then the cumulative effect will be negative in all the parallel contests, and as a result, the subgame perfect equilibrium with pure strategies exists. However, a sufficient number of highly-reputable agents is not a necessary condition for the cumulative effect to be negative. Our requirements that $R_{k,m,s} = |f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s) - f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_{s-1})|$ satisfies that for any given reputation vector $\overline{\alpha}_k^m$, $m = 2, 3, ...$ and for every vectors of s and $s-1$ additional agents with a reputation type of α, as well as the fact that $\lim_{m\to\infty} R_{k,m,s} = 0$ are necessary conditions for the cumulative effect to be negative, although they are not sufficient. For each value function $f_k(v_k, \bar{\alpha}_k)$, there are different sufficient conditions for the cumulative effect to be negative for any nominal prize v_k and any given reputation vector $\overline{\alpha}_k$. Obviously, there are some value functions for which sufficient conditions do not exist, but in the following, we illustrate two examples in which the value functions have negative cumulative effects under certain conditions regardless of the distribution of the agents among the contests and their reputation types.

Claim 1 *Consider the model of two-stage parallel contests with reputation. Assume that contest k has the following value function*

$$f(v_k, \bar{\alpha}_k) = h(v_k, \alpha_{\min}).$$

where $\alpha_{\min} = \min\{\alpha : \alpha \in \bar{\alpha}_k\}$ and $h : R_+^2 \to R_+$ is a monotonically increasing function in both v_k and $\alpha_{\min}$. Then, the value function $f(v_k, \bar{\alpha}_k)$ has a negative cumulative effect for any number of agents n_k, any nominal prize v_k and any given reputation vector $\overline{\alpha}_k$.

This result is straightforward because the entry of any agent either reduces or does not change the reputation effect, but it increases the competitiveness effect. As a result, the addition of any new agent reduces the cumulative effect, implying that the value function $h(v_k, \alpha_{\min})$ has a negative cumulative effect regardless of the agents' reputation types. While the previous claim demonstrated that a value function could have a negative cumulative effect without any conditions, the following claim demonstrates that for the value function given in Example 2, a modest level of agents' heterogeneity in each contest is sufficient to guarantee that it will have a negative cumulative effect.

Claim 2 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function, $c_i(x) = c_j(x)$ for all $i, j \in N$ and for all $x \geq 0$. The set of agents includes h-type and l-type agents with reputation types of α_h and α_l respectively where $\alpha_h > \alpha_l$. Let h and l be the number of h-type and l-type agents in the competition, where $h + l = n$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

Then, if $\alpha_h \leq 3\alpha_l$, this value function has a negative cumulative effect for any nominal prize v_k and any given reputation vector $\overline{\alpha}_k$.

When the introduction of a new highly reputable agent into a contest has a negative cumulative effect, the option to move between contests in loops is removed. Then, in contrast to Example 1, agents will not follow the movements of highly reputable agents. Even if the agent with the highest reputation joins a contest and increases its value function, his negative cumulative effect will reduce the expected payoffs of the agents in that contest and thus the incentive of other agents to enter this contest. With this property of negative cumulative effect, we can implement an algorithm and show that the competition will reach a situation where no agent moves. In such a case, we say that the algorithm has reached equilibrium, and in particular, the model of two-stage parallel contests with reputation has a subgame perfect equilibrium.

Algorithm 1 *The algorithm has the following four steps:*

Step 1 *Begin with all contests empty and deny any agent access to the contests.*

Step 2 *Assign the agent with the lowest reputation who has not yet been assigned to the contest in which he has the highest expected payoff.*

Step 3 *Allow agents who have already been assigned to contests to switch contests if they so desire.*

Step 4 *Repeat steps 2-3 until all the agents have been assigned to a contest and no agent wants to move in step 3.*

Algorithm 1 assigns agents in ascending order of reputation, beginning with the agent with the least reputable reputation and ending with the agent with the most reputable reputation. Despite the fact that each new additional agent offers the opportunity to switch contests, the agents remain in the contest to which they were initially assigned. The following theorem demonstrates that this algorithm converges to equilibrium and that agents do not move during its execution.

Theorem 1 *Consider the model of two-stage parallel contests with reputation consisting of m contests and n agents where all the agents have the same cost function, $c_i(x) = c_j(x)$ for all $i, j \in N$ and for all $x \geq 0$. Then, if any additional agent to any contest has a negative cumulative effect, Algorithm 1 reaches equilibrium such that there is a subgame perfect equilibrium with pure strategies.*

Theorem 1 proves that if the agents have a negative cumulative effect on the payoff in each of the contests they enter, our model of parallel contests with reputation has a subgame perfect equilibrium. For example, according to Claim 1, if the value function in each contest is the minimum function $f(v_k, \bar{\alpha}_k) = \min\{\alpha : \alpha \in \bar{\alpha}_k\}$, then we have a subgame perfect equilibrium in pure strategies. Similarly, according to Claim 2, if the value function in each of two parallel contests with two types of agents is the average function $f^A(v_k, \bar{\alpha}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}$, then we have a subgame perfect equilibrium in pure strategies if the difference in the agents' reputation types is not too large.

5 The first stage - case B

So far, we have assumed that the cost functions of all the agents are the same. We will show in this section that this is not a limiting assumption in our model and that the results, particularly the existence of subgame perfect equilibrium with pure strategies, can be obtained even under weaker conditions when the agents have asymmetric cost functions. Thus, we consider here a set of agents $N = \{1, 2, ..., n\}$ with n agents of two types, highly-reputable agent (h) and low-reputable agent (l), such that their reputation types satisfy $\alpha_h \geq \alpha_l$ and their cost functions satisfy $c_h(x) \leq c_l(x)$ for all $x \geq 0$. In other words, an agent's reputation and ability are related.

Cohen et al. (2023) demonstrated the existence of subgame perfect equilibrium in a model of two-stage parallel contests with asymmetric, non-linear cost functions and a value function based on a fixed nominal prize. We extend their work by considering value functions that are not fixed and are endogenously defined by the nominal prize and the reputation types of the agents who chose to compete in the same contest.

We assumed in the previous section that the agents' competitiveness effect was the same because their cost function (or ability) was the same. In this case, the competitiveness effect is not homogeneous, as highly reputable agents incur lower costs for each level of effort. Thus, the competitiveness effect of including highly reputable agents in the contest has a greater negative impact, implying that the cumulative effect, which is the sum of the competitiveness and reputation effects, will be lower (more negative) in that case. Thus, as in the previous section, we make the same assumption that the value functions have a negative cumulative effect. Then, we can implement an algorithm and demonstrate that the competition will eventually reach a state in which no agents move, i.e., the algorithm has reached equilibrium.

Algorithm 2 *The algorithm has the following four steps:*

Step 1 *Allocate the agents randomly across the contests.*

Step 2 *Let the h-type agents switch contests if they want.*

Step 3 *Let a single l-type agent move between the contests.*

Step 4 *Repeat steps 2-3 until, in step 3, no l-type agent wants to move and every agent is assigned to a contest.*

The next theorem shows that this algorithm converges to equilibrium and, more specifically, to an allocation of the agents across the contests such that no agent wants to move from his location to another one.

Theorem 2 *Consider the model of two-stage parallel contests with m contests and two types of agents h and l where their reputation types satisfy $\alpha_h \geq \alpha_l$ and their cost functions satisfy $c_h(x) \leq c_l(x)$ for all $x \geq 0$. If any additional h-type agent has a negative cumulative effect on the contest, algorithm 2 reaches equilibrium such that there is a subgame perfect equilibrium with pure strategies.*

6 Discussion

In this section, we investigate the robustness and strength of our main findings. According to Theorems 1 and 2, in our model of two-stage parallel contests, a sufficient condition for the existence of a subgame perfect equilibrium with pure strategies is that any additional agent to any contest has a negative cumulative effect. In the following, we want to highlight that this sufficient condition for the existence of equilibrium is far more likely to hold in an environment where the agents' skills and reputations are correlated. In Example 1, we show that when agents have different reputation types but the same cost of effort, namely, the agents' skills and reputations are not correlated, the subgame perfect equilibrium with pure strategies does not exist. Example 3 is slightly different from Example 1, such that the agents' skills and reputations are correlated, that is, highly reputable agents have a lower cost than less reputable agents. As a result, the negative cumulative effect is more likely to hold, implying the existence of a subgame perfect equilibrium based on pure strategies.

Example 3 *As in Example 1, consider a set of three agents $N = \{h, l, l\}$ with one highly-reputable agent (h-type) and two less-reputable agents (l-type), and a set of two contests $M = \{1, 2\}$ with nominal prizes of $v_1 = v_2 = 10$. The h-type agent has a reputation level of $\alpha_h = 30$, while the l-type agent has a reputation level of $\alpha_l = 1$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha_k}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

For both types of agents, the cost function is $c_i(x_i) = c_i x_i$, where $c_l = 1$ and $c_h = 0.85$. As shown in Example 1, only three permutations are relevant due to the contest's symmetry. However, because each type of agent has a distinct cost function, the expected payoff and effort will differ between them. The two tables below show the realized prize, expected payoff, and effort from the perspective of an h-type (l-type) agent for all permutations. The h-type agents can be found alone, with one, or two l-type agents. Similarly, a l-type agent can compete alone, with another l-type agent, another h-type agent, or with all agents participating in the same contest.

		Realized Prize	Effort	Payoff	
1	$\{h, l, l\}$	106.66	30.20	17.36	
2	$\{h, l\}$	155	45.28	45.28	
3	$\{h\}$	300	-	300	

Table 2: The expected payoff, effort and realized prize for the h-type agent.

		Realized Prize	Effort	Payoff	
1	$\{h, l, l\}$	106.66	22.32	9.48	
2	$\{h, l\}$	155	38.49	32.72	
3	$\{l, l\}$	10	2.5	2.5	
4	$\{l\}$	10	-	10	

Table 3: The expected payoff, effort and realized prize for the l-type agents.

Starting with all agents selecting the same contest, the h-type agent gets a payoff of 17.36 (Table 2, case 1). He will increase his payoff to 300 by switching to the other contest (Table 2, case 3). Contest 2 now contains only one h-type agent, whereas Contest 1 contains two l-type agents. In this case, one l-type agent *from contest 1 who received a payoff of 2.5 (Table 3, case 3) would prefer to move to contest 2, where he will receive a payoff of 32.72 (Table 2, case 2). The h-type agent has no incentive to move because both contests have exactly one l-type agent, and the l-type agent who is alone in contest 1 would prefer a payoff of 10 (Table 3, case 4) over 9.48 if he switched to the other contest (Table 3, case 1). So we have reached a point of equilibrium.*

As demonstrated in Theorem 2, the requirement for a negative cumulative effect is a sufficient condition for the existence of a subgame perfect equilibrium with pure strategies in our model of parallel contests with reputation. However, it is sometimes a necessary condition. According to Proposition 3, as the number of highly reputable agents increases, the condition of a negative cumulative effect is met because the marginal effect of each agent on the value function is negative.

