Contests with Prize Contracts

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Abstract

We analyze two-stage contests with symmetric players. During the first stage, each player can receive a prize contract, which includes both a losing and winning prize with the same prize sum. After receiving their prize contracts, and no player wishes to change them due to the allocation of the other prize contracts, the players compete in the standard Tullock contest during the second stage. We show that in the subgame perfect equilibrium, if players participate in the second stage, they should receive multiple types of prize contracts. While the designer cannot achieve the optimal value of the players' total effort in equilibrium, we demonstrate that if there are at least five players, he can achieve the optimal value of the players' highest effort in equilibrium by using two types of prize contracts.

KEYWORDS: Contests, prize contracts, risk

JEL classification: D44, C72, O31, O32

1 Introduction

The extensive literature on prize allocation has shown that contest prizes based on performance rank can be used to generate effective incentives.[1] Usually, in the literature on prize allocation for all contest models, whether with one or more stages and single or multiple prizes, the value of the prizes is assumed to be endogenous for the contest designers but exogenous for the contestants. The players may receive a variety

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[1] For the first works on labor tournament prizes based on performance rank, see Lazear and Rosen (1981), Green and Stokey (1983), and Nalebuff and Stiglitz (1983).

of prizes, but they have no control over the value of their prizes during the competition. In other words, contestants are unable to negotiate the value of the prizes awarded during the contest.

However, in some contests, as in many other organizations, the prizes awarded to contestants are endogenous. In these contests, a prize contract is formed between an employer and an employee, or between the manager and the contestant. This prize contract includes both a fixed and an uncertain payment, which is dependent on the employee's performance or the outcome of that performance. A fixed payment is a predetermined amount paid by an employer to an employee who meets the regular job requirements, regardless of performance. Variable payment, on the other hand, is compensation given to an employee who goes above and beyond their regular job requirements and is based on the employee's performance. In sports competitions, for example, a player and his manager may agree on a prize contract for one or more games. This type of prize contract is typically divided into two parts: an initial sum awarded regardless of the outcome of the contest, and a subsequent sum determined by the player's or team's performance in the competition. This also applies to other types of competitions, such as research and development competitions, in which participants are compensated for their time and effort, with the possibility of further compensation if their research is successful. In all of these scenarios, different prize contracts are offered to players based on their skills or reputation.

In contests, this type of prize contract between the contest designer and the contestants is equivalent to one that includes both a prize for winning and a prize for losing. When a contest designer and contestants sign a prize contract, there is undoubtedly a conflict. To maximize contestants' efforts (outputs), the contest designer may raise the proportion of uncertain payments based on the player's performance, specifically the winning prize. In contrast, the players may prefer to increase the proportion of fixed payment, namely the prize for losing, though this may not apply to all of them, especially if the players are asymmetric; the strongest players may increase their uncertain payment, namely the prize for winning. One of the objectives of this study is to characterize an equilibrium in which the contest designer does not impose prize contracts on the contestants but instead offers some prize contracts, and the contestants have the option of receiving their prize contracts in such a way that no one wants to change his prize contract given the other players' prize contracts, and these prize contracts are optimal for the contest designer. In many real-life situations, prize contracts are determined by negotiation between the two parties, one awarding the prizes and the other receiving them. However, the manner in which the prize contracts are determined, whether by negotiation or any other way, is not relevant to this paper. What matters is that the outcome of the players' prize contracts is in equilibrium, so that no player wants to change his prize contract based on his opponents' prize contracts.

We study a multi-stage competition. First, the contest designer offers prize contracts for the contestants, each of which includes both a losing prize, which is equivalent to a fixed payment, and a winning prize, the difference between the winning and losing prizes being equivalent to an uncertain payment based on the outcome. Following the allocation of their prize contracts, the players participate in the standard Tullock contest, with the winner receiving his winning prize and the other contestants receiving their losing prizes.[2] We assume that the contest designer's goal is to maximize either the contestants' total effort or their highest effort, and he provides a variety of prize contracts to accomplish this goal. If contestants have asymmetric abilities, it is natural for the contest designer to provide different prize contracts based on their abilities. However, we consider symmetric contestants with the same abilities (effort costs) and demonstrate that the contest designer wants to allocate different prize contracts such that in equilibrium, no contestant wishes to change his prize contract based on the prize contracts of the other contestants.

First, we consider two-player contests in which there are two possible prize contracts, each with a winning and a losing prize, and the prize sum is the same for both prize contracts. Two prize contracts will be classified as more or less risky based on the difference between the winning and losing prizes; the larger the difference, the more risky the prize contract. We demonstrate that for any two prize contracts, there exists a unique subgame-perfect equilibrium in which one contestant receives the prize contract with the lower risk and the other contestant receives the prize contract with the higher risk. We also show that there exists a unique subgame perfect equilibrium when the number of prize contracts is infinite but the prize sum is fixed. In this equilibrium, one contestant receives the prize contract with the highest risk (i.e., the one with no losing prize), while the other receives the prize contract with the lowest risk (i.e., the one with the greatest possible losing prize). We also show that this outcome holds true even when the prize sums for each prize contract

[2] More about Tullock contest can be found, among others, in Tullock (1980), Skaperdas (1996), and Baye and Hoppe (2003).

are not identical.

Next, we consider a competition with two types of prize contracts and any finite number of contestants. We later show that the assumption of two types of prize contracts is not a limitation because if there are more than two types of prize contracts, the contestants prefer to receive the extreme ones, namely the one with the highest risk (the greatest difference between the winning and losing prizes) and the one with the lowest risk (the smallest difference between the winning and losing prizes). We show that in any equilibrium, no more than one contestant will receive the prize contract with the highest risk.

To maximize the contestants' total effort, the contest designer must ensure that all contestants receive the prize contract with the greatest risk. However, our findings indicate that such a scenario is not possible when contestants are given the option of receiving one of two prize contracts. However, we demonstrate that if there are at least five contestants, the contest designer can offer two different prize contracts, one of which has the highest risk (no losing prize), so that one player receives the contract with the highest risk and all other players receive the contract with the lower risk, and this prize contract allocation is in equilibrium, resulting in the optimal contestants' highest effort. Otherwise, if there are fewer than five contestants, the contest designer will be unable to obtain the optimal contestants' highest effort.

1.1 Related literature

The optimal prize allocation has been extensively researched in the contest literature, particularly in winner-take-all contests such as all-pay contests,[3] Tullock contests,[4] and, rank-order tournaments.[5] In this contest literature, it is assumed that the prizes are exogenous to the contestants and are either determined by the contest designer or given as winning values. In our model, the contest designer can present a menu of prize contracts from which contestants can receive their preferred prize contract by negotiating with the

[3] For the literature on the optimal prize allocation in all-pay contests, see, for example, Clark and Riis (1998), Moldovanu and Sela (2001, 2006, 2012), Liu and Lu (2019), and Fang et al. (2020).

[4] For the literature on the optimal prize allocation in Tullock contests, see, for example, Clark and Riis (1996), Sheremeta (2011), Fu and Lu (2012), Feng and Lu (2018), Cason et al. (2020), Sela (2020), Alshech and Sela (2021), Fu et al. (2021), and Lu et al. (2022).

[5] For the literature on the optimal prize allocation in rank-order tournaments, see, for example, Rosen (1986), and Akerlof and Holden (2012).

designer. Furthermore, the assumption that there are only one, or even multiple prizes for winning contests is extremely naive; contestants should be able to benefit from their participation even if they do not win, and in this case, the contestants may receive different benefits. As a result, we assume that this is not a winner-take-all competition and that players are rewarded for both winning and losing, even if the latter clearly outweighs the former. The prizes for winning and losing have already been discussed in contest literature (e.g., Hillman and Riley 1989 and Baye, Kovenock, and de Vries 2012), but unlike our model, these prizes are not endogenous for contestants.

Contests, by definition, involve risk because either winning or losing is possible. The impact of risk in competitions has been studied in a variety of ways. Skaperdas and Gan (1995), for example, find that in equilibrium, the more risk-averse player makes a smaller investment than the less risk-averse player under some restrictive assumptions on the contest success function and the utility function. Treich (2010) shows that risk aversion always reduces effort when compared to risk neutrality. Liu et al. (2018) show that when only the winners bear the cost of effort, risk aversion has the opposite effect than when both winners and losers bear the cost. In our model, the contestants are risk-neutral, but they still take different risks in the contest because the prize contract of each of them is considered more risky if the difference between winning and losing prizes is larger. It is interesting that players with similar risk tolerances receive different prize contracts with varying risks.