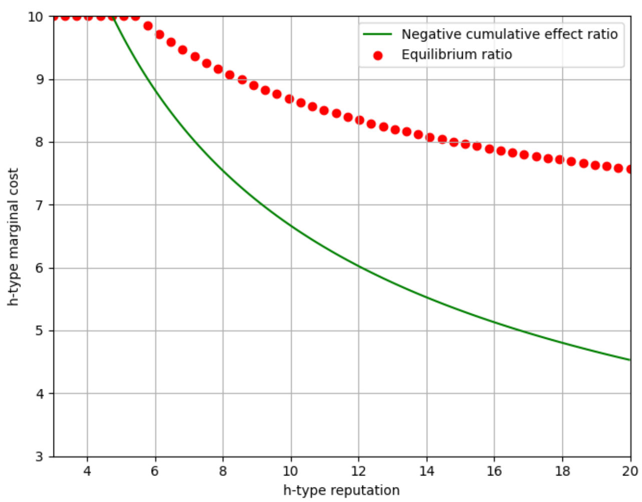

Figure 1: Cumulative negative effect as a sufficient condition

Figure 1 elucidates the relationship between the negative cumulative effect condition and the existence of pure-strategy equilibrium. This figure depicts a situation with seven agents, five of which are low-type agents and two of which are high-type agents. There are two contests with the same nominal prize (normalized to 1), and the value function for each contest is $f(v_k, \bar{\alpha}_k) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}$. We also assume that low-type agents have a low reputation ($\alpha_l = 1$) and a high marginal cost of effort ($c_l = 10$), whereas the reputations and marginal costs of effort of high-type agents vary and are shown in Figure 1.[4]

According to Claim 2, when $\alpha_h \leq 3\alpha_l$, there is a subgame perfect equilibrium with pure strategies even if the marginal cost of effort of both types of agents is identical. However, in this case with heterogeneous marginal costs of effort, we examine for each value of the high-type's reputation, the maximum marginal cost of effort (so that the high-type agents are as weak as possible) such that there is still an equilibrium allocation of the agents (the curve of red dots). On the same graph, we have added the appropriate value of the cost of effort for each value of the high-types' reputation, so that the value function has a negative cumulative effect (the green curve). There is a gap between the red and green curves, which is due to the fact that the condition of the negative cumulative effect is sufficient for the existence of equilibrium. However, it is not necessary.

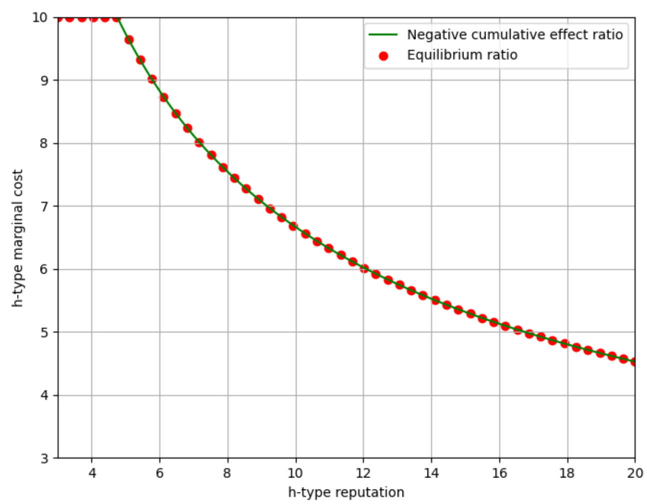

Figure 2: Cumulative negative effect as a necessary condition

Figure 2 depicts the same situation as Figure 1 but with a different number of agents. There is one high-type agent and three low-type agents in Figure 2. We can see that when the number of agents is small, as in this case, the condition of the negative cumulative effect merges with the condition of equilibrium's existence. In other words, the negative cumulative effect condition becomes a necessary and sufficient condition for the

[4]The method for numerically determining, for a given reputation, the maximum cost that will still result in an equilibrium existence in each scenario is outlined in the appendix.

existence of a subgame perfect equilibrium with pure strategies.

7 Conclusion

In this paper, we studied parallel contests in which the agents' reputations play a crucial role in determining the outcome of the competition. We analyzed a model of two-stage parallel contests where, in the first stage, agents simultaneously select the contests in which they want to compete, and in the second stage, agents who selected the same contest compete against each other for a single prize determined not only by a nominal value but also by the reputation of the competing agents.

We initially assumed that the parallel contests include heterogeneous agents with distinct reputation types and identical cost functions. Then, we showed that there are situations in which there is no subgame perfect equilibrium with pure strategies, implying that some agents always prefer to leave the contest they chose to enter since by doing so they can increase their expected payoffs. Then, we provided a sufficient condition on the agents' value function that guaranteed the existence of the pure-strategy equilibrium, and we demonstrated that this sufficient condition is feasible for several forms of the agents' value function.

Then, we assumed that there are n heterogeneous agents with two possible types, each with a different nonlinear cost of effort and a different reputation, and that the non-linear costs of effort are not explicitly given. Although it is impossible to characterize the subgame perfect equilibrium in this case, we established a sufficient condition for the existence of a subgame perfect equilibrium with pure strategies and provided an algorithm for calculating this equilibrium using revealed preference techniques. Later, we demonstrated using computer simulations that when the number of agents is small, our sufficient condition is also necessary for the existence of subgame perfect equilibrium with pure strategies, but as the number of agents increases, our condition is sufficient but no longer necessary.

We found that the alignment between an agent's skills (cost functions) and reputation plays a crucial role in determining the likelihood of achieving a subgame perfect equilibrium with pure strategies in such contests. When the reputations of agents are closely related to their skill levels, the model tends to achieve stability with pure strategy equilibria. When reputations and skills are misaligned, resulting in low-ranked agents having high reputations, the model becomes unstable, and pure-strategy equilibria become elusive.

In this paper, we improve our understanding of the complexities of reputation-driven parallel contests. Understanding the implications of reputation in parallel contests provides a useful lens through which to examine the dynamics of other competitive environments, as reputation continues to have a significant impact on a variety of social and economic phenomena.

8 Appendix

8.1 Proof of Lemma 1

Assume that $i, j \in N_k$, and $\alpha_i > \alpha_j$, and that agent i does not want to move from contest k to contest s, while agent j wants to make this movement. We will show that such a scenario leads to a contradiction. By (5), agent i's expected payoff in contest k is

$$f_k(v_k, \bar{\alpha}_k)\left(1 - \frac{(n_k - 1)c_i'}{\sum_{j \in N_k} c_j'}\right)^2.$$

By symmetry of the agents' cost functions, we obtain that all the agents in contest k have the same expected payoff, which is given by

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2}.$$

An i-type agent will not switch from contest k to contest s if the expected payoff in contest k is greater than the expected payoff in contest s after he joins this contest. This holds true if

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2} \geq \frac{f_s(v_s, \bar{\alpha}_s, \alpha_i)}{(n_s + 1)^2}.$$

Similarly, j-type agent wants to switch from contest k to contest s if

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2} < \frac{f_s(v_s, \bar{\alpha}_s, \alpha_j)}{(n_s + 1)^2}.$$

The last two inequalities imply that

$$f_s(v_s, \bar{\alpha}_s, \alpha_j) > f_s(v_s, \bar{\alpha}_s, \alpha_i).$$

The monotonicity of $f_s(.)$ in each reputation type and our presumption that $\alpha_i > \alpha_j$ are incompatible with this inequality. As a result, if the i-type agent from contest k does not want to move to contest s, the j-type agent from contest k will not either.

8.2 Proof of Proposition 2

By (5), if all the agents chose to compete in contest k, the expected payoff of agent i is

$$f_k(v_k, \bar{\alpha})\left(1 - \frac{(n-1)c_i'}{\sum_{j \in N} c_j'}\right)^2,$$

where $\bar{\alpha}$ is the vector of reputation types of all the agents in contest k. Because all the agents have the same cost function $c_i(x) = c_j(x), \forall i, j \in N$, $x \geq 0$, agent i's winning probability is $1/n$. This means that if all the agents choose to compete in contest k, the expected payoff of agent i is

$$\frac{f_k(v_k, \bar{\alpha})}{n^2}$$

We can conclude from Lemma 1 that the allocation of agents in which all the agents compete in contest k is in equilibrium if there is no other contest $s \in M, s \neq k$, that the highest reputable agent, agent 1, with a reputation type of α_1, would prefer over contest k. Agent 1 prefers contest s over contest k if

$$f_s(v_s, \alpha_1) < \frac{f_k(v_k, \bar{\alpha})}{n^2},$$

or, alternatively, if

$$\frac{\sqrt{f_k(v_k, \bar{\alpha})}}{\sqrt{f_s(v_s, \alpha_1)}} < n.$$

Thus, there is an equilibrium in which all the agents choose to compete in contest k, if

$$\frac{\sqrt{f_k(v_k, \bar{\alpha})}}{\sqrt{f_s(v_s, \alpha_1)}} \geq n.$$

8.3 Proof of Proposition 3

Consider a set of agents $N = \{1, 2, ..., n\}$ and a set of contests $M = \{1, 2, ..., m\}$. Let the γ-type agents be the highest reputable agents with a reputation type of α_γ, $\alpha_\gamma \geq \alpha_i, \forall i \in N$. Let Γ be the number of γ-type agents in the competition where $\Gamma > m$, and where Γ_s is the number of γ-type agents allocated in contest s.

We begin by allocating all types except the γ-type agents to the contests in any way, while the γ-type agents will be allocated later. Let ζ_s be the number of agents with different types than γ-type agents that are allocated to contest s. Then, we allocate the γ-type agents equally across the m contests. Now when all the agents, both γ-type and other types of agents, have been allocated, we reach an equilibrium point if no agent wishes to move to another contest. Otherwise, If any agent from contest $i \in M$ wishes to move to another contest j, according to Lemma 1, the γ-type agent from that contest also wishes to move from contest i to contest j.

Remember that we assumed that the number of competing agents reduces the impact of an additional agent on the agents' value function f_k, where formally. $\lim_{m \to \infty} R_{k,m,s} = \lim_{m \to \infty} |f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s) - f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_{s-1})| =$

0 for any s additional agents with reputation types $\alpha_s = (a)_{i=1}^s$, particularly for γ-type agents. This implies that

$$\lim_{\Gamma_j \to \infty} \frac{f_j(v_j, \bar{\alpha}_j)}{(\Gamma_j + \zeta_j)^2} = 0$$

Thus, if there are a sufficient number of γ-type agents in contest j, entering a new agent to the contest reduces the payoff of the other agents in that contest. This means that if there are sufficient number of γ-type agents in the competition and they spread across the contests, no agent will want to join contest j. Hence, there is an equilibrium allocation of agents across the contests.

8.4 Proof of Claim 2

The number of l-type and h-type agents who decide to participate in contests $i \in M$ in the first stage will be denoted as l_i and h_i respectively. According to Proposition 1, the expected payoff of the agents who chose to compete in contest i is

$$v_i \frac{h_i \alpha_h + l_i \alpha_l}{(h_i + l_i)^3}.$$

An entry of an additional l-type agent has a negative cumulative effect on a contest, as it increases the contest's competitiveness and decreases (or has no effect on) the average reputation in the contest. However, an additional h-type agent has two opposing effects. After another h-type agent entered the contest, the expected payoff of each agent in that contest is

$$\frac{v_i((h_i + 1)\alpha_h + l_i \alpha_l)}{(h_i + 1 + l_i)^3}$$

We compare the previous payoff of the agents to the new one after the h-type agent joined the contest to determine whether the cumulative effect is positive or negative. Thus, the cumulative effect is negative if

$$\frac{v_i(h_i \alpha_h + l_i \alpha_l)}{(h_i + l_i)^3} - \frac{v_i((h_i + 1)\alpha_h + l_i \alpha_l)}{(h_i + 1 + l_i)^3} \geq 0.$$

Let $\alpha_h = \gamma \alpha_l$, where $\gamma > 1$. Then the condition that the cumulative effect is negative is

$$\frac{v_i(h_i \gamma \alpha_l + l_i \alpha_l)}{(h_i + l_i)^3} - \frac{v_i((h_i + 1)\gamma \alpha_l + l_i \alpha_l)}{(h_i + 1 + l_i)^3} \geq 0,$$

or, alternatively,

$$\frac{(h_i + 1 + l_i)^3}{(h_i + l_i)^3} \geq \frac{\gamma(h_i + 1) + l_i}{\gamma h_i + l_i}.$$

It can be verified that the last inequality is satisfied for every $\gamma \leq 3$. This means that if $\alpha_h \leq 3\alpha_l$, the cumulative effect of an h-type is negative, namely, contest i will become less attractive for new agents of any type.

8.5 Proof of Theorem 1

Algorithm 1 begins with all the agents unassigned and all contests empty. Beginning with step 2, the agent with the lowest reputation is assigned to a contest. He will not want to change contests in step 3. Returning to step 2, the agent with the second lowest reputation will be assigned to the contest in which he has the highest expected payoff. If this is a different contest than the previous agent's contest, the previous agent will not join the new agent in step 3. This is because a new agent has a negative cumulative effect on any contest he enters. If, on the other hand, the new agent will be assigned to the same contest of the previous agent, according to Lemma 1, the previous agent will not want to switch contests. This argument holds for subsequent steps, and no contest switching will occur throughout the algorithm's steps. Therefore, once all of the agents are assigned to the contests, no one will want to move, and the algorithm will reach an equilibrium.

8.6 Proof of Theorem 2

We start by allocating the agents randomly across the contests. In step 2, we let only h-type agents move until they could not increase the expected payoff. In step 3, a single l-type agent has the option to switch between the contests. Assume the l-type agent changed locations from contest i to j. This means that his expected payoff in contest j is greater than his current expected payoff in contest i. No h-type agent will join contest j as a result of the l-type move. Because of his movement, the l-type agent reduces contest j's value function. This is because his action has a negative cumulative effect on contest j. Back in step 2, as a response, h-type agents who initially did not wish to move (in the previous iteration of step 2), may now wish to move to contest i, but not to contest j. Thus, no h-type will ever join contest j and no l-type agent will join contest i. As the algorithm continues, the agents will not join more and more contests, until the competition reaches an equilibrium.

9 The process of creating graphs

We implemented a computer simulation to produce the graphs. The simulation is conducted in several steps. All of the agent permutations among the contests are created by the simulation in the first step. For each allocation in each permutation, the simulation then calculates the equilibrium values of the second stage, namely, the prize, based on the agents' reputations, efforts, winning probabilities, and expected payoffs. The simulation then produces a map of "adjacent permutations." A pair of permutations is referred to as "adjacent permutations" if, with the exception of the allocation of one agent between the contests, the distribution of agents in each permutation is the same. Then, for each permutation i, it is examined if there is an adjacent permutation j where the agent who switches locations receives a greater expected payoff than in permutation i. If this is the case, then permutation i is not an equilibrium permutation. If there are no adjacent permutations for which the expected payoff of the agent changing his location is higher than his prior expected payoff, the permutation is an equilibrium permutation.

With this simulation algorithm, we can examine the existence of equilibrium. Given that there are two different types of agents, we can determine the high-type agents' maximum fixed marginal cost while maintaining equilibrium. We employ a binary search algorithm variant for this purpose (Lehmer 1960; Bottenbruch 1962). The algorithm receives potential upper and lower bounds where the maximum cost of the high-type agent exists in equilibrium and reduces the range of possibilities as follows (pseudo code):

Function: `find_max_cost_in_equilibrium(model)`

```
// The assumed range for the cost of the high-type agent
cost_min = 0  // Minimum cost value
cost_max = 100  // Maximum cost value (adjust as needed)
precision = 0.001  // Desired precision
// Perform a binary search for the maximum cost value
WHILE cost_max - cost_min ≥ precision:
    cost_to_test = (cost_min + cost_max) / 2  // Midpoint of the range
    equilibrium_exists = run_simulation(model, cost_to_test)
    IF equilibrium_exists
```

```
        // If there is an equilibrium allocation, move the lower bound up
        cost_min = cost_to_test
    ELSE:
        // If there is no equilibrium allocation, move the upper bound down
        cost_max = cost_to_test
/* The maximum cost value of the strong agent
   for this model is stored in cost_min */
RETURN cost_min
```

Thus, it is possible to find the point where the cost of the strong agent is maximal and there is still equilibrium.

References

[1] Azmat, G., Möller, M. (2009). Competition among contests. *The RAND Journal of Economics*, 40 (4), 743–768.

[2] Azmat, G., Möller, M. (2018). The distribution of talent across contests. *The Economic Journal*, 128 (609), 471–509.

[3] Baye, M. R., Hoppe, H. C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior*, 44 (2), 217–226.

[4] Bhattacharya, H., Dugar, S. (2014). Partnership formation: The role of social status. *Management Science* 60(5), 1130–1147.

[5] Bottenbruch, H. (1962). Structure and use of ALGOL 60. *Journal of the ACM (JACM)*, 9(2), 161-221.

[6] Charness, G., Masclet, D., Villeval, M. C. (2014). The dark side of competition for status. *Management Science* 60, 38–55.

[7] Cohen, C., Rabi, I., Sela, A. (2023). An Algorithmic Analysis of Parallel Contests. Working paper.

[8] Cripps, M. W., Mailath, G. J., Samuelson, L. (2004). Imperfect monitoring and impermanent reputations. *Econometrica*, 72(2), 407-432.

[9] Cripps, M. W., Mailath, G. J., Samuelson, L. (2007). Disappearing private reputations in long-run relationships. *Journal of Economic Theory*, 134(1), 287-316.

[10] Damiano, E., Li, H., Suen, W. (2010). First in village or second in Rome? *International Economic Review*, 51 (1), 263–288.

[11] Damiano, E., Li, H., Suen, W. (2012). Competing for talents. *Journal of Economic Theory*, 147 (6), 2190–2219.

[12] Drugov, M., Ryvkin, D. (2020). Tournament rewards and heavy tails. *Journal of Economic Theory* 190, 105116.

[13] Dubey, P., Geanakopolos, J. (2010). Grading exams: 100, 99, 98,..., A, B, C ? *Games and Economic Behavior* 69(1), 72-94.

[14] Einy, E., Haimanko, O., Moreno, D., Sela, A., Shitovitz, B. (2015). Equilibrium existence in Tullock contests with incomplete information. *Journal of Mathematical Economics*, 61, 241–245.

[15] Ewerhart, C. (2015). Mixed equilibria in Tullock contests. *Economic Theory*, 60, 59–71.

[16] Ewerhart, C., Quartieri, F. (2020). Unique equilibrium in contests with incomplete information. *Economic Theory*, 70, 243–271.

[17] Ely, J., Fudenberg, D., Levine, D. K. (2008). When is reputation bad?. *Games and Economic Behavior*, 63(2), 498-526.

[18] Hopkins, E., Kornienko, T. (2004). Running to keep the same place: consumer choice as a game of status. *American Economic Review* 94(4), 1085-1107.

[19] Juang, W. T., Sun, G. Z., Yuan, K. C. (2020). A model of parallel contests. *International Journal of Game Theory*, 49 (2), 651–672.

[20] Konrad, K. A., Kovenock, D. (2012). The lifeboat problem. *European Economic Review*, 56 (3), 552–559.

[21] Kosefeld, M., Neckermann, S. (2011). Getting more work for nothing? Symbolic awards and worker performance. *American Economic Journal: Microeconomics* 3(3), 86-99.

[22] Kreps, D. M., Milgrom, P., Roberts, J., & Wilson, R. (1982). Rational cooperation in the finitely repeated prisoners' dilemma. *Journal of Economic theory*, 27(2), 245-252.

[23] Kreps, D. M., & Wilson, R. (1982). Reputation and imperfect information. *Journal of economic theory*, 27(2), 253-279.

[24] Lehmer, D. H. (1960). Teaching combinatorial tricks to a computer. *Combinatorial Analysis*, 179-193.

[25] Mailath, G. J., Samuelson, L. (2015). Reputations in repeated games. Handbook of Game Theory with Economic Applications, 4, 165-238.

[26] Milgrom, P., & Roberts, J. (1982). Predation, reputation, and entry deterrence. *Journal of economic theory*, 27(2), 280-312.

[27] Moldovanu B., Sela, A., Shi, X. (2007). Contests for status, *Journal of Political Economy*, 115(2), 338-363.

[28] Morgan, J., Sisak, D., Vardy, F. (2018). The ponds dilemma. *The Economic Journal, 128* (611), 1634–1682.

[29] Origgi, G. (2019). Reputation: What it is and why it matters. Princeton University Press.

[30] Sela, A. (2020). Optimal allocations of prizes and punishments in Tullock contests. *International Journal of Game Theory*, 49, 749-771.

[31] Skaperdas, S. (1996). Contest success functions. *Economic Theory,* 7, 283-290.

[32] Szidarovszky, F., Okuguchi, K. (1997). On the existence and uniqueness of pure Nash equilibrium in rent-seeking games. *Games and Economic Behavior,* 18 (1), 135–140.

[33] Tullock, G. (1980). Efficient Rent Seeking. In J. M. Buchanan, R. D. Tollison, & G. Tullock, Toward a theory of the rent-seeking society. Texas A & M University.

[34] Wilson, R. (1985). Reputations in games and markets. Game-theoretic models of bargaining, 96, 27-62.

[35] Yamazaki, T. (2008). On the existence and uniqueness of pure-strategy Nash equilibrium in asymmetric rent-seeking contests. *Journal of Public Economic Theory,* 10 (2), 317–327.

Reputation in Contests

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Abstract

We investigate two-stage parallel contests with a finite number of heterogeneous agents with various skills (cost of efforts) and reputations, as well as a finite number of heterogeneous contests. Each agent chooses which contest to enter in the first stage, and in the second stage, the agents who chose the same contest compete against one another for a single prize, the value of which is a combination of a nominal value and the reputation of the competing agents. We demonstrate that reputation in contests may lead to an unstable environment. We present a sufficient condition for resolving this instability, which results in the existence of a subgame perfect equilibrium with pure strategies. We then provide an algorithm for calculating it. We demonstrate that our equilibrium is far more likely to hold in an environment where the agents' skills and reputations are correlated

Keywords: Parallel contests; reputation; heterogenous agents, non-linear effort costs.

JEL classification: D44, D72, D82, J31

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1 Introduction

"Reputation, reputation, reputation! Oh, I have lost my reputation! I have lost the immortal part of myself, and what remains is bestial." This scene from the play "Othello" by William Shakespeare begins the book of Origgi (2019) to emphasize why reputation is so important in our lives, both personally and socially. In any situation, an agent's reputation, which refers to his general quality or ability as perceived or assessed by other agents, primarily has a direct impact on his opponents. For example, in the well-known biblical tale of "David and Goliath" (1 Samuel 17), Saul, the king of Israel, and his soldiers are battling the Philistines in the Valley of Elah. Goliath, the Philistine champion, challenges the Israelites to send out a champion of their own to decide the outcome in single combat twice daily for 40 days, but Saul and all of his soldiers are afraid to face Goliath because of his reputation as an unconquerable warrior. Finally, David accepts the challenge, defeats the opponent, and establishes his reputation as a valiant warrior before ascending to the throne of Israel. This story shows that, in any competition, an agent's value of winning depends on the reputation of his opponents; the higher their reputation, the higher an agent's value of winning.

In this paper, we assume that each agent has his own reputation and that an agent's winning value in a contest is influenced by the reputations of all the agents in the contest. We study the issue of reputation in a two-stage model of parallel contests. In the first stage, agents simultaneously select the contest in which they want to compete, with the restriction that each agent may only compete in a single contest. The agents in each contest compete against one another in the second stage after choosing their contests, and the winner of each contest is awarded a prize based on a nominal value and the reputation of the contending agents.