Similar to our model, the theory of procurement contracts suggest that the buyer screens the seller by providing a menu of contracts from which the seller receives one (see Laffont and Tirole 1993). In procurement literature, contracts can be simple fixed price contracts in which the buyer offers the seller a predetermined price for completing the project or cost-plus contracts in which the buyer reimburses the costs plus a stipulated fee (see Bajari and Tadelis 2001). In our model, the prize contracts are more like cost-plus contracts than fixed-price contracts. In procurement auctions (see Che and Gale 2003), project allocation is determined not only by prices, but also by the quality of bidders' proposals. The main idea in this literature is to reduce the multi-dimensional bidding problem to a single-dimensional bidding on score problem. Similarly, in our model, a player must receive two parameters, the winning and losing prizes, but because we assume that their sum is fixed, he can only receive one: the difference between these prizes.

We assume that the contest designer wants to maximize the highest level of effort. The literature typically assumes that the designer wishes to maximize total effort or, equivalently, average effort; however, several works assume that the designer wishes to maximize the highest effort. For example, Moldovanu and Sela (2006) show that in a simultaneous all-pay auction with incomplete information, the optimal expected highest effort can be obtained for any number of contestants. Segev and Sela (2014) show that the expected highest effort in the multi-stage sequential all-pay contest is not necessarily monotonic in the number of contestants (stages), and Cohen et al. (2009) show that for a designer who maximizes the highest effort, the optimal reward function decreases in effort, that is, a larger effort decreases the size of the reward gained by winning. Several studies assume that the contest designer wishes to maximize the winner's efforts (Serena 2017, Barbieri, and Serena 2021, 2024). If the contest success function is deterministic, as in the all-pay contest, the winner has the highest effort, but when the contest success function is stochastic, as in the Tullock contest, the winner's effort is not always the highest.

The remainder of the paper is organized as follows: Section 2 investigates the subgame perfect equilibrium in our two-player contest with prize contracts, whereas Section 3 investigates the subgame perfect equilibrium with more than two players. Section 4 analyzes the optimal contest with prize contracts that maximizes the contestants' highest effort. Section 5 discusses the robustness of the findings, and Section 6 concludes. All of the proofs are in the appendix.

2 The model

Consider a two-stage contest with n symmetric players. The designer has a menu of m types of prize contracts $M = \{(w_k, l_k)\}_{k=1}^{m}$where each prize contract $k, k = 1, 2, ..., m$ contains two prizes, a winning prize of w_k and a losing prize of l_k such that w_k is greater than l_k, $w_k > l_k$, and their sum remains constant, $w_k + l_k = v$, $k = 1, 2, ..., m$. If $w_i > w_j$ and $l_i < l_j$, the prize contract (w_i, l_i) is considered riskier than $(w_j.l_j)$. In the first stage, the designer and the players negotiate over the prize contracts. The designer offers each of the players one of the m possible prize contracts, and the players who observe all of the other players' prize contracts have the option of accepting or rejecting any prize contract suggested by the designer. The first stage of negotiation concludes when no player wishes to change his prize contract based on the prize contracts of the other players. If, however, the negotiation does not end, the players do not proceed to the second stage, and we conclude that our two-stage contest does not have a subgame perfect equilibrium. In the second stage, after each player has observed the allocation of prize contracts and accepted his own prize contract, the players compete in the Tullock contest. In this contest, if player i exerts an effort of $x_i, i = 1, 2, ..., n$, according to his prize contract, he obtains his winning prize with a probability of $\frac{x_i}{\sum_{j=1}^{n} x_j}$, and obtains his losing prize with a probability of $1 - \frac{x_i}{\sum_{j=1}^{n} x_j}$. Player i's effort cost function is $c_i(x) = x$, $i = 1, 2, ..., n$. This two-stage competition will be referred to as a contest with prize contracts.

3 Two-player contests with prize contracts

Consider a two-player contest with prize contracts. In order to analyze the subgame perfect equilibrium, we begin with the second stage of the contest and go backwards to the previous stage. Suppose that player i has the prize contract $(w_i, l_i), i = 1, 2$. Then, player i's maximization problems is

$$\max_{x_i} w_i \frac{x_i}{x_i + x_j} + l_i \frac{x_j}{x_i + x_j} - x_i.$$

Let $d_i = w_i - l_i$. Then, according to the analysis of the one-stage Tullock contest (see, Tullock 1980), the players' equilibrium strategies in the second stage are

$$x_i = \frac{d_i^2 d_j}{(d_i + d_j)^2}, x_j = \frac{d_i d_j^2}{(d_i + d_j)^2}.$$

The equilibrium strategies in the second stage are unique (see, Szidarovszky and Okuguchi 1997, and Yamazaki 2008). The players' expected payoffs in the second stage are

$$\begin{aligned} u_i &= \frac{(d_i)^3}{(d_i + d_j)^2} + l_i \\ u_j &= \frac{(d_j)^3}{(d_i + d_j)^2} + l_j. \end{aligned} \tag{1}$$

Note that

$$u_i - u_j = \frac{(d_i - d_j)}{(d_i + d_j)^2} \left(d_i^2 + d_i d_j + d_j^2 \right) + (l_i - l_j). \tag{2}$$

Assume that the set of prize contracts includes only two possible prize contracts, where (w_1, l_1) is the prize contract with the higher risk and (w_2, l_2) is the prize contract with the lower risk, that is, $w_1 > w_2$

and $l_1 < l_2$. The following result indicates that if one of the players has the prize contract with the lower risk (w_2, l_2), the other one wants the more risky prize contract (w_1, l_1).

Proposition 1 *Consider a two-player contest with two possible prize contracts (w_1, l_1) and (w_2, l_2) where $w_1 > w_2$. Then. there is a unique subgame perfect equilibrium in which, in the first stage, one player receives the prize contract with the lower risk (w_2, l_2), and the other player receives the prize contract with the higher risk (w_1, l_1).*

In contests, it is well known that when players are symmetric, they exert the highest total effort, and as a result, their expected payoffs are relatively small. This is one of the reasons why players prefer asymmetric prize contracts over symmetric ones. The example below illustrates an equilibrium prize contract allocation when the set of prize contracts is finite but greater than two.

Example 1 *Consider a two-player contest with the following set of prize contracts:*

$$M = \{(w, l) : l = 0, 1, 2, 3, 4\ ,\ w = 10 - l\}.$$

The payoff matrix for each pair of prize contracts is shown below, with the first row representing player 2's possible prize contracts and the first column representing player 1's possible prize contracts.

(w, l)	$(6, 4)$	$(7, 3)$	$(8, 2)$	$(9, 1)$	$(10, 0)$
$(6, 4)$	$4.5, 4.5$	$4.22, 4.77$	$4.12, 5.37$	$4.08, 6.12$	$4.05, 6.94$
$(7, 3)$	$4.77, 4.22$	$4, 4$	$3.64, 4.16$	$3.44, 4.55$	$3.32, 5.10$
$(8, 2)$	$5.37, 4.12$	$4.16, 3.64$	$3.5, 3.5$	$3.10, 3.61$	$2.84, 3.90$
$(9, 1)$	$6.12, 4.08$	$4.55, 3.44$	$3.61, 3.10$	$3, 3$	$2.58, 3.08$
$(10, 0)$	$6.94, 4.05$	$5.10, 3.32$	$3.90, 2.84$	$3.08, 2.58$	$2.5, 2.5$

We can see that there are two equilibrium points, in which one player receives the prize contract with the lowest risk $(w, l) = (6, 4)$ and his opponent receives the prize contract with the highest risk $(w, l) = (10, 0)$. It is important to note that symmetric prize contract allocation is the worst strategy for both players, and any deviation from a symmetric point is profitable for both players.

Assume now that the set of prize contracts is infinite and contains any prize contract from the set $\{(w, l) :$

$w + l = v$ and $0 \leq l < \frac{v}{2}\}$. The following result generalizes the previous example's findings for any number of prize contracts, including a set of infinitely many prize contracts.

Proposition 2 *Consider a two-player contest with an infinite set of prize contracts $M = \{(w, l) : w + l = v$ and $0 \leq l \leq s < \frac{v}{2}\}$. Then, for every $0 < s < \frac{v_i}{2}$, there exists a unique subgame perfect equilibrium in which in the first stage player i receives the prize contract with the highest risk $(w_i, l_i) = (v, 0)$ while his opponent receives the prize contract with the lowest risk $(w_j, l_j) = (v - s, s)$. The expected payoff of the player who receives the prize contract with the highest risk is always larger than the expected payoff of the player who receives the prize contract with the lowest risk.*

So far, we have assumed that each player has a prize contract with the same total prize. Theorem 2 can be extended for players with asymmetric total prizes as shown below.