Given that an agent's expected payoff depends on the nominal prize as well as the endogenous allocation of agents in each contest and their reputations, choosing which contest to enter may not be an easy decision. An agent wants to compete against highly reputable opponents because it increases his winning value, which is dependent on his opponents' reputations, but it also makes winning the prize more difficult because there is a correlation between agents' reputations and their abilities. For instance, different universities have different reputations around the world, and the value of attending a particular university increases with its reputation. On the other hand, it is undoubtedly more challenging for young researchers to obtain tenure at universities with a strong reputation. Similarly, professional tennis players must decide which tournaments they want to enter throughout the year, some of which take place at the same time. A tournament with highly reputable players will increase the value of winning, but it will also be much more difficult to win.

We first consider $m \geq 2$ parallel contests, each of which has a generalized Tullock contest success function (Tullock 1980). We assume that the parallel contests include n heterogeneous agents with different reputation types and identical effort cost functions. Then, there are circumstances in which there is no pure strategy equilibrium, meaning that some agents always prefer to leave the contest they chose to enter since by doing so they can increase their expected payoffs. This is because the value of the payoff for the agents is endogenously defined by the reputations of the agents in the contest. We actually provide a simple illustration of how a highly reputable agent is constantly evading the less reputable agents, such that a pure strategy equilibrium does not exist. Then we present a sufficient condition that, if met, ensures a subgame perfect equilibrium with pure strategies. We use a variety of examples throughout the analysis to demonstrate the viability of the sufficient condition.

Later, we assume that there are n heterogeneous agents with two possible types, each with a different non-linear cost of effort and a different reputation. Furthermore, the costs of effort are non-linear and not explicitly given. All we know about these costs of effort is that they are increasing and weakly convex in effort. We also assume some ranking, including the agents' costs of effort, that clarifies the relative strength of the agents of different types, and we assume the same ranking for reputation, meaning that agents with low costs of effort are more reputable. In that case, it is impossible to characterize the subgame perfect equilibrium since the costs of effort are not explicitly given, so the agents' expected payoffs over the parallel contests cannot be calculated. To overcome this obstacle, we use revealed preference techniques to demonstrate the existence of a subgame perfect equilibrium with pure strategies and provide an algorithm that enables us to calculate this equilibrium, namely, the allocation of agents across the contests. The algorithm is based on finite sequential steps, each of which includes an instruction that enables some types of agents to move across the contests. Our parallel contests have a subgame perfect equilibrium with pure strategies when the algorithm reaches a point at which no agent wants to change his position. At this point, the algorithm is said to have reached an equilibrium.

To summarize, we demonstrate that reputation in contests may lead to an unstable environment. Our goal is to determine when the reputation effect on contests leads to a stable outcome, that is, the conditions under which a subgame perfect equilibrium exists. We establish sufficient conditions that, in some cases, become necessary for the existence of this equilibrium, and we demonstrate that achieving these conditions is significantly more likely when agents' skills (costs of effort) and reputations are closely aligned. On the other hand, when agents have reputations that are unrelated to their skill sets (a low rank for an agent's cost of effort and a high rank for his reputation), the entire model becomes unstable.

1.1 Related literature

There are several papers dealing with parallel contests. These include the works of Konrad and Kovenock (2012) and Juang et al. (2020) who examine models of parallel all-pay contests with homogeneous players, and Azmat and Möller (2009), who study two parallel Tullock contests with homogeneous players. Azmat and Möller (2018) investigate parallel all-pay contests with a continuum of heterogeneous agents, and Morgan et al. (2018) examine two parallel contests involving a continuum of heterogeneous agents in which each agent's performance is deterministic but noisy. In each of the aforementioned models, the authors presumptively assume that agents are homogenous and have a linear cost of effort. Those who take into account heterogeneous agents also presumptively assume a continuum of agents. Cohen et al. (2023), in contrast, deal with a finite number of heterogeneous agents whose effort costs are non-linear. The current model deviates from Cohen et al. (2023) by assuming that the prizes are not constant and depend on the types (reputations) of the participating agents. This modification is crucial because, in contrast to all the papers mentioned above, an additional contestant does not necessarily disadvantage the other contestants.

In the second stage of our model of parallel contests, each contest takes place using a Tullock CSF (see, for example, Skaperdas 1996, Baye and Hoppe 2003, Ewerhart 2015, and Sela 2020). Based on the uniqueness of the equilibrium in one-stage generalized Tullock contests either in complete or incomplete information (see Szidarovszky and Okuguchi 1997, Yamazaki 2008, Einy et al. 2015, and Ewerhart and Quartieri 2020), our algorithm converges to the equilibrium in the case of non-linear costs of effort. The uniqueness of the equilibrium is important because it implies that there will always be a single equilibrium in every contest in the second stage for any change in the allocation of agents determined by the algorithm in the first stage.

The theoretical economic literature (e.g., Wilson 1985; Mailath and Samuelson 2015) primarily focuses on reputation effects in repeated games where each player builds his own reputation based on prior actions. Some of these models, for example, assume that a long-run player playing against a series of short-lived opponents can develop a reputation for playing in a certain way and thus benefit from commitment power (see, for instance, Kreps and Wilson, 1982; Kreps et al., 1982; Milgrom and Roberts, 1982; Cripps et al., 2004; Ely et al., 2008). However, in our two-stage model, an agent's reputation is fixed and unaffected by his actions. The common theme in our paper and the related literature is that interactions with highly reputable agents are beneficial to an agent.

The model by Damiano et al. (2010) considers a continuum of agents with heterogeneous types (ability or reputation) who must choose between two organizations of fixed sizes. Their interpretation of reputation is similar to that in our model. They assume that an agent's payoff increases with the average type of organization they join, but this payoff also increases with their ranking within the organization. Damiano et al. (2012) expand on their previous model by incorporating a competition between the two organizations, with each organization having the power to decide how to award prizes. Each organization benefits from the average type, and the agent's payoff increases with the nominal prize as well as the average type in the organization he chooses. In both of these models, agents simultaneously select the organization they wish to join. However, unlike our model, in which agents compete against one another in each contest (organization) and the outcome is determined by their actions, in their models, competitions among agents within these organizations are solely based on their types; that is, the agents are not active during that stage of the competition.

The literature on status in contests is very similar to that on reputation in contests (see, for example, Hopkins and Kornienko 2004, Kosfeld and Neckermann 2011, Bhattacharya and Dugar 2014, and Charness et al. 2014). Both situations are similar in that an agent prefers to be surrounded by high-type agents. However, in our model of contests with reputation, an agent wants to be in a group with high-type (highly reputable) agents, and the distribution of the agents in the other groups does not affect his payoff, while in status contests, an agent wants to be in the highest status class with the high-type agents, but the distribution of the agents in the other groups have a significant impact on his payoff. For example, Moldovanu et al. (2007), Dubey and Geanakoplos (2010), and Drugov and Ryvkin (2020), assume that each agent cares about the number of agents in the categories above and below him. In particular, agents receive a positive utility that is proportional to the number of agents in lower status categories and a negative utility that is proportional to the number of agents in higher status categories.

The rest of the paper is organized as follows: We describe the model of two-stage parallel contests with reputation in Section 2. We analyze the second stage in Section 3. We analyze the first stage of our model in Section 4, where agents have asymmetric reputations but similar cost functions. In Section 5, we analyze the first stage of our model, where agents have asymmetric cost functions as well as asymmetric reputations. Section 6 discusses the robustness and strength of our results, while Section 7 concludes. The appendix contains the proofs.

2 The model

Consider a set of n asymmetric agents, $N = \{1, 2, \ldots, n\}$, and a set of m asymmetric contests, $M = \{1, 2, \ldots, m\}$, where $m < n$. Each agent $i \in N$ has a reputation type α_i where $\alpha_i \geq \alpha_{i+1}$ for $i = 1, 2, ..., n-1$, and each contest $k \in M$ has a nominal prize v_k. In the first stage, each agent chooses a contest $k \in M$ to compete in where the set of agents who choose to compete in contest k is denoted by N_k, and n_k is the number of agents who choose to compete in contest k. In the second stage, after all of the agents have selected a contest to compete in, agents who have chosen the same contest compete against one another to win the contest. The agents in the contest k simultaneously exert their efforts. Let x_i denote the effort of agent $i \in N_k$. The probability that agent i will win contest k is $p_i = \frac{g_i(x_i)}{\sum_{j \in N_k} g_j(x_j)}$ where $g_i(x_i)$, which is called the production function, is assumed to be zero at $x_i = 0$, continuous, strictly increasing, and weakly concave in x_i for all $x_i \in \mathbb{R}_+$. Agent i who exerts an effort of x_i bears the cost of his effort $\hat{c}_i(x_i)$ regardless of whether he wins or loses, where $\hat{c}_i(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is the effort cost function, which is strictly increasing and weakly convex in the effort x_i. In addition, the winning value for the winner of contest k is a function $f_k(v_k, \bar{\alpha}_k)$ where v_k is the nominal prize of contest k and $\bar{\alpha_k} = (\alpha_i)_{i \in N_k}$ is the vector of all reputation types of the agents who have chosen contest k. The value function $f_k(v_k, \bar{\alpha}_k)$ is increasing in both the nominal prize v_k and the reputation types of the agents in contest k, $(\alpha_i)_{i\in N_k}$.The assumption that the value function is based on all contestants' reputation types is not critical and is made to simplify the calculations because, under this assumption, all contestants in the same contest have the same value, whereas if a contestant's value function is based on all contestants' reputations in the contest, excluding his own reputation, the contestants in the same contest have different value functions.[1]

We assume that adding an agent to a contest has a decreasing marginal effect on the value function f_k as the number of agents increases. Formally, denote by $f_k(v_k, \overline{\alpha}_k^m)$ the value function in contest k with m agents of reputation types $\overline{\alpha}_k^m = (\alpha_i)_{i=1}^m$ and by $f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s)$ the value function in contest k with m agents of the reputation types $\overline{\alpha}_k^m = (\alpha_i)_{i=1}^m$ plus $s > 0$ agents of the reputation types $\overline{\alpha}_s = (\alpha)_{j=1}^s$, namely, the additional s agents have the same reputation type α. Denote $R_{k,m,s} = |f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s) - f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_{s-1})|$ where $\overline{\alpha}_s$ is a vector of length of s and $\overline{\alpha}_{s-1}$ is a vector of length of $s-1$. Then we assume that $R_{k,m,s} > R_{k,m,s+1}$ for any given reputation vector $\overline{\alpha}_k^m$, $m = 2, 3, ...$ and for every reputation vectors $\overline{\alpha}_s$ and $\overline{\alpha}_{s-1}$. In particular, we assume that $\lim_{m\to\infty} R_{k,m,s} = 0$. In other words, the number of agents in the contest reduces the impact of an additional agent on the value function.

The expected payoff of agent $i \in N_k$ is:

$$f_k(v_k, \bar{\alpha}_k)\frac{g_i(x_i)}{\sum_{j\in N_k} g_j(x_j)} - \hat{c}_i(x_i). \tag{1}$$

Let $y_j = g_j(x_j)$ for all $j \in N_k$. Then, the expected payoff given by (1) is equivalent to

$$f_k(v_k, \bar{\alpha}_k)\frac{y_i}{\sum_{j\in N_k} y_j} - c_i(y_i), \tag{2}$$

where $c_i = g_i^{-1} \circ \hat{c}_i$.[2] The assumptions on g_i and $\hat{c}_i$ imply that c_i is a well-defined function with $c_i(0) = 0$, and that c_i is strictly increasing and weakly convex with respect to y_i. Thus, we will consider the maximization function given by (2) and assume that the agents' costs of effort can be ranked such that for all $x \geq 0$, $c_i(x) \leq c_{i+1}(x)$, $i = 1, ..., n-1$. Then, we can say that an agent with a lower cost function has a higher ability than other agents with higher cost functions. This competition will be known as the model of two-stage parallel contests with reputation.[3]

[1]It is worth noting that our main findings in this paper about the existence of equilibrium are valid even if we assume that an agent's value function is solely determined by the reputation of the other agents in the contest.

[2]This transformation was proposed by Szidarovszky and Okuguchi (1997), and Yamazaki (2008).

[3]We can assume a much more general contest success function here than the one given by (1). However, we also require the

To study the dynamics of the competition, we first analyze the second stage, in which each agent has already chosen the specific contest in which he competes, and now each agent i in each contest k chooses the effort level to compete with the other agents who have chosen the same contest. Following that, we analyze the competition in the first stage, in which the agents choose the contest, based on the analysis of the second stage.

3 The second stage

There are m contests taking place at the same time in the second stage. At this stage of the competition, analyzing contest $k \in M$ is sufficient because the m contests are independent of one another. The maximization problem of agent i in contest k is

$$\max_{x_i} f_k(v_k, \bar{\alpha_k}) \frac{x_i}{\sum_{j \in N_k} x_j} - c_i(x_i). \tag{3}$$

The FOC (first-order condition) of this maximization problem is

$$f_k(v_k, \bar{\alpha}_k) \frac{\sum_{j \in N_k} x_j - x_i}{\left(\sum_{j \in N_k} x_j\right)^2} \leq c_i'(x_i).$$

Without loss of generality, it is assumed that all the agents are active in contest k. If there are inactive agents, only the active agents will be considered. Following that, solving the maximization problem (3) yields the well-known result:

Proposition 1 *In the second stage of the model of two-stage parallel contests with reputation, if $i \in N_k$, then agent i's equilibrium effort is*

$$x_i = p_i \sum_{j \in N_k} x_j,$$

where p_i is agent i's winning probability and is given by

$$p_i = 1 - \frac{(n_k - 1)c_i'(x_i)}{\sum_{j \in N_k} c_j'(x_j)}, \tag{4}$$

and agent i's expected payoff is given by

$$u_i = f_k(v_k, \bar{\alpha}_k)(p_i)^2. \tag{5}$$

uniqueness of the equilibrium in the second stage of each contest, so we consider the contest success function given by (1).

So far, we have analyzed the second stage of the model of two-stage parallel contests with reputation. In the following section, we will analyze the first stage of this competition, focusing on the effect of reputation on the agents' strategies in the first stage. The first stage analysis will be divided into two cases. In case A, we will assume that all the agents have the same cost functions, whereas in case B, we will assume that agents have different cost functions that can be ranked as described above.

4 The first stage - case A

Given the analysis of the second stage, in which the agents choose pure strategies (efforts) according to Proposition 1, we will say that our model of two-stage parallel contests with reputation has a subgame perfect equilibrium in pure strategies if, in the first stage, the agents choose their contests using pure strategies such that, given the choices of the other agents, no agent wishes to switch to a different contest.

It is unclear whether an agent would prefer to compete against high-type or low-type agents. This is because highly reputable agents increase the winning value of the contest, whereas low-cost (high-ability) agents are difficult to compete with. These two effects must be considered by each agent when determining his best strategy. In the following section, we will assume that all the agents have the same cost function, $c_i(x) = c_j(x), \forall i, j \in N$, and analyze their dynamics solely based on their reputation, whereas in the following section, we will look at a more general situation in which cost functions differ between agents. Even when agents have the same cost function, the two effects exist; however, in terms of competitiveness, there is no difference between competing against a highly reputable agent and competing against a less-reputable agent. Thus, agents have an incentive to follow highly reputable agents because the winning value is higher with these agents, and the chances of winning are independent of agent type. As such, we will analyze the competition dynamics solely based on the reputations of the agents, and we will specifically find subgame perfect equilibrium with pure strategies for these competitions.

Even if all the agents' cost functions are symmetric, the analysis of the model of parallel contests with reputation is complex. Furthermore, it is clear in some cases that there is no subgame perfect equilibrium in pure strategies. This is especially true when there is no clear relationship between the agents' reputations and abilities (cost of effort), as in the case under consideration. The reason for this is that a highly reputable agent can significantly increase the value of the prize and, in particular, the expected payoffs of the other agents in the contest in which he competes, providing an incentive for other agents to follow him to every contest he chooses. Such a situation is depicted in the following example.

Example 1 *Consider a set of three agents $N = \{h, l, l\}$ with one highly-reputable agent (h-type) and two less-reputable agents (l-type), and a set of two contests $M = \{1, 2\}$ with nominal prizes of $v_1 = v_2 = 10$. The h-type agent has a reputation level of $\alpha_h = 30$, while the l-type agent has a reputation level of $\alpha_l = 1$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha_k}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

For both types of agents, the cost function is $c_i(x_i) = x_i$. To understand the contest dynamics, we look at every possible agent allocation across the two contests and calculate the agents' expected payoffs in the second stage. Following that, we will be able to identify subgame perfect equilibria in pure strategies, specifically, agents' allocations across the two contests such that no agent wishes to move to another contest. There are six possible permutations for the agents' allocation across the contests; however, due to the symmetry of the contests, only three are relevant. Because of the symmetry of the cost functions, the effort, payoff, and realized prize (the level of the value function given the allocation of agents) are the same for all the agents in the same contest. The table below depicts all possible allocations and the corresponding expected payoffs for the agents.

	Contest 1	*Contest 2*	*Contest* 1			*Contest* 2		
			Realized Prize	*Effort*	*Payoff*	*Realized Prize*	*Effort*	*Payoff*
1	$\{h, l, l\}$	$\emptyset$	*106.66*	*23.70*	*11.85*	-	-	-
2	$\{h, l\}$	$\{l\}$	*155*	*38.75*	*38.75*	*10*	-	*10*
3	$\{h\}$	$\{l, l\}$	*300*	-	*300*	*10*	*2.5*	*2.5*

Table 1: The expected payoff, effort and realized prize for the agents.

Beginning with allocation 1, in which all the agents chose the same contest, the h-type agent with a payoff of 11.85 will increase his payoff by switching to the other contest, bringing us to allocation 3. However, in *allocation 3, one l-type from contest 2, who now receives a payoff of 2.5, would prefer to move to contest 1, which brings us to allocation 2, where he will receive a payoff of 38.75. Now, in allocation 2, the l-type agent who competes in contest 2 and has a payoff of* 10 *would choose to move to contest 1, bringing us to allocation 1 in which he receives a payoff of 11.85. As a result, there is no allocation from which no agent wishes to move on, and thus no subgame perfect equilibrium in pure strategies exists in this case.*

The example 1 shows that a subgame perfect equilibrium in pure strategies does not always exist in our model of two-stage parallel contests with reputation. This occurs when there is a movement loop, as in the preceding example, specifically when a highly-reputable agent moves from contest i to contest j, which is followed by the movement of less-reputable agents and a chain reaction that returns us to an infinite loop. Identifying the conditions under which these loops do not exist will provide us the sufficient conditions for the existence of equilibrium. First, we order the agents' incentives to shift from place to place based on their level of reputation.

Lemma 1 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function $c_i(x) = c(x)$ for all $i \in N$ and for all $x \geq 0$. Let $i, j \in N_k$ and $\alpha_i \geq \alpha_j$. Then if agent i does not want to switch from contest k to contest s, neither does agent j.*

Lemma 1 demonstrates that if highly-reputable agents do not want to move from one contest to another, less-reputable agents in the same contest do not want to move either. In particular, if the agent with the highest reputation in a contest does not want to move to a different contest, then all the agents in the same contest will not want to move either. This enables us to analyze the corner equilibrium points as follows:

Proposition 2 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function, $c_i(x) = c(x)$ for all $i \in N$ and for all $x \geq 0$. Let $\bar{\alpha}$ be the vector of all the agents reputation types in N and α_1 be the highest reputation type. Then, if for all $s \in M, s \neq k$:*

$$\frac{\sqrt{f_k(v_k, \bar{\alpha})}}{\sqrt{f_s(v_s, \alpha_1)}} \geq n,$$

there is a subgame perfect equilibrium in which all the agents choose to compete in contest k.

According to proposition 2, when all the agents are allocated to contest k and the other contests remain empty, if the highest-reputable agent does not want to move to another contest, then there is a subgame perfect equilibrium in which, in the first stage, all the agents choose to compete in contest k. It is important to note that a corner equilibrium can exist when all the agents prefer to compete in contest k rather than contest s even if $v_k < v_s$. Using the same approach, we can identify not only corner equilibria but also interior equilibria. In particular, we will have such an equilibrium when the most reputable agent in every contest does not want to move.

Proposition 3 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function, $c_i(x) = c(x)$ for all $i \in N$ and for all $x \geq 0$. Let $\alpha_1 \geq \alpha_j$ for all $j \in N$. If there are a sufficient number of agents with the highest reputation type of α_1, then there is a subgame perfect equilibrium with pure strategies.*

Proposition 3 demonstrates that if there are a sufficient number of highly reputable agents in the competition, a pure strategy equilibrium must exist. Of course, the critical number of highly reputable agents is determined by the shape of the value function. The following example illustrates this phenomenon by examining a scenario with two types of agents, one with a low reputation and one with a high reputation. This example particularly illustrates that the number of highly reputable agents does not need to be very large for an equilibrium to exist.

Example 2 *Consider the set of agents $N = \{1, 2, 3, ..., n\}$ with $h > 1$ highly-reputable (h-type) and $l > 1$ less-reputable (l-type) agents, that is, $\alpha_h > \alpha_l$. There are two parallel contests, $M = \{1, 2\}$ with nominal prizes of $v_1 > v_2$. All the agents regardless of their reputation have the same cost function $c(x_i) = x_i$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

In the first stage, both h-type and l-type agents decide which contest to enter. The number of l-type and h-type agents in contests $i, i = 1, 2$, is denoted as l_i and h_i respectively. According to Proposition 1, the expected payoff of the agents who chose to compete in contest i is

$$v_i \frac{h_i \alpha_h + l_i \alpha_l}{(h_i + l_i)^3}.$$

Assume, without loss of generality, that h-type agents from contest 2 want to switch to contest 1. In this case, we let h-type agents move from contest 2 to contest 1 one by one. If there is at least one h-type agent *left in contest 2 who has no incentive to move to contest 1 then by Lemma 1, all the l-type agents in contest 2 do want to move either to contest 1 and we actually have an equilibrium allocation of the agents between the two contests. Accordingly, assume that $h-1$ h-type agents compete in contest 1, meaning that there is exactly one h-type agent left in contest 2. This agent will have no incentive to move to contest 1 if his expected payoff in contest 2 is greater than his expected payoff in contest 1 when he joins this contest. This occurs when*

$$v_1 \frac{h\alpha_h + l_1\alpha_l}{(h+l_1)^3} \leq v_2 \frac{\alpha_h + l_1\alpha_l}{(1+l_1)^3}, \tag{6}$$

which yields

$$\frac{(h+l_1)^3}{(l_2+1)^3} \geq \frac{v_1(h\alpha_h + l_1\alpha_l)}{v_2(\alpha_h + l_2\alpha_l)}.$$

Now, since

$$\frac{v_1(h\alpha_h + l_1\alpha_l)}{v_2(\alpha_h + l_2\alpha_l)} < \frac{v_1\alpha_h(h+l_1)}{v_2\alpha_l(1+l_2)},$$

a sufficient condition for the last h-type in contest 2 to stay there is

$$\frac{(h+l_1)^3}{(l_2+1)^3} > \frac{v_1\alpha_h(h+l_1)}{v_2\alpha_l(1+l_2)},$$

or, alternatively,

$$\left(\frac{h+l_1}{l_2+1}\right)^2 > \frac{v_1\alpha_h}{v_2\alpha_l}.$$

This holds if

$$h > \sqrt{\frac{v_1\alpha_h}{v_2\alpha_l}}(l_2+1) - l_1. \tag{7}$$

We can see from (7), that as the number of h-type agents increases, an equilibrium with pure strategies will be reached.