Proposition 3 *Consider a two-player contest with infinite sets of prize contracts for both players $M_i = \{(w_k, l_k) : w_k + l_k = v_i$ and $0 \leq l_k \leq s < \frac{v}{2}\}, i = 1, 2$ where $v_i \geq v_j$. Then, for every $0 < s < \frac{v_i}{2}$ there exists a subgame perfect equilibrium in which in the first stage player i receives the prize contract with the highest risk $(w_i, l_i) = (v_i, 0)$ while player j receives the prize contract with the lowest risk $(w_j, l_j) = (v_j - s, s)$. This equilibrium is unique if $\frac{(v_i)^3}{(v_i+v_j)^2} \geq \frac{(v_i-2s)^3}{(v_i-2s+v)^2} + s$; otherwise, there also exists another subgame perfect equilibrium in which in the first stage player i receives the prize contract with the lowest risk $(w_i, l_i) = (v_i - s, s)$ while player j receives the prize contract with the highest risk $(w_j, l_j) = (v_j, 0)$.*

According to Proposition 3, there exists an equilibrium in which the weaker player (player j, the player with the lower total prize) receives the prize contract with the highest risk and the stronger player (player i, the player with the higher total prize) receives the prize contract with the lowest risk, as long as the asymmetry is relatively weak, that is, the difference of $v_i - v_j$ is small enough. In particular, the weak player's probability of winning the contest might be higher than that of the strong one. The reason is that when $v_i - v_j$ is small enough, the players are quite symmetric. Then they try to avoid a contest between symmetric players because their expected payoffs are relatively low.

3.1 Convex cost functions

In the following example, we show that the players' risk attitudes have no effect on the prize contracts they ultimately receive. We show that two risk-averse players with quadratic cost functions rather than linear ones still have two distinct prize contracts; in particular, one of the two players continues to receive the prize contract with the highest risk.

Example 2 *Consider a two-player contest with two possible prize contracts* (w_1, l_1) *and* (w_2, l_2) *where* $w_1 > w_2$*. Each player has a convex cost of effort such that if players* i*,* $i = 1, 2$*, exerts an effort of* x_i *his cost of effort is* $c(x_i) = (x_i)^2$*. Then, the maximization problem of player* 1 *who receives the prize contract* (w_1, l_1) *is*

$$\max_{x_1} w_i \frac{x_1}{x_1 + x_2} + l_1 \frac{x_1}{x_1 + x_2} - (x_1)^2, \tag{3}$$

and the maximization problem for player 2 is similar. Let $d_i = w_i - l_i$*,* $i = 1, 2$*. The solution of (3) yields the equilibrium efforts,*

$$x_i = \sqrt{2} \frac{d_i}{2d_i + 2d_j + 4d_i \sqrt{\frac{1}{d_i} d_j}} \sqrt{2d_j + d_i \sqrt{\frac{1}{d_i} d_j} + d_j \sqrt{\frac{1}{d_i} d_j}}, \; i, j \in \{1, 2\}. \tag{4}$$

When $d_1 = d_2 = d$*, the symmetric equilibrium effort is*

$$x = \frac{1}{4} \sqrt{2} \sqrt{d}.$$

By (3), and (4), for $i, j \in \{1, 2\}$*, player* i*'s expected payoff is*

$$u_i = \frac{1}{2} \frac{d_i}{\left(\sqrt{\frac{1}{d_i} d_j} + 1\right) \left(d_i + d_j + 2d_i \sqrt{\frac{1}{d_i} d_j}\right)^2} \left(2d_i^2 + d_j^2 + 9d_i d_j + 7d_i^2 \sqrt{\frac{1}{d_i} d_j} + 5d_i d_j \sqrt{\frac{1}{d_i} d_j}\right) + l_i. \tag{5}$$

When $d_1 = d_2 = d$*, the symmetric expected payoff is*

$$u_s = \frac{3}{8} d \; .$$

By (5), If the two players receive the prize contract (w_2, l_2)*, a player will want the prize contract* (w_1, l_1)

because that

$$\begin{aligned}
\Delta &= u_s - u_1 = \frac{3}{8}d_2 + l_2 - \frac{1}{2}\frac{d_1}{\left(\sqrt{\frac{1}{d_1}d_2}+1\right)\left(d_1+d_2+2d_1\sqrt{\frac{1}{d_1}d_2}\right)^2} \times \\
&\quad \left(2d_1^2 + d_2^2 + 9d_1d_2 + 7d_1^2\sqrt{\frac{1}{d_1}d_2} + 5d_1d_2\sqrt{\frac{1}{d_1}d_2}\right) - l_1 \\
&\leq \frac{3}{8}d_2 - \frac{1}{2}\frac{d_1}{\left(\sqrt{\frac{1}{d_1}d_2}+1\right)\left(d_1+d_2+2d_1\sqrt{\frac{1}{d_1}d_2}\right)^2} \times \\
&\quad \left(2d_1^2 + d_2^2 + 9d_1d_2 + 7d_1^2\sqrt{\frac{1}{d_1}d_2} + 5d_1d_2\sqrt{\frac{1}{d_1}d_2}\right) + \frac{1}{2}(d_1 - d_2) \\
&= -\frac{1}{8}\frac{(d_1-d_2)^2}{\left(\sqrt{\frac{1}{d_1}d_2}+1\right)\left(d_1+d_2+2d_1\sqrt{\frac{1}{d_1}d_2}\right)^2}\left(4d_1 + 5d_2 + 8d_1\sqrt{\frac{1}{d_1}d_2} + d_2\sqrt{\frac{1}{d_1}d_2}\right) \leq 0.
\end{aligned}$$

Thus, one of the players wants to receive the prize contract with the higher risk (w_1, l_1). By the same argument, it can be shown that if the two players receive the prize contract with the higher risk (w_1, l_1), one of the players will want the prize contract with the lower risk (w_2, l_2). We can conclude that when the players have a quadratic effort cost function, then there is a unique subgame perfect equilibrium in which one player receives the prize contract with the higher risk (w_1, l_1) and the other player receives the prize contract with the lower risk (w_2, l_2).

It is worth noting that the maximization problem (3) is equivalent to

$$\max_{y_i} w_i \frac{(y_i)^{0.5}}{(y_i)^{0.5} + (y_j)^{0.5}} + l_i \frac{(y_j)^{0.5}}{(y_i)^{0.5} + (y_j)^{0.5}} - y_i,$$

where $y_i = (x_i)^2, i = 1, 2$. Thus, the result in Example 2 is valid for competitions using a generalized Tullock contest success function of the form $\frac{(x_i)^2}{\sum_{j=1}^{2}(x_j)^2}$ and linear cost functions.

3.2 All-pay contests with prize contracts

Assume that both players receive prize contracts and then compete in an all-pay contest. Then, if players i and j receive the prize contracts (w_i, l_i) and (w_j, l_j) and they exert efforts of x_i, x_j respectively, the expected payoff of player i is given by

$$u_i(x_i, x_j) = \begin{cases} l_i - x_i & \text{if } x_i < x_j \\ \frac{1}{2}(w_i + l_i) - x_i & \text{if } x_i = x_j \\ w_i - x_i & \text{if } x_i > x_j. \end{cases} .$$

According to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), if $d_i > d_j$, there is a unique mixed-strategy equilibrium in which player i and player j randomize on the interval $[0, w_j - l_j]$ according to their effort distribution functions $F_i(x), F_j(x)$ which are given by

$$\begin{aligned} w_i F_j(x) + l_i(1 - F_j(x)) - x &= w_i - w_j + l_j \\ w_j F_i(x) + l_j(1 - F_i(x)) - x &= l_j. \end{aligned}$$

Thus, the expected payoffs of the players are

$$\begin{aligned} u_i &= w_i - w_j + l_j \\ u_j &= l_j. \end{aligned}$$

Assume that the set of prize contracts is infinite and contains any prize contract from the set $M = \{\ (w,l) : w+l = v \text{ and } l \leq s < \frac{v}{2}\}$. Then, if $w_i > w_j$, player i's expected payoff is independent of his losing prize l_i. As a result, this player will receive the prize contract with the highest risk $(w_i, l_i) = (v, 0)$. In contrast, player j aims to maximize his losing prize l_j by receiving the prize contract with the lowest risk $(w_j, l_j) = (v - s, s)$. As a result, we find that

Proposition 4 *Consider a two-player all-pay contest with the set of prize contracts $M = \{\ (w,l) : w+l = v$ and $l \leq s < \frac{v}{2}\}$. Then there is a unique subgame perfect equilibrium in which one player receives the prize contract with the highest risk while the other player receives the prize contract with the lowest risk.*