When a new agent enters a contest, it has two simultaneous effects on the value of the other agents in that contest. The reputation and competitiveness effects. The reputation effect can increase or decrease the value of the prize depending on the new agent's reputation in comparison to the reputations of the other agents in the contest. However, because there is one more agent to compete against for winning, the competitiveness effect reduces the expected payoff of the agents in the contest. According to Proposition 3, if there are a sufficient number of highly-reputable agents, the competitiveness effect will have a greater absolute value than the reputation effect, and no agent will be motivated to leave one contest and move to another. In that case, we say that an additional agent in a contest has a negative cumulative (competitiveness plus reputation) effect. The cumulative effect is defined formally as follows:

Definition 1 *We say that the introduction of a new agent into a contest has a positive (or negative) cumulative effect if, as a result of the new agent's entry into the contest, the expected payoff increases (decreases) for all other agents in that contest. Formally, if all the agents have the same cost function* $c_i(x) = c_j(x), \forall i, j \in N$*, a* β*-type agent, with reputation type of* α_β *has a negative cumulative effect on contest* k *if, given the the value function* f_k *of the nominal prize* v_k*, and the reputation vector* $\bar{\alpha}_k$ *of the* n_k *agents, there exists*

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2} > \frac{f_k(v_k, \bar{\alpha}_k, \alpha_\beta)}{(n_k + 1)^2}. \tag{8}$$

When a new agent enters a competition, the cumulative effect, which takes reputation and competitiveness effects into account, can be either positive or negative. When the cumulative effect is negative, either the agent's reputation is low relative to the reputations of the agents already competing, or the agent's reputation is high, but the competitiveness effect has a greater impact, resulting in a lower expected payoff for the agents who already compete. On the other hand, when the cumulative effect is positive, it shows that the reputation effect outweighs the competitiveness effect, which is always negative. In our model of two-stage parallel contests with reputation, the negative cumulative effect of all agents is critical to the existence of equilibrium. This means that even if the most reputable agent enters a competition, the expected payoff of the agents currently competing in that competition will decrease. Then, we will show that a pure-strategy equilibrium will exist.

According to Proposition 3, if there are a sufficient number of highly reputable agents dispersed among the contests, then the cumulative effect will be negative in all the parallel contests, and as a result, the subgame perfect equilibrium with pure strategies exists. However, a sufficient number of highly-reputable agents is not a necessary condition for the cumulative effect to be negative. Our requirements that $R_{k,m,s} = |f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s) - f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_{s-1})|$ satisfies that for any given reputation vector $\overline{\alpha}_k^m$, $m = 2, 3, ...$ and for every vectors of s and $s-1$ additional agents with a reputation type of α, as well as the fact that $\lim_{m\to\infty} R_{k,m,s} = 0$ are necessary conditions for the cumulative effect to be negative, although they are not sufficient. For each value function $f_k(v_k, \bar{\alpha}_k)$, there are different sufficient conditions for the cumulative effect to be negative for any nominal prize v_k and any given reputation vector $\overline{\alpha}_k$. Obviously, there are some value functions for which sufficient conditions do not exist, but in the following, we illustrate two examples in which the value functions have negative cumulative effects under certain conditions regardless of the distribution of the agents among the contests and their reputation types.

Claim 1 *Consider the model of two-stage parallel contests with reputation. Assume that contest k has the following value function*

$$f(v_k, \bar{\alpha}_k) = h(v_k, \alpha_{\min}).$$

where $\alpha_{\min} = \min\{\alpha : \alpha \in \bar{\alpha}_k\}$ and $h : R_+^2 \to R_+$ is a monotonically increasing function in both v_k and $\alpha_{\min}$. Then, the value function $f(v_k, \bar{\alpha}_k)$ has a negative cumulative effect for any number of agents n_k, any nominal prize v_k and any given reputation vector $\overline{\alpha}_k$.

This result is straightforward because the entry of any agent either reduces or does not change the reputation effect, but it increases the competitiveness effect. As a result, the addition of any new agent reduces the cumulative effect, implying that the value function $h(v_k, \alpha_{\min})$ has a negative cumulative effect regardless of the agents' reputation types. While the previous claim demonstrated that a value function could have a negative cumulative effect without any conditions, the following claim demonstrates that for the value function given in Example 2, a modest level of agents' heterogeneity in each contest is sufficient to guarantee that it will have a negative cumulative effect.

Claim 2 *Consider the model of two-stage parallel contests with reputation where all the agents have the same cost function, $c_i(x) = c_j(x)$ for all $i, j \in N$ and for all $x \geq 0$. The set of agents includes h-type and l-type agents with reputation types of α_h and α_l respectively where $\alpha_h > \alpha_l$. Let h and l be the number of h-type and l-type agents in the competition, where $h + l = n$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

Then, if $\alpha_h \leq 3\alpha_l$, this value function has a negative cumulative effect for any nominal prize v_k and any given reputation vector $\overline{\alpha}_k$.

When the introduction of a new highly reputable agent into a contest has a negative cumulative effect, the option to move between contests in loops is removed. Then, in contrast to Example 1, agents will not follow the movements of highly reputable agents. Even if the agent with the highest reputation joins a contest and increases its value function, his negative cumulative effect will reduce the expected payoffs of the agents in that contest and thus the incentive of other agents to enter this contest. With this property of negative cumulative effect, we can implement an algorithm and show that the competition will reach a situation where no agent moves. In such a case, we say that the algorithm has reached equilibrium, and in particular, the model of two-stage parallel contests with reputation has a subgame perfect equilibrium.

Algorithm 1 *The algorithm has the following four steps:*

Step 1 *Begin with all contests empty and deny any agent access to the contests.*

Step 2 *Assign the agent with the lowest reputation who has not yet been assigned to the contest in which he has the highest expected payoff.*

Step 3 *Allow agents who have already been assigned to contests to switch contests if they so desire.*

Step 4 *Repeat steps 2-3 until all the agents have been assigned to a contest and no agent wants to move in step 3.*

Algorithm 1 assigns agents in ascending order of reputation, beginning with the agent with the least reputable reputation and ending with the agent with the most reputable reputation. Despite the fact that each new additional agent offers the opportunity to switch contests, the agents remain in the contest to which they were initially assigned. The following theorem demonstrates that this algorithm converges to equilibrium and that agents do not move during its execution.

Theorem 1 *Consider the model of two-stage parallel contests with reputation consisting of m contests and n agents where all the agents have the same cost function, $c_i(x) = c_j(x)$ for all $i, j \in N$ and for all $x \geq 0$. Then, if any additional agent to any contest has a negative cumulative effect, Algorithm 1 reaches equilibrium such that there is a subgame perfect equilibrium with pure strategies.*

Theorem 1 proves that if the agents have a negative cumulative effect on the payoff in each of the contests they enter, our model of parallel contests with reputation has a subgame perfect equilibrium. For example, according to Claim 1, if the value function in each contest is the minimum function $f(v_k, \bar{\alpha}_k) = \min\{\alpha : \alpha \in \bar{\alpha}_k\}$, then we have a subgame perfect equilibrium in pure strategies. Similarly, according to Claim 2, if the value function in each of two parallel contests with two types of agents is the average function $f^A(v_k, \bar{\alpha}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}$, then we have a subgame perfect equilibrium in pure strategies if the difference in the agents' reputation types is not too large.

5 The first stage - case B

So far, we have assumed that the cost functions of all the agents are the same. We will show in this section that this is not a limiting assumption in our model and that the results, particularly the existence of subgame perfect equilibrium with pure strategies, can be obtained even under weaker conditions when the agents have asymmetric cost functions. Thus, we consider here a set of agents $N = \{1, 2, ..., n\}$ with n agents of two types, highly-reputable agent (h) and low-reputable agent (l), such that their reputation types satisfy $\alpha_h \geq \alpha_l$ and their cost functions satisfy $c_h(x) \leq c_l(x)$ for all $x \geq 0$. In other words, an agent's reputation and ability are related.

Cohen et al. (2023) demonstrated the existence of subgame perfect equilibrium in a model of two-stage parallel contests with asymmetric, non-linear cost functions and a value function based on a fixed nominal prize. We extend their work by considering value functions that are not fixed and are endogenously defined by the nominal prize and the reputation types of the agents who chose to compete in the same contest.

We assumed in the previous section that the agents' competitiveness effect was the same because their cost function (or ability) was the same. In this case, the competitiveness effect is not homogeneous, as highly reputable agents incur lower costs for each level of effort. Thus, the competitiveness effect of including highly reputable agents in the contest has a greater negative impact, implying that the cumulative effect, which is the sum of the competitiveness and reputation effects, will be lower (more negative) in that case. Thus, as in the previous section, we make the same assumption that the value functions have a negative cumulative effect. Then, we can implement an algorithm and demonstrate that the competition will eventually reach a state in which no agents move, i.e., the algorithm has reached equilibrium.

Algorithm 2 *The algorithm has the following four steps:*

Step 1 *Allocate the agents randomly across the contests.*

Step 2 *Let the h-type agents switch contests if they want.*

Step 3 *Let a single l-type agent move between the contests.*

Step 4 *Repeat steps 2-3 until, in step 3, no l-type agent wants to move and every agent is assigned to a contest.*

The next theorem shows that this algorithm converges to equilibrium and, more specifically, to an allocation of the agents across the contests such that no agent wants to move from his location to another one.

Theorem 2 *Consider the model of two-stage parallel contests with m contests and two types of agents h and l where their reputation types satisfy $\alpha_h \geq \alpha_l$ and their cost functions satisfy $c_h(x) \leq c_l(x)$ for all $x \geq 0$. If any additional h-type agent has a negative cumulative effect on the contest, algorithm 2 reaches equilibrium such that there is a subgame perfect equilibrium with pure strategies.*

6 Discussion

In this section, we investigate the robustness and strength of our main findings. According to Theorems 1 and 2, in our model of two-stage parallel contests, a sufficient condition for the existence of a subgame perfect equilibrium with pure strategies is that any additional agent to any contest has a negative cumulative effect. In the following, we want to highlight that this sufficient condition for the existence of equilibrium is far more likely to hold in an environment where the agents' skills and reputations are correlated. In Example 1, we show that when agents have different reputation types but the same cost of effort, namely, the agents' skills and reputations are not correlated, the subgame perfect equilibrium with pure strategies does not exist. Example 3 is slightly different from Example 1, such that the agents' skills and reputations are correlated, that is, highly reputable agents have a lower cost than less reputable agents. As a result, the negative cumulative effect is more likely to hold, implying the existence of a subgame perfect equilibrium based on pure strategies.

Example 3 *As in Example 1, consider a set of three agents $N = \{h, l, l\}$ with one highly-reputable agent (h-type) and two less-reputable agents (l-type), and a set of two contests $M = \{1, 2\}$ with nominal prizes of $v_1 = v_2 = 10$. The h-type agent has a reputation level of $\alpha_h = 30$, while the l-type agent has a reputation level of $\alpha_l = 1$. The value function in both contests is*

$$f^A(v_k, \bar{\alpha_k}) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}.$$

For both types of agents, the cost function is $c_i(x_i) = c_i x_i$, where $c_l = 1$ and $c_h = 0.85$. As shown in Example 1, only three permutations are relevant due to the contest's symmetry. However, because each type of agent has a distinct cost function, the expected payoff and effort will differ between them. The two tables below show the realized prize, expected payoff, and effort from the perspective of an h-type (l-type) agent for all permutations. The h-type agents can be found alone, with one, or two l-type agents. Similarly, a l-type agent can compete alone, with another l-type agent, another h-type agent, or with all agents participating in the same contest.

		Realized Prize	Effort	Payoff	
1	$\{h, l, l\}$	106.66	30.20	17.36	
2	$\{h, l\}$	155	45.28	45.28	
3	$\{h\}$	300	-	300	

Table 2: The expected payoff, effort and realized prize for the h-type agent.

		Realized Prize	Effort	Payoff	
1	$\{h, l, l\}$	106.66	22.32	9.48	
2	$\{h, l\}$	155	38.49	32.72	
3	$\{l, l\}$	10	2.5	2.5	
4	$\{l\}$	10	-	10	

Table 3: The expected payoff, effort and realized prize for the l-type agents.

Starting with all agents selecting the same contest, the h-type agent gets a payoff of 17.36 (Table 2, case 1). He will increase his payoff to 300 by switching to the other contest (Table 2, case 3). Contest 2 now contains only one h-type agent, whereas Contest 1 contains two l-type agents. In this case, one l-type agent *from contest 1 who received a payoff of 2.5 (Table 3, case 3) would prefer to move to contest 2, where he will receive a payoff of 32.72 (Table 2, case 2). The h-type agent has no incentive to move because both contests have exactly one l-type agent, and the l-type agent who is alone in contest 1 would prefer a payoff of 10 (Table 3, case 4) over 9.48 if he switched to the other contest (Table 3, case 1). So we have reached a point of equilibrium.*

As demonstrated in Theorem 2, the requirement for a negative cumulative effect is a sufficient condition for the existence of a subgame perfect equilibrium with pure strategies in our model of parallel contests with reputation. However, it is sometimes a necessary condition. According to Proposition 3, as the number of highly reputable agents increases, the condition of a negative cumulative effect is met because the marginal effect of each agent on the value function is negative.

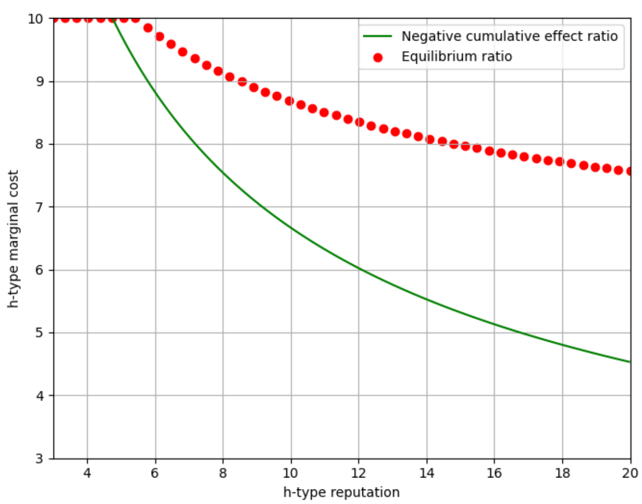

Figure 1: Cumulative negative effect as a sufficient condition

Figure 1 elucidates the relationship between the negative cumulative effect condition and the existence of pure-strategy equilibrium. This figure depicts a situation with seven agents, five of which are low-type agents and two of which are high-type agents. There are two contests with the same nominal prize (normalized to 1), and the value function for each contest is $f(v_k, \bar{\alpha}_k) = v_k \frac{\sum_{i \in N_k} \alpha_i}{n_k}$. We also assume that low-type agents have a low reputation ($\alpha_l = 1$) and a high marginal cost of effort ($c_l = 10$), whereas the reputations and marginal costs of effort of high-type agents vary and are shown in Figure 1.[4]

According to Claim 2, when $\alpha_h \leq 3\alpha_l$, there is a subgame perfect equilibrium with pure strategies even if the marginal cost of effort of both types of agents is identical. However, in this case with heterogeneous marginal costs of effort, we examine for each value of the high-type's reputation, the maximum marginal cost of effort (so that the high-type agents are as weak as possible) such that there is still an equilibrium allocation of the agents (the curve of red dots). On the same graph, we have added the appropriate value of the cost of effort for each value of the high-types' reputation, so that the value function has a negative cumulative effect (the green curve). There is a gap between the red and green curves, which is due to the fact that the condition of the negative cumulative effect is sufficient for the existence of equilibrium. However, it is not necessary.

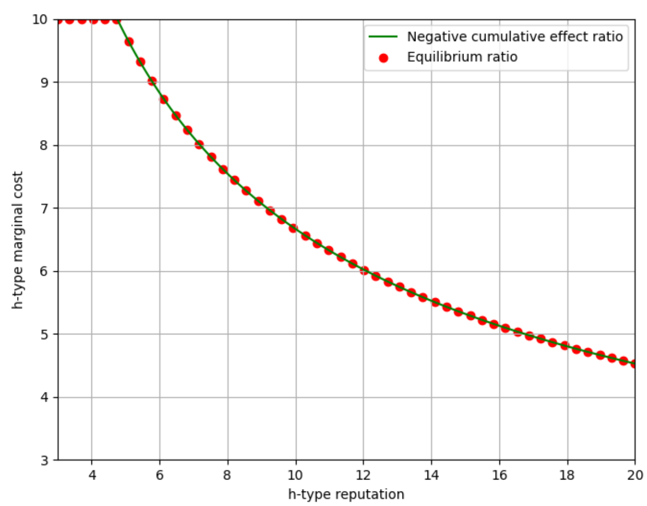

Figure 2: Cumulative negative effect as a necessary condition

Figure 2 depicts the same situation as Figure 1 but with a different number of agents. There is one high-type agent and three low-type agents in Figure 2. We can see that when the number of agents is small, as in this case, the condition of the negative cumulative effect merges with the condition of equilibrium's existence. In other words, the negative cumulative effect condition becomes a necessary and sufficient condition for the

[4]The method for numerically determining, for a given reputation, the maximum cost that will still result in an equilibrium existence in each scenario is outlined in the appendix.

existence of a subgame perfect equilibrium with pure strategies.

7 Conclusion

In this paper, we studied parallel contests in which the agents' reputations play a crucial role in determining the outcome of the competition. We analyzed a model of two-stage parallel contests where, in the first stage, agents simultaneously select the contests in which they want to compete, and in the second stage, agents who selected the same contest compete against each other for a single prize determined not only by a nominal value but also by the reputation of the competing agents.

We initially assumed that the parallel contests include heterogeneous agents with distinct reputation types and identical cost functions. Then, we showed that there are situations in which there is no subgame perfect equilibrium with pure strategies, implying that some agents always prefer to leave the contest they chose to enter since by doing so they can increase their expected payoffs. Then, we provided a sufficient condition on the agents' value function that guaranteed the existence of the pure-strategy equilibrium, and we demonstrated that this sufficient condition is feasible for several forms of the agents' value function.

Then, we assumed that there are n heterogeneous agents with two possible types, each with a different nonlinear cost of effort and a different reputation, and that the non-linear costs of effort are not explicitly given. Although it is impossible to characterize the subgame perfect equilibrium in this case, we established a sufficient condition for the existence of a subgame perfect equilibrium with pure strategies and provided an algorithm for calculating this equilibrium using revealed preference techniques. Later, we demonstrated using computer simulations that when the number of agents is small, our sufficient condition is also necessary for the existence of subgame perfect equilibrium with pure strategies, but as the number of agents increases, our condition is sufficient but no longer necessary.

We found that the alignment between an agent's skills (cost functions) and reputation plays a crucial role in determining the likelihood of achieving a subgame perfect equilibrium with pure strategies in such contests. When the reputations of agents are closely related to their skill levels, the model tends to achieve stability with pure strategy equilibria. When reputations and skills are misaligned, resulting in low-ranked agents having high reputations, the model becomes unstable, and pure-strategy equilibria become elusive.

In this paper, we improve our understanding of the complexities of reputation-driven parallel contests. Understanding the implications of reputation in parallel contests provides a useful lens through which to examine the dynamics of other competitive environments, as reputation continues to have a significant impact on a variety of social and economic phenomena.

8 Appendix

8.1 Proof of Lemma 1

Assume that $i, j \in N_k$, and $\alpha_i > \alpha_j$, and that agent i does not want to move from contest k to contest s, while agent j wants to make this movement. We will show that such a scenario leads to a contradiction. By (5), agent i's expected payoff in contest k is

$$f_k(v_k, \bar{\alpha}_k)\left(1 - \frac{(n_k - 1)c_i'}{\sum_{j \in N_k} c_j'}\right)^2.$$

By symmetry of the agents' cost functions, we obtain that all the agents in contest k have the same expected payoff, which is given by

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2}.$$

An i-type agent will not switch from contest k to contest s if the expected payoff in contest k is greater than the expected payoff in contest s after he joins this contest. This holds true if

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2} \geq \frac{f_s(v_s, \bar{\alpha}_s, \alpha_i)}{(n_s + 1)^2}.$$

Similarly, j-type agent wants to switch from contest k to contest s if

$$\frac{f_k(v_k, \bar{\alpha}_k)}{n_k^2} < \frac{f_s(v_s, \bar{\alpha}_s, \alpha_j)}{(n_s + 1)^2}.$$

The last two inequalities imply that

$$f_s(v_s, \bar{\alpha}_s, \alpha_j) > f_s(v_s, \bar{\alpha}_s, \alpha_i).$$

The monotonicity of $f_s(.)$ in each reputation type and our presumption that $\alpha_i > \alpha_j$ are incompatible with this inequality. As a result, if the i-type agent from contest k does not want to move to contest s, the j-type agent from contest k will not either.

8.2 Proof of Proposition 2

By (5), if all the agents chose to compete in contest k, the expected payoff of agent i is

$$f_k(v_k, \bar{\alpha})\left(1 - \frac{(n - 1)c_i'}{\sum_{j \in N} c_j'}\right)^2,$$

where $\bar{\alpha}$ is the vector of reputation types of all the agents in contest k. Because all the agents have the same cost function $c_i(x) = c_j(x), \forall i, j \in N$, $x \geq 0$, agent i's winning probability is $1/n$. This means that if all the agents choose to compete in contest k, the expected payoff of agent i is

$$\frac{f_k(v_k, \bar{\alpha})}{n^2}$$

We can conclude from Lemma 1 that the allocation of agents in which all the agents compete in contest k is in equilibrium if there is no other contest $s \in M, s \neq k$, that the highest reputable agent, agent 1, with a reputation type of α_1, would prefer over contest k. Agent 1 prefers contest s over contest k if

$$f_s(v_s, \alpha_1) < \frac{f_k(v_k, \bar{\alpha})}{n^2},$$

or, alternatively, if

$$\frac{\sqrt{f_k(v_k, \bar{\alpha})}}{\sqrt{f_s(v_s, \alpha_1)}} < n.$$

Thus, there is an equilibrium in which all the agents choose to compete in contest k, if

$$\frac{\sqrt{f_k(v_k, \bar{\alpha})}}{\sqrt{f_s(v_s, \alpha_1)}} \geq n.$$

8.3 Proof of Proposition 3

Consider a set of agents $N = \{1, 2, ..., n\}$ and a set of contests $M = \{1, 2, ..., m\}$. Let the γ-type agents be the highest reputable agents with a reputation type of α_γ, $\alpha_\gamma \geq \alpha_i, \forall i \in N$. Let Γ be the number of γ-type agents in the competition where $\Gamma > m$, and where Γ_s is the number of γ-type agents allocated in contest s.

We begin by allocating all types except the γ-type agents to the contests in any way, while the γ-type agents will be allocated later. Let ζ_s be the number of agents with different types than γ-type agents that are allocated to contest s. Then, we allocate the γ-type agents equally across the m contests. Now when all the agents, both γ-type and other types of agents, have been allocated, we reach an equilibrium point if no agent wishes to move to another contest. Otherwise, If any agent from contest $i \in M$ wishes to move to another contest j, according to Lemma 1, the γ-type agent from that contest also wishes to move from contest i to contest j.