4 Contests with more than two players

Consider a contest with prize contracts and $r \geq 3$ symmetric players. We assume that there are only two types of prize contracts; however, we will argue later that this assumption is unnecessary (see the Discussion section), because even if players had more than two prize contract options, they would receive only two, those with the highest and lowest risk. Assume that n players receive the prize contract with the higher risk (w_1, l_1), and $m = r - n$ players receive the prize contract with the lower risk (w_2, l_2), that is, $w_1 > w_2$ and $w_1 + l_1 = w_2 + l_2$. In order to analyze the subgame perfect equilibrium, we begin with the second stage of the contest and go backwards to the previous stage. In the second stage, the maximization problem of player i, $i = 1, ..., n$ who receives the prize contract (w_1, l_1) is

$$\max_{x_i} w_1 \frac{x_i}{\sum_{j=1}^{n} x_j + \sum_{j=1}^{m} y_j} + l_1(1 - \frac{x_i}{\sum_{j=1}^{n} x_j + \sum_{j=1}^{m} y_j}) - x_i, \tag{6}$$

where $x_i, i = 1, ..., n$ is the effort of player i who receives the prize contract (w_1, l_1), and y_j, $j = 1, ..., m$ is the effort of player j who receives the prize contract (w_2, l_2). Similarly, the maximization problem of player j, $j = 1, ..., m$ who receives the prize contract (w_2, l_2) is

$$\max_{y_j} w_2 \frac{y_j}{\sum_{j=1}^{n} x_j + \sum_{j=1}^{m} y_j} + l_2(1 - \frac{y_j}{\sum_{j=1}^{n} x_j + \sum_{j=1}^{m} y_j}) - y_j. \tag{7}$$

By symmetry of the players who receive the same prize contract, we have $x = x_i, i = 1, ..., n$ and $y = y_j, j = 1, ..., m$. Then, the first-order conditions (FOC) are

$$\begin{aligned} d_1 \frac{(n-1)x + my}{(nx + my)^2} &= 1 \\ d_2 \frac{nx + (m-1)y}{(nx + my)^2} &= 1, \end{aligned} \tag{8}$$

where $d_i = w_i - l_i, i = 1, 2$. By (8), we have

$$y = x \frac{d_2 n - d_1(n-1)}{d_1 m - d_2(m-1)}.$$

This yields that the equilibrium strategies in the second stage are

$$x = \frac{d_1 d_2}{(nd_2 + md_1)^2} \left(md_1 - (m-1)d_2\right)(m + n - 1), \tag{9}$$

and

$$y = \frac{d_1 d_2}{(nd_2 + md_1)^2} \left(nd_2 - (n-1)d_1\right)(m + n - 1). \tag{10}$$

We can see that the players who receive the prize contract with the higher risk (w_1, l_1) are always active, while those who receive the less risky prize contract (w_2, l_2) are active only if

$$d_2 \geq \frac{n-1}{n} d_1. \tag{11}$$

The following result limits the number of players who can receive the contract with the higher risk when the other players who receive the contract with the lower risk are inactive.

Proposition 5 *Consider a two-player contest with two prize contracts (w_1, l_1) and (w_2, l_2) where $w_1 > w_2$. For every $n > 1$, if the players who receive the prize contract with the lower risk are not active, that is, $d_2 < \frac{n-1}{n}d_1$, there is no equilibrium in which n or more players receive the prize contract with the higher risk (w_1, l_1). In particular, if $d_2 < \frac{1}{2}d_1$ there is no equilibrium where multiple players receive the prize contract with the higher risk.*

In the following, we only consider scenarios in which all players are active, that is, $d_2 \geq \frac{n-1}{n}d_1$. By (6), (7), (9), and (10), the expected payoff of each player who receives the prize contract with the higher risk (w_1, l_1) is

$$u_1 = \frac{d_1}{(nd_2 + md_1)^2}(md_1 - (m-1)d_2)^2 + l_1, \tag{12}$$

and the expected payoff of each player who receives the prize contract with the lower risk (w_2, l_2) is

$$u_2 = \frac{d_2}{(nd_2 + md_1)^2}(nd_2 - (n-1)d_1)^2 + l_2. \tag{13}$$

Based on the second-stage equilibrium analysis, the following result demonstrates that when all players are active, there is never an equilibrium in a contest in which more than one player receives the prize contract with the greater risk in the first stage.

Proposition 6 *Consider a contest with $r > 2$ players and two types of prize contracts $(w_i, l_i), i = 1, 2$ where $w_1 > w_2$. Then, if d_2 is sufficiently close to d_1, no player will receive the prize contract with the higher risk (w_1, l_1). However, for any other value of d_2 where all the players are active, no more than one player will receive the prize contract with the higher risk (w_1, l_1).*

Proposition 6 states that in situations where d_2 is small enough, but still all the players are active, it is advantageous for one player to take the prize contract with the higher risk (w_1, l_1), while in situations where d_2 is large enough, namely, it is close enough to d_1, no player wants to take a chance by choosing the prize contract with the higher risk (w_1, l_1). The explanation for these results is that when the difference between d_1 and d_2 is small enough, competition intensifies because all players are nearly symmetric and compete equally in the contest. In this case, each player prefers the prize contract with the lower risk. When d_2 is small and all players receive the prize contract with the lower risk, a player has an incentive to receive the prize contract with the higher risk. If he is the only one who receives this prize contract, his chances of winning the contest are fairly high. According to Proposition 6, if multiple players receive the prize contract with the higher risk, the chance of winning decreases significantly, making it unfavorable to receive the higher risk contract.

5 The optimal allocation of prize contracts

The contest designer may have various goals. The most common design goal is to maximize the players' total effort. In our model, when n players receive the prize contract (w_1, l_1) and m players receive the prize contract (w_2, l_2), then, by (9) and (10), the player's total effort is given by

$$\begin{aligned} TE &= nx + my = n\frac{d_1 d_2}{(nd_2 + md_1)^2}\left(md_1 - (m-1)d_2\right)(m+n-1) \\ &\quad + m\frac{d_1 d_2}{(nd_2 + md_1)^2}\left(nd_2 - (n-1)d_1\right)(m+n-1) \\ &= \frac{d_1 d_2}{md_1 + nd_2}(m+n-1)\,. \end{aligned}$$

Because this total effort increases in both d_1 and d_2, we can see that for a designer who wants to maximize the players' total effort, it is optimal that all the players will receive the most risky prize contract $(v, 0)$ in which $d = d_1 = d_2 = v$. However, the optimal total effort is impossible in equilibrium because, according to Proposition 6, if there are at least two types of prize contracts, no more than one player will receive the one with the highest risk. Thus, there is no equilibrium in which the optimal total effort is obtained.

However, the contest designer may want to maximize the highest effort rather than the total effort. In that case, it is obvious that it is not optimal for all players to receive the prize contract with the highest risk, because this results in a symmetric contest that increases total effort while decreasing average effort, which is the same as the highest effort. On the other hand, it is clearly not optimal for all players to receive the contract with the lowest risk; this will result in the least total effort as well as the least highest effort. Thus, in that case, a designer seeks an asymmetric equilibrium in which players can receive between both types of prize contracts. The following result demonstrates that when there are two types of prize contracts, there are only two options for maximizing the highest effort: either one or two players should receive the prize contract with the higher risk.

Proposition 7 *Consider a contest with $r > 2$ players and two types of prize contracts $(w_i, l_i), i = 1, 2$ where $w_1 > w_2$. If the difference between d_1 and d_2 is small enough, it is optimal for a designer to maximize the highest effort by having only one player receives the prize contract with the higher risk, while all other players receive the prize contract with the lower risk. When d_2 is small enough, it may be optimal for two players to receive the prize contract with the highest risk.*

The intuition behind Proposition 7 is that when the value of d_2 is sufficiently small, the players who receive the prize contract with the lower risk either exert very low or no effort. In such a case, to make any competition, two players should receive the contract with the higher risk and compete against each other. On the other hand, when the value of d_2 is large or the difference $d_1 - d_2$ is small, there is intense competition. Then, to maximize the highest effort, only one player should receive the prize contract with the highest risk.

There is a conflict between Propositions 7 and 6. According to Proposition 7, If the difference of $d_1 - d_2$ is sufficiently small, then it is optimal for a designer who wants to maximize the highest effort that only one player receives the prize contract with the higher risk, but according to Proposition 6, if $d_1 - d_2$ is sufficiently small no player will receive the prize contract with the higher risk (w_1, l_1). Likewise, according to Proposition 7, if d_2 is sufficiently small, then it might be optimal that two players receive the prize contract with the higher risk, while according to Proposition 6, no more than one player will receive the prize contract with the higher risk (w_1, l_1).

According to the last argument, it is unclear whether the contest designer can achieve the optimal highest value in any equilibrium. However, if the designer can choose the values of the prize contracts, the following result shows which prize contracts maximize the highest effort.

Proposition 8 *Consider a contest with $r > 2$ players and two types of prize contracts (w_i, l_i), where $w_i + l_i = v$, $i = 1, 2$. Then, the highest effort is obtained when one player receives the prize contract with the higher risk that satisfies*

$$d_1 = w_1 - l_1 = v.$$

and all the other players receive the prize contract with the lower risk that satisfies

$$d_2 = w_2 - l_2 = d_1 \frac{m}{2m - 1},$$

In that case, the highest effort is equal to $\frac{1}{4}d_1$.

The relevant question is whether the situation in Proposition 8, which yields the optimal value of the highest effort, is achieved in equilibrium. The following result demonstrates that a minimum of five players are required for this purpose; otherwise, no equilibrium can implement the optimal highest effort.

Proposition 9 *Consider a contest with* r *players and two types of prize contracts* $(w_i, l_i), i = 1, 2$ *where* $(w_1, l_1) = (v, 0)$ *and* $(w_2, l_2) = (v\frac{3m-1}{4m-2}, v\frac{m-1}{4m-2})$. *If* $r \geq 5$, *an equilibrium exists where one player receives the contract with the highest risk* (w_1, l_1) *and all other players receive the contract with the lower risk* (w_2, l_2). *In this equilibrium, the players' highest effort is optimal and equals* $\frac{d_1}{4}$.

6 Discussion

So far, we have assumed that players can receive two types of prize contracts. We will explain why having only two prize contracts is not a constraint. The reason for this is that even if there are more than two types of prize contracts, players will only want two: the one with the highest risk and the one with the lowest risk.

Assume, without loss of generality, that there are n different prize contracts such that the difference between their winning and losing prizes satisfy $d_1 > d_2 > .... > d_n$. Each of n players receives a different type of a prize contract such that player i receives the prize contract for which the difference of the prizes is d_i, $i = 1, 2, ..., n$. Then (see, Corchon and Serena, 2018), player i's equilibrium effort is

$$x_i = p_i \sum_{j=1}^{n} x_j,$$

where p_i is agent i's winning probability and is given by

$$p_i = \frac{d_i}{\sum_{j=1}^{n} d_j},$$

and agent i's expected payoff is

$$u_i = d_i(p_i)^2 + l_i = \frac{(d_i)^2}{\sum_{j=1}^{n} d_j} + l_i,$$

or, alternatively,

$$u_i = \frac{(w_i - l_i)^2}{\sum_{\substack{j=1 \\ j \neq i}}^{n} d_j + (w_i - l_i)} + l_i.$$

Then, the marginal effect of player i's winning prize on his expected payoff is

$$\begin{aligned}\frac{du}{dw_i} &= \frac{d}{dw_i}\left(\frac{(w_i - l_i)^2}{\sum_{\substack{j=1\\ j\neq i}}^{n} d_j + (w_i - l_i)} + l_i\right) \\ &= \frac{d}{dw_i}\left(\frac{(w_i - l_i)^3}{(k_{-i} + (w_i - l_i))^2} + l_i\right) = \frac{(d_i)^2}{(k_{-i} + d_i)^3}\left(3k_{-i} + d_i\right),\end{aligned}$$

where $k_{-i} = \sum_{\substack{j=1\\ j\neq i}}^{n} d_j$. Similarly, the marginal effect of player i's losing prize on his expected payoff is

$$\begin{aligned}\frac{du}{dl_i} &= \frac{d}{dl_i}\left(\frac{(w_i - l_i)^3}{\sum_{\substack{j=1\\ j\neq i}}^{n} d_j + (w_i - l_i)} + l_i\right) \\ &= \frac{d}{dl_i}\left(\frac{(w_i - l_i)^3}{(k_{-i} + (w_i - l_i))^2} + l_i\right) = \frac{k_{-i}^2}{(k_{-i} + d_i)^3}\left(k_{-i} + 3d_i\right).\end{aligned}$$

The difference of these marginal effects of the prizes for winning and losing on player i's expected payoff is

$$\begin{aligned}\Delta_i &= \frac{du}{dw_i} - \frac{du}{dl_i} = \frac{(d_i)^2}{(k_{-i} + d_i)^3}\left(3k_{-i} + d_i\right) - \frac{k_{-i}^2}{(k_{-i} + d_i)^3}\left(k_{-i} + 3d_i\right) \\ &= \frac{1}{(k_{-i} + d_i)^3}\left(d_i(d_i^2 - 3k_{-i}) - k_{-i}(k_{-i}^2 - 3d_i)\right).\end{aligned}$$

Then, we have for any value of d_i,

$$\frac{d\Delta_i}{dd_i} = \frac{3}{(d_i + k_{-i})^4}\left(k_{-i}^2 + d_i^2\right)k_{-i} > 0.$$

As a result, if player i has the option of selecting the prize contract, depending on the value of Δ_i, he would prefer to receive the prize contract with the highest risk and the greatest difference between the prizes for winning and losing, or the one with the lowest risk and the smallest difference between the prizes for winning and losing.

7 Conclusion

We consider contests where the contest designer and contestants negotiate on the prize contracts, which include winning and losing prizes. We assume that the contest designer has a menu of multiple prize contracts from which each contestant can receive his prize contract. The manner in which the prize contracts are distributed among the contestants, whether by a negotiation or through any other process, is unimportant; however, the outcome of the final distribution of prize contracts is important because it must be in equilibrium so that no contestant wishes to change his prize contract given the other contestants' prize contracts.

We show that by offering two types of prize contracts, one of which has the highest risk, namely, no losing prize but only a winning prize, the contest designer can maximize the contestants' highest effort. In that case, one of the contestants receives the prize contract with the highest risk, while the remaining contestants receive the prize contract with a lower risk, and no contestant wishes to change his prize contract; thus, the contestants' prize contract distribution is in equilibrium.

Our findings add another dimension by demonstrating that in any competition with two or more players—clearly, even with only two—some players will receive the prize contract with the lowest risk. As a result, if contestants are allowed to negotiate their own prize contracts, there will never be an optimal contest in which all participants receive only a winning prize and no losing prize, maximizing their total effort. In other words, we cannot expect all contestants to give their all in a competition.

We demonstrate that in equilibrium, symmetric contestants with linear effort costs receive asymmetric equilibrium prize contracts. Furthermore, we demonstrate that there is no symmetric equilibrium in which all contestants receive the same prize contract. Our model's generalization to contestants with asymmetric costs of effort is quite complex, but we believe that, similar to our model for symmetric contestants, the contest designer should offer two or more types of prize contracts to maximize the contestants' highest effort or any other goal.

8 Appendix

8.1 Proof of Proposition 1

Suppose that the two players receive the prize contract with the higher risk (w_1, l_1). Then, by (2), a player who receives the prize contract with the higher risk (w_1, l_1) will prefer to receive the prize contract with the lower risk (w_2, l_2) if

$$\Delta = (\frac{(d_1)}{4} + l_1) - (\frac{(d_2)^3}{(d_1 + d_2)^2} + l_2) < 0. \tag{14}$$

Because that

$$d_2 - d_1 = (w_2 - w_1) + (l_1 - l_2),$$

and

$$w_1 + l_1 = w_2 + l_2,$$

we obtain that

$$(d_2 - d_1) = 2(l_1 - l_2).$$

Then, by (14), we have

$$\begin{aligned} \Delta &= \frac{(d_1)}{4} - \frac{(d_2)^3}{(d_1 + d_2)^2} + \frac{1}{2}(d_2 - d_1) \\ &= -\frac{1}{4}(d_1 - d_2)^2 \frac{d_1 + 2d_2}{(d_1 + d_2)^2} < 0. \end{aligned}$$

Thus, at least one player wants to receive the prize contract with the lower risk (w_2, l_2).

Now, suppose that the two players receive the prize contract with the lower risk (w_2, l_2). Then, by (2), a player who receives the prize contract with the lower risk (w_2, l_2) will prefer to receive the prize contract with the higher risk (w_1, l_1) if

$$\Delta = (\frac{(d_2)}{4} + l_2) - (\frac{(d_1)^3}{(d_1 + d_2)^2} + l_1) < 0. \tag{15}$$

Because that

$$(d_1 - d_2) = 2(l_2 - l_1),$$

by (15), we have

$$\begin{aligned} \Delta &= \frac{(d_2)}{4} - \frac{(d_1)^3}{(d_1 + d_2)^2} + \frac{1}{2}(d_1 - d_2) \\ &= -\frac{1}{4}(d_1 - d_2)^2 \frac{2d_1 + d_2}{(d_1 + d_2)^2} < 0. \end{aligned}$$

Thus, at least one player wants to receive the prize contract with the higher risk (w_1, l_1).

The argument above demonstrates that there is a unique subgame perfect equilibrium in which one player receives the prize contract with the lower risk (w_2, l_2), and the other player receives the prize contract with the higher risk (w_1, l_1).

8.2 Proof of Proposition 2

Consider a two-player contest with an infinite set of prize contracts $M = \{(w, l) : w + l = v \text{ and } 0 \leq l \leq s < \frac{v}{2}\}$. Assume that players i and j receive the prize contracts (w_i, l_i) and (w_j, l_j), respectively. Then, we will show that

a) if $w_j < w_i$, then player i's expected payoff increases in w_i, and later that, b) if $w_i > w_j$, then player j's expected payoff decreases in w_j.

Given the prize contract of player j, by (1), if player i receives a prize contract with a winning prize of $v - x$ and a loosing prize of x, his expected payoff is

$$u_i(x) = \frac{(v-2x)^3}{(v-2x+d)^2} + x,$$

where $d = d_j = w_j - l_j$. Then, we have

$$\begin{aligned} \frac{d}{dx}u_i(x) &= \frac{d}{dx}(\frac{(v-2x)^3}{(v-2x+d)^2} + x) \qquad (16) \\ &= \frac{1}{(d+v-2x)^3}\begin{pmatrix} 3d^2v - 6d^2x - 3dv^2 + 12dvx - 12dx^2 \\ +d^3 - v^3 + 6v^2x - 12vx^2 + 8x^3 \end{pmatrix}. \end{aligned}$$

If $d_i = v - 2x \geq d_j$, we obtain that

$$(d - v + 2x)^3 = d^3 - 3d^2v + 6d^2x + 3dv^2 - 12dvx + 12dx^2 - v^3 + 6v^2x - 12vx^2 + 8x^3 \leq 0,$$

or alternatively,

$$A = d^3 - v^3 + 6v^2x - 12vx^2 + 8x^3 \leq 3d^2v - 6d^2x - 3dv^2 + 12dvx - 12dx^2 = B.$$

Note that both sides of the last inequality are non-positive because that

$$B = 3d^2v - 6d^2x - 3dv^2 + 12dvx - 12dx^2 = 3d(v-2x)(d-(v-2x)) \leq 0.$$

Therefore, by (16), because that $A \leq B \leq 0$, we have,

$$\frac{d}{dx}u_i(x) = \frac{1}{(d+v-2x)^3}(A+B) \leq 0. \qquad (17)$$

Hence, we obtain that if $w_i > w_j$, which implies that $d_i > d_j$, the prize contract with the highest risk $(v, 0)$ yields the highest expected payoff for player i than any other prize contract (w_i, l_i) where $w_i + l_i \leq v$.

Now, if $d = d_j \geq v - 2x = d_i$, we have

$$B = 3d(v-2x)(d-(v-2x)) \geq 0,$$

and because that $d \geq v - 2x$ we have

$$A = d^3 - v^3 + 6v^2x - 12vx^2 + 8x^3 \geq (v-2x)^3 - v^3 + 6v^2x - 12vx^2 + 8x^3 = 0.$$

The last two inequalities imply that

$$\frac{d}{dx}u_i(x) = \frac{1}{(d+v-2x)^3}(A+B) \geq 0. \tag{18}$$

Therefore, we obtain that if $w_j < w_i$ which implies that $d_j < d_i$, the prize contract with the lowest risk $(v-s, s)$ yields the highest expected payoff for player j than any other prize contract (w_j, l_j) where $w_j + l_j \leq v$.

Now, we show that in the two-player contest with prize contracts, if a player has higher values of winning and losing than his opponent he also has a higher expected payoff than his opponent. If, on the other hand players have the same sum of values then the player with the higher value of winning has a higher expected payoff.

Suppose that $w_i > w_j$ and $l_i > l_j$, then by (2),

$$u_j - u_i = (w_i - w_j)(-1 + \frac{d_i d_j}{(d_i + d_j)^2}) - (l_i - l_j)\frac{d_i d_j}{(d_i + d_j)^2}.$$

Since $\frac{d_i d_j}{(d_i + d_j)} < 1$, we obtain that $u_j - u_i < 0$.

Suppose now that the both players have the same sum of values, i.e., $w_i + l_i = w_j + l_j$, or, alternatively, $w_i - w_j = l_j - l_i$, Then, we obtain that the difference of the players' expected payoffs is

$$u_j - u_i = -(w_i - w_j) < 0.$$

8.3 Proof of Proposition 3

Consider a two-player contest with infinite sets of prize contracts for both players $M_i = \{(w_k, l_k) : w_k + l_k = v_i$ and $0 \leq l_k \leq s < \frac{v}{2}\}, i = 1, 2$ where $v_i \geq v_j$. By (1), if player i receives a winning prize of $v_i - x$ and a losing prize of x, his expected payoff is

$$u_i(x) = \frac{(v_i - 2x)^3}{(v_i - 2x + d)^2} + x,$$

where $d = d_j = w_j - l_j$. Then, similarly to (17), we have

$$\frac{d}{dx}u_i(x) = \frac{1}{(d+v_i-2x)^3}(A+B)$$

where

$$A = d^3 - v_i^3 + 6v^2x - 12v_ix^2 + 8x^3 \leq 3d^2v_i - 6d^2x - 3dv_i^2 + 12dv_ix - 12dx^2 = B.$$

If $d \leq v_i - 2x$, we obtain that $\frac{d}{dx}u_i(x) \leq 0$, and similarly, $\frac{d}{dx}u_j(x) \geq 0$. Therefore, the prize contracts $(w_i, l_i) = (v_i, 0)$ and $(w_j, l_j) = (v_j - s, s)$ are the players' equilibrium selections in the first stage of the two-player contest.

Now, suppose that player j receives the prize contract with the highest risk $(w_j, l_j) = (v_j, 0)$. Then, if player i receives a losing prize of x such that $d_j \geq d_i = v_i - 2x$, we obtain that $\frac{d}{dx}u_i(x) \geq 0$, and therefore the prize contract $(w_i, l_i) = (v_i - s, s)$ is the best response to player i. Otherwise, if player i receives a losing prize of x such that $d_j \leq d_i = v_i - 2x$, we obtain that $\frac{d}{dx}u_u(x) \leq 0$ and then the prize contract with the highest risk $(w_i, l_i) = (v_i, 0)$ is the best response to player i. Hence, given that player j receives the prize contract $(w_j, l_j) = (v_j, 0)$, if

$$u_i(x)_{|x=0} = \frac{(v_i)^3}{(v_i + v_j)^2} \leq \frac{(v_i - 2s)^3}{(v_i - 2s + v_j)^2} + s_i = u_i(x)_{x=s},$$

there exists a subgame perfect equilibrium in which in the first stage player i receives the prize contract with the lowest risk $(w_i, l_i) = (v_i - s, s)$ and player j receives the prize contract with the highest risk $(w_j, l_j) = (v_j, 0)$.

8.4 Proof of Proposition 5

Assume now that $d_2 < \frac{n-1}{n}d_1$ and $n > 1$. We want to show that there is no equilibrium in which n players receive the prize contract with the higher risk (w_1, l_1). If n players receive the prize contract with the higher risk $(w_1.l_1)$, and the other m players receive the prize contract with the lower risk (w_2, l_2) are not active, by (9), the expected payoff of each of the n players who receive the contract with the higher risk is

$$u_1 = \frac{d_1}{n^2} + l_1.$$

If one of these n players will receive the prize contract with the lower risk over the one with the higher risk, by (10), his expected payoff will satisfy,

$$u_2 > l_2.$$

Then, since $d_2 < \frac{n-1}{n} d_1$ we have

$$\begin{aligned} u_1 - u_2 &< \frac{d_1}{n^2} + l_1 - l_2 = \frac{d_1}{n^2} - \frac{d_1 - d_2}{2} = \frac{1}{2} d_2 - \frac{n^2 - 2}{2n^2} d_1 \\ &\leq \frac{n-1}{2n} d_1 - \frac{n^2-2}{2n^2} d_1 \leq 0. \end{aligned}$$

Thus, if $d_2 < \frac{n-1}{n} d_1$, one of the n players who receive the higher risk contract would prefer to switch to the lower risk contract. This result is obviously true when more than n players receive the higher risk contract. As a result, we do not have an equilibrium when n or more players receive the prize contract with the higher risk.

8.5 Proof of Proposition 6

Assume that n players receive the prize contract (w_1, l_1) and m players receive the prize contract (w_2, l_2) where $w_1 > w_2$. According to (12) and (13), a player who receives the prize contract with the lower risk (w_2, l_2) prefers to receive the prize contract with the higher risk (w_1, l_1) iff

$$\begin{aligned} \Delta &= u_2(n,m) - u_1(n+1, m-1) = \frac{d_2}{(nd_2 + md_1)^2} (nd_2 - (n-1)d_1)^2 \\ &\quad - \frac{d_1}{((n+1)d_2 + (m-1)d_1)^2} ((m-1)d_1 - (m-2)d_2)^2 + (l_2 - l_1) < 0, \end{aligned} \tag{19}$$

where $u_2(n,m)$ is the expected payoff of this player when he receives with the other $m-1$ players the prize contract (w_2, l_2), and $u_1(n+1, m-1)$ is this player's expected payoff if he receives with the other n players the prize contract (w_1, l_1) instead of the prize contract (w_2, l_2). Below, we examine under which conditions Δ is negative. Because that

$$d_1 - d_2 = (w_1 - w_2) + (l_2 - l_1),$$

and

$$w_1 + l_1 = w_2 + l_2,$$

we have

$$(d_1 - d_2) = 2(l_2 - l_1).$$

Thus, by (19), we obtain that

$$\begin{aligned}
\Delta &= u_2(n,m) - u_1(n+1,m-1) = \frac{d_2}{(nd_2+md_1)^2}(nd_2-(n-1)d_1)^2 \\
&\quad - \frac{d_1}{((n+1)d_2+(m-1)d_1)^2}((m-1)d_1-(m-2)d_2)^2 + (l_2-l_1) \\
&= \frac{d_2}{(nd_2+md_1)^2}(nd_2-(n-1)d_1)^2 - \\
&\quad \frac{d_1}{((n+1)d_2+(m-1)d_1)^2}((m-1)d_1-(m-2)d_2)^2 + \frac{1}{2}(d_1-d_2),
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
\Delta &= \frac{1}{2}\frac{d_1-d_2}{(m^2d_1^2+2mnd_1d_2-md_1^2+md_1d_2+n^2d_2^2-nd_1d_2+nd_2^2)^2} \\
&\quad \left(\begin{array}{c}
-m^4d_1^4+2m^4d_1^3d_2+4m^3nd_1^2d_2^2+2m^3d_1^4-6m^3d_1^3d_2+2m^2n^2d_1^3d_2 \\
+2m^2n^2d_1^2d_2^2+2m^2n^2d_1d_2^3-2m^2nd_1^3d_2-10m^2nd_1^2d_2^2-m^2d_1^4+6m^2d_1^3d_2 \\
+m^2d_1^2d_2^2+4mn^3d_1^2d_2^2-4mn^2d_1^3d_2-2mn^2d_1^2d_2^2-6mn^2d_1d_2^3 \\
+6mnd_1^3d_2+4mnd_1^2d_2^2+2mnd_1d_2^3-4md_1^3d_2+2n^4d_1d_2^3-n^4d_2^4-4n^3d_1^2d_2^2 \\
+2n^3d_1d_2^3-2n^3d_2^4+2n^2d_1^3d_2+n^2d_1^2d_2^2+4n^2d_1d_2^3-n^2d_2^4 \\
-4nd_1^3d_2+4nd_1^2d_2^2-4nd_1d_2^3+2d_1^3d_2-2d_1^2d_2^2
\end{array}\right).
\end{aligned}$$

When $n=0$, namely, no one receives the prize contract (w_1, l_1), we obtain that

$$\begin{aligned}
\Delta &= \frac{1}{2}\frac{d_1-d_2}{(m^2d_1^2-md_1^2+md_1d_2)^2} \\
&\quad \left(\begin{array}{c}
-m^4d_1^4+2m^4d_1^3d_2+2m^3d_1^4-6m^3d_1^3d_2-m^2d_1^4 \\
+6m^2d_1^3d_2+m^2d_1^2d_2^2-4md_1^3d_2+2d_1^3d_2-2d_1^2d_2^2
\end{array}\right).
\end{aligned}$$

Denote

$$\begin{aligned}
G &= -m^4d_1^4+2m^4d_1^3d_2+2m^3d_1^4-6m^3d_1^3d_2-m^2d_1^4 \\
&\quad +6m^2d_1^3d_2+m^2d_1^2d_2^2-4md_1^3d_2+2d_1^3d_2-2d_1^2d_2^2
\end{aligned}$$

Then, $G<0$ implies that $\Delta<0$. When $d=d_1=d_2>0$, we obtain that for $m>2$,

$$G = md_1^4\left(m^3-4m^2+6m-4\right) \geq 0,$$

and for $m = 2$, we have $G = 0$. In addition,

$$\frac{dG}{dd_2} = 2d_1^2(d_1(m^4 - 3m^3 + 3m^2 - 2m + 1) + d_2(m^2 - 2)) > 0.$$

Thus, when $m > 2$ and d_2 is sufficiently large, or, alternatively, the difference $d_1 - d_2$ is sufficiently small, we obtain that $G > 0$ which implies that $\Delta > 0$. In that case, $n = 0$, namely, no one wants to receive the prize contract with the higher risk (w_1, l_1).

On the other hand, when $d_2 = \frac{1}{2}d_1$, for all $m \geq 1$ we have

$$G = -\frac{1}{4}d_1^4\left(4m^3 - 9m^2 + 8m - 2\right) < 0.$$

Thus, in that case, when d_2 is sufficiently small, in particular, smaller than $\frac{1}{2}d_1$, $G < 0$ for all $m \geq 2$, which implies that $\Delta < 0$. Then, $n > 0$, and at least one player wants the prize contract with the higher risk (w_1, l_1).

Now, assume that $n = 1$, namely, one player receives the prize contract (w_1, l_1). Then, we have

$$\begin{aligned}\Delta \quad = \quad & \frac{1}{2}\frac{d_1 - d_2}{\left(m^2d_1^2 + 2md_1d_2 - md_1^2 + md_1d_2 + d_2^2 - d_1d_2 + d_2^2\right)^2} \\ & \left(\begin{array}{c} -m^4d_1^4 + 2m^4d_1^3d_2 + 4m^3d_1^2d_2^2 + 2m^3d_1^4 - 6m^3d_1^3d_2 + 2m^2d_1^3d_2 \\ +2m^2d_1^2d_2^2 + 2m^2d_1d_2^3 - 2m^2d_1^3d_2 - 10m^2d_1^2d_2^2 - m^2d_1^4 + 6m^2d_1^3d_2 \\ +m^2d_1^2d_2^2 + 4md_1^2d_2^2 - 4md_1^3d_2 - 2md_1^2d_2^2 - 6md_1d_2^3 \\ +6md_1^3d_2 + 4md_1^2d_2^2 + 2md_1d_2^3 - 4md_1^3d_2 + 2d_1d_2^3 - d_2^4 - 4d_1^2d_2^2 \\ +2d_1d_2^3 - 2d_2^4 + 2d_1^3d_2 + d_1^2d_2^2 + 4d_1d_2^3 - d_2^4 \\ -4d_1^3d_2 + 4d_1^2d_2^2 - 4d_1d_2^3 + 2d_1^3d_2 - 2d_1^2d_2^2 \end{array}\right).\end{aligned}$$

Denote by

$$S=\left(\begin{array}{c}-m^4d_1^4+2m^4d_1^3d_2+4m^3d_1^2d_2^2+2m^3d_1^4-6m^3d_1^3d_2+2m^2d_1^3d_2\\+2m^2d_1^2d_2^2+2m^2d_1d_2^3-2m^2d_1^3d_2-10m^2d_1^2d_2^2-m^2d_1^4+6m^2d_1^3d_2\\+m^2d_1^2d_2^2+4md_1^2d_2^2-4md_1^3d_2-2md_1^2d_2^2-6md_1d_2^3\\+6md_1^3d_2+4md_1^2d_2^2+2md_1d_2^3-4md_1^3d_2+2d_1d_2^3-d_2^4-4d_1^2d_2^2\\+2d_1d_2^3-2d_2^4+2d_1^3d_2+d_1^2d_2^2+4d_1d_2^3-d_2^4\\-4d_1^3d_2+4d_1^2d_2^2-4d_1d_2^3+2d_1^3d_2-2d_1^2d_2^2\end{array}\right).$$

Because $d_2\geq\frac{n-1}{n}d_1$ the minimum value of d_2 is equal to $\frac{1}{2}d_1$, and then we have

$$S=\frac{1}{2}m^2d_1^4.$$

We also have

$$\frac{dS}{dd_2}=d_1^3(2m^4-6m^3+6m^2-2m)+d_1^2d_2(8m^3-14m^2+12m-2)+d_1d_2^2(6m^2-12m+12)-16d_2^3$$

Thus, for all $m\geq 1$, since $d_1>d_2$ we obtain that

$$\frac{dS}{dd_2}>4d_1^3+30d_1^2d_2+12d_1d_2^2-16d_2^3>0$$

This implies that for all $d_1>d_2\geq\frac{1}{2}d_1$ we obtain that $\Delta>0$, that is, we will never have a case in which two players will receive the prize contract with the higher risk (w_1,l_1).

So far we showed that we will never have a situation in which two players receive the prize contract with the higher risk. In order to show that no more than two players will receive the more risky prize contract, define for $n>1$,

$$\begin{aligned}\Delta &= u_1(n,m)-u_1(n+1,m-1)\\&= \frac{d_1}{(nd_2+md_1)^2}\left(md_1-(m-1)d_2\right)^2+l_1\\&\quad-\frac{d_1}{((n+1)d_2+(m-1)d_1)^2}\left((m-1)d_1-(m-2)d_2\right)^2+l_1\\&= -d_1d_2\left(d_1-d_2\right)\frac{m+n-1}{\left(m^2d_1^2+2mnd_1d_2-md_1^2+md_1d_2+n^2d_2^2-nd_1d_2+nd_2^2\right)^2}\times\\&\quad\left(d_1d_2-d_2^2-2m^2d_1^2+2md_1^2+md_2^2-3nd_2^2+2m^2d_1d_2-5md_1d_2+nd_1d_2+2mnd_2^2-2mnd_1d_2\right),\end{aligned}$$

and define

$$
\begin{aligned}
Q &= d_1 d_2 - d_2^2 - 2m^2 d_1^2 + 2md_1^2 + md_2^2 - 3nd_2^2 \\
&\quad +2m^2 d_1 d_2 - 5md_1 d_2 + nd_1 d_2 + 2mnd_2^2 - 2mnd_1 d_2 \\
&= d_1^2(-2m^2 + 2m) + d_2^2(2mn - 3n + m - 1) + d_1 d_2(2m^2 - 2mn - 5m + n + 1).
\end{aligned}
$$

If $Q < 0$ then $\Delta > 0$ which means that, for all $n > 1$, no more than n players want to receive the prize contract with the highest risk (w_1, l_1). Because that $d_1 > d_2$ we have

$$
\begin{aligned}
Q &< d_1^2(-2m^2 + 2m) + d_1^2(2mn - 3n + m - 1) + d_1 d_2(2m^2 - 2mn - 5m + n + 1) \\
&= -d_1^2(2m^2 - 2m - 2mn + 3n - m + 1) + d_1 d_2(2m^2 - 2mn - 5m + n + 1).
\end{aligned}
$$

If $n > m$, then,

$$
(2m^2 - 2m - 2mn + 3n - m + 1) > (2m^2 - 2mn - 5m + n + 1),
$$

which implies that $Q < 0$ and therefore $\Delta > 0$. On the other hand, when $n \leq m$, then $2m^2 - 2mn - 5m + n + 1 > 0$, and we have

$$
\begin{aligned}
Q &< d_1^2(-2m^2 + 2m) + d_1^2(2mn - 3n + m - 1) + d_1^2(2m^2 - 2mn - 5m + n + 1) \\
&= -2d_1^2\,(m + n) < 0.
\end{aligned}
$$

Thus, for all $n > 1$ we obtain that $Q < 0$ which implies that $\Delta > 0$. In other words, no more than n players want to receive the prize contract with the higher risk (w_1, l_1). We also proved it for $n = 1$, so if all the players are active, no more than one player prefers to receive the prize contract with the higher risk.

8.6 Proof of Proposition 7

When n players receive the prize contract (w_1, l_1) and m players receive the prize contract (w_2, l_2) where $w_1 > w_2$, the highest effort equals the equilibrium effort of the players who receive the prize contract with the highest risk. By (9), for all $n \geq 1$, $m \geq 2$, the equilibrium effort of the players who receive the prize contract with the highest risk satisfies

$$\begin{aligned}
\Delta &= x(n+1,m-1)-x(n,m) \\
&= \frac{d_1 d_2}{((n+1)d_2+(m-1)d_1)^2}\left((m-1)d_1-(m-2)d_2\right)(m+n-1) \\
&\quad -\frac{d_1 d_2}{(nd_2+md_1)^2}\left(md_1-(m-1)d_2\right)(m+n-1) \\
&= -d_1 d_2\left(d_1-d_2\right)\frac{m+n-1}{\left(m^2 d_1^2+2mnd_1 d_2-md_1^2+md_1 d_2+n^2 d_2^2-nd_1 d_2+nd_2^2\right)^2}\times \\
&\quad \left(-m^2 d_1^2+2m^2 d_1 d_2+2mnd_2^2+md_1^2-4md_1 d_2+md_2^2+n^2 d_2^2-2nd_2^2+d_1 d_2-d_2^2\right).
\end{aligned}$$

Denote

$$\begin{aligned}
Q &= \left(-m^2 d_1^2+2m^2 d_1 d_2+2mnd_2^2+md_1^2-4md_1 d_2+md_2^2+n^2 d_2^2-2nd_2^2+d_1 d_2-d_2^2\right) \\
&= d_1^2(-m^2+m)+d_2^2(2mn+m+n^2-2n-1)+d_1 d_2(2m^2-4m+1).
\end{aligned}$$

We want to show that $Q>0$, which implies that $\Delta<0$. It is sufficient to show that

$$Q>d_1^2(-m^2+m)+d_2^2(2mn+m-1)+d_1 d_2(2m^2-4m+1)>0.$$

Assuming all players are active, $\frac{n}{n-1}\, d_2>d_1$. For $n>1$, the last inequality holds because

$$-d_1^2 m^2+d_1 d_2 2m^2>0 and d_2^2 2mn-4d_1 d_2 m>0.$$

Thus, in order to maximize effort, no more than two players should receive the contract with the higher risk.

We want to show now that, if all the players are active, then, in order to maximize the highest effort, only one player should receive the contract with the higher risk; that is, we need to show

$$x(2,m-1)-x(1,m)<0.$$

When $n=1$, it is sufficient to show that

$$Q=d_1^2(-m^2+m)+d_2^2(3m-2)+d_1 d_2(2m^2-4m+1)>0,$$

where $d_2>0$. Note that when d_2 converges to zero we obtain that $Q<0$. On the other hand, when $d_2=d_1$ we obtain that

$$\begin{aligned}
Q &= d_1^2(-m^2+m)+d_1^2(3m-2)+d_1 d_1(2m^2-4m+1) \\
&= d_1^2\left(m^2-1\right)>0.
\end{aligned}$$

When $d_2 = \frac{1}{2}d_1$ then

$$\begin{aligned}
Q &= d_1^2(-m^2+m) + (\frac{1}{2}d_1)^2(3m-2) + \frac{1}{2}d_1^2(2m^2-4m+1) \\
&= -\frac{1}{4}md_1^2 < 0
\end{aligned}$$

Note that for all $m \geq 2$,

$$\begin{aligned}
\frac{dQ}{dd_2} &= \frac{d}{dd_2}(d_1^2(-m^2+m) + d_2^2(3m-2) + d_1d_2(2m^2-4m+1)) \\
&= d_1(2m^2-4m+1) + d_2(6m-4) > 0
\end{aligned}$$

Thus, we obtained that if $d_2 - d_1$ is sufficiently small then it is optimal that one player will receive the contract with higher risk, while if d_2 is sufficiently small then it is optimal that two players will receive the contract with the higher risk.

When $n = 0$, we obtain that

$$\begin{aligned}
& x(1, m-1) - y(0, m) \\
= & \frac{d_1 d_2}{(d_2 + (m-1)d_1)^2}\left((m-1)d_1 - (m-2)d_2\right)m - \frac{1}{m^2}d_2\left(m-1\right) \\
= & \frac{1}{m^2}\frac{d_2}{(d_2 - d_1 + md_1)^2} \times \\
& ((m^4 - 2m^3)(d_1^2 - d_1d_2) + m^2(3d_1^2 - 2d_1d_2) + m(4d_1d_2 - d_2^2) + (d_1 - d_2)^2) > 0
\end{aligned}$$

where $y(0, m)$ is the highest effort when all the players receive the prize contract with the lower risk, and $x(1, m-1)$ is the highest effort when one player receives the prize contract with the higher risk. The difference between these terms is positive for all $m \geq 2$, and therefore at least one player should receive the prize contract with the higher risk.

We can conclude that the optimal number of players who receive the prize contract with the higher risk is either one or two. It could be two players only if d_2 is sufficiently small since then the players who receive the contract with the lower