Remember that we assumed that the number of competing agents reduces the impact of an additional agent on the agents' value function f_k, where formally. $\lim_{m \to \infty} R_{k,m,s} = \lim_{m \to \infty} |f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_s) - f_k(v_k, \overline{\alpha}_k^m, \overline{\alpha}_{s-1})| =$

0 for any s additional agents with reputation types $\alpha_s = (a)_{i=1}^{s}$, particularly for γ-type agents. This implies that

$$\lim_{\Gamma_j \to \infty} \frac{f_j(v_j, \bar{\alpha}_j)}{(\Gamma_j + \zeta_j)^2} = 0$$

Thus, if there are a sufficient number of γ-type agents in contest j, entering a new agent to the contest reduces the payoff of the other agents in that contest. This means that if there are sufficient number of γ-type agents in the competition and they spread across the contests, no agent will want to join contest j. Hence, there is an equilibrium allocation of agents across the contests.

8.4 Proof of Claim 2

The number of l-type and h-type agents who decide to participate in contests $i \in M$ in the first stage will be denoted as l_i and h_i respectively. According to Proposition 1, the expected payoff of the agents who chose to compete in contest i is

$$v_i \frac{h_i \alpha_h + l_i \alpha_l}{(h_i + l_i)^3}.$$

An entry of an additional l-type agent has a negative cumulative effect on a contest, as it increases the contest's competitiveness and decreases (or has no effect on) the average reputation in the contest. However, an additional h-type agent has two opposing effects. After another h-type agent entered the contest, the expected payoff of each agent in that contest is

$$\frac{v_i((h_i + 1)\alpha_h + l_i \alpha_l)}{(h_i + 1 + l_i)^3}$$

We compare the previous payoff of the agents to the new one after the h-type agent joined the contest to determine whether the cumulative effect is positive or negative. Thus, the cumulative effect is negative if

$$\frac{v_i(h_i \alpha_h + l_i \alpha_l)}{(h_i + l_i)^3} - \frac{v_i((h_i + 1)\alpha_h + l_i \alpha_l)}{(h_i + 1 + l_i)^3} \geq 0.$$

Let $\alpha_h = \gamma \alpha_l$, where $\gamma > 1$. Then the condition that the cumulative effect is negative is

$$\frac{v_i(h_i \gamma \alpha_l + l_i \alpha_l)}{(h_i + l_i)^3} - \frac{v_i((h_i + 1)\gamma \alpha_l + l_i \alpha_l)}{(h_i + 1 + l_i)^3} \geq 0,$$

or, alternatively,

$$\frac{(h_i + 1 + l_i)^3}{(h_i + l_i)^3} \geq \frac{\gamma(h_i + 1) + l_i}{\gamma h_i + l_i}.$$

It can be verified that the last inequality is satisfied for every $\gamma \leq 3$. This means that if $\alpha_h \leq 3\alpha_l$, the cumulative effect of an h-type is negative, namely, contest i will become less attractive for new agents of any type.

8.5 Proof of Theorem 1

Algorithm 1 begins with all the agents unassigned and all contests empty. Beginning with step 2, the agent with the lowest reputation is assigned to a contest. He will not want to change contests in step 3. Returning to step 2, the agent with the second lowest reputation will be assigned to the contest in which he has the highest expected payoff. If this is a different contest than the previous agent's contest, the previous agent will not join the new agent in step 3. This is because a new agent has a negative cumulative effect on any contest he enters. If, on the other hand, the new agent will be assigned to the same contest of the previous agent, according to Lemma 1, the previous agent will not want to switch contests. This argument holds for subsequent steps, and no contest switching will occur throughout the algorithm's steps. Therefore, once all of the agents are assigned to the contests, no one will want to move, and the algorithm will reach an equilibrium.

8.6 Proof of Theorem 2

We start by allocating the agents randomly across the contests. In step 2, we let only h-type agents move until they could not increase the expected payoff. In step 3, a single l-type agent has the option to switch between the contests. Assume the l-type agent changed locations from contest i to j. This means that his expected payoff in contest j is greater than his current expected payoff in contest i. No h-type agent will join contest j as a result of the l-type move. Because of his movement, the l-type agent reduces contest j's value function. This is because his action has a negative cumulative effect on contest j. Back in step 2, as a response, h-type agents who initially did not wish to move (in the previous iteration of step 2), may now wish to move to contest i, but not to contest j. Thus, no h-type will ever join contest j and no l-type agent will join contest i. As the algorithm continues, the agents will not join more and more contests, until the competition reaches an equilibrium.

9 The process of creating graphs

We implemented a computer simulation to produce the graphs. The simulation is conducted in several steps. All of the agent permutations among the contests are created by the simulation in the first step. For each allocation in each permutation, the simulation then calculates the equilibrium values of the second stage, namely, the prize, based on the agents' reputations, efforts, winning probabilities, and expected payoffs. The simulation then produces a map of "adjacent permutations." A pair of permutations is referred to as "adjacent permutations" if, with the exception of the allocation of one agent between the contests, the distribution of agents in each permutation is the same. Then, for each permutation i, it is examined if there is an adjacent permutation j where the agent who switches locations receives a greater expected payoff than in permutation i. If this is the case, then permutation i is not an equilibrium permutation. If there are no adjacent permutations for which the expected payoff of the agent changing his location is higher than his prior expected payoff, the permutation is an equilibrium permutation.

With this simulation algorithm, we can examine the existence of equilibrium. Given that there are two different types of agents, we can determine the high-type agents' maximum fixed marginal cost while maintaining equilibrium. We employ a binary search algorithm variant for this purpose (Lehmer 1960; Bottenbruch 1962). The algorithm receives potential upper and lower bounds where the maximum cost of the high-type agent exists in equilibrium and reduces the range of possibilities as follows (pseudo code):

Function: `find_max_cost_in_equilibrium(model)`

```
    // The assumed range for the cost of the high-type agent
    cost_min = 0  // Minimum cost value
    cost_max = 100  // Maximum cost value (adjust as needed)
    precision = 0.001  // Desired precision
    // Perform a binary search for the maximum cost value
    WHILE cost_max - cost_min ≥ precision:
        cost_to_test = (cost_min + cost_max) / 2  // Midpoint of the range
        equilibrium_exists = run_simulation(model, cost_to_test)
        IF equilibrium_exists
```

```
            // If there is an equilibrium allocation, move the lower bound up
            cost_min = cost_to_test
        ELSE:
            // If there is no equilibrium allocation, move the upper bound down
            cost_max = cost_to_test
    /* The maximum cost value of the strong agent
       for this model is stored in cost_min */
    RETURN cost_min
```

Thus, it is possible to find the point where the cost of the strong agent is maximal and there is still equilibrium.

References

[1] Azmat, G., Möller, M. (2009). Competition among contests. *The RAND Journal of Economics*, 40 (4), 743–768.

[2] Azmat, G., Möller, M. (2018). The distribution of talent across contests. *The Economic Journal*, 128 (609), 471–509.

[3] Baye, M. R., Hoppe, H. C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior*, 44 (2), 217–226.

[4] Bhattacharya, H., Dugar, S. (2014). Partnership formation: The role of social status. *Management Science* 60(5), 1130–1147.

[5] Bottenbruch, H. (1962). Structure and use of ALGOL 60. *Journal of the ACM (JACM)*, 9(2), 161-221.

[6] Charness, G., Masclet, D., Villeval, M. C. (2014). The dark side of competition for status. *Management Science* 60, 38–55.

[7] Cohen, C., Rabi, I., Sela, A. (2023). An Algorithmic Analysis of Parallel Contests. Working paper.

[8] Cripps, M. W., Mailath, G. J., Samuelson, L. (2004). Imperfect monitoring and impermanent reputations. *Econometrica*, 72(2), 407-432.

[9] Cripps, M. W., Mailath, G. J., Samuelson, L. (2007). Disappearing private reputations in long-run relationships. *Journal of Economic Theory*, 134(1), 287-316.

[10] Damiano, E., Li, H., Suen, W. (2010). First in village or second in Rome? *International Economic Review*, 51 (1), 263–288.

[11] Damiano, E., Li, H., Suen, W. (2012). Competing for talents. *Journal of Economic Theory*, 147 (6), 2190–2219.

[12] Drugov, M., Ryvkin, D. (2020). Tournament rewards and heavy tails. *Journal of Economic Theory* 190, 105116.

[13] Dubey, P., Geanakopolos, J. (2010). Grading exams: 100, 99, 98,..., A, B, C ? *Games and Economic Behavior* 69(1), 72-94.

[14] Einy, E., Haimanko, O., Moreno, D., Sela, A., Shitovitz, B. (2015). Equilibrium existence in Tullock contests with incomplete information. *Journal of Mathematical Economics*, 61, 241–245.

[15] Ewerhart, C. (2015). Mixed equilibria in Tullock contests. *Economic Theory*, 60, 59–71.

[16] Ewerhart, C., Quartieri, F. (2020). Unique equilibrium in contests with incomplete information. *Economic Theory*, 70, 243–271.

[17] Ely, J., Fudenberg, D., Levine, D. K. (2008). When is reputation bad?. *Games and Economic Behavior*, 63(2), 498-526.

[18] Hopkins, E., Kornienko, T. (2004). Running to keep the same place: consumer choice as a game of status. *American Economic Review* 94(4), 1085-1107.

[19] Juang, W. T., Sun, G. Z., Yuan, K. C. (2020). A model of parallel contests. *International Journal of Game Theory*, 49 (2), 651–672.

[20] Konrad, K. A., Kovenock, D. (2012). The lifeboat problem. *European Economic Review*, 56 (3), 552–559.

[21] Kosefeld, M., Neckermann, S. (2011). Getting more work for nothing? Symbolic awards and worker performance. *American Economic Journal: Microeconomics* 3(3), 86-99.

[22] Kreps, D. M., Milgrom, P., Roberts, J., & Wilson, R. (1982). Rational cooperation in the finitely repeated prisoners' dilemma. *Journal of Economic theory*, 27(2), 245-252.

[23] Kreps, D. M., & Wilson, R. (1982). Reputation and imperfect information. *Journal of economic theory*, 27(2), 253-279.

[24] Lehmer, D. H. (1960). Teaching combinatorial tricks to a computer. *Combinatorial Analysis*, 179-193.

[25] Mailath, G. J., Samuelson, L. (2015). Reputations in repeated games. Handbook of Game Theory with Economic Applications, 4, 165-238.

[26] Milgrom, P., & Roberts, J. (1982). Predation, reputation, and entry deterrence. *Journal of economic theory*, 27(2), 280-312.

[27] Moldovanu B., Sela, A., Shi, X. (2007). Contests for status, *Journal of Political Economy*, 115(2), 338-363.

[28] Morgan, J., Sisak, D., Vardy, F. (2018). The ponds dilemma. *The Economic Journal, 128* (611), 1634–1682.

[29] Origgi, G. (2019). Reputation: What it is and why it matters. Princeton University Press.

[30] Sela, A. (2020). Optimal allocations of prizes and punishments in Tullock contests. *International Journal of Game Theory*, 49, 749-771.

[31] Skaperdas, S. (1996). Contest success functions. *Economic Theory,* 7, 283-290.

[32] Szidarovszky, F., Okuguchi, K. (1997). On the existence and uniqueness of pure Nash equilibrium in rent-seeking games. *Games and Economic Behavior,* 18 (1), 135–140.

[33] Tullock, G. (1980). Efficient Rent Seeking. In J. M. Buchanan, R. D. Tollison, & G. Tullock, Toward a theory of the rent-seeking society. Texas A & M University.

[34] Wilson, R. (1985). Reputations in games and markets. Game-theoretic models of bargaining, 96, 27-62.

[35] Yamazaki, T. (2008). On the existence and uniqueness of pure-strategy Nash equilibrium in asymmetric rent-seeking contests. *Journal of Public Economic Theory,* 10 (2), 317–327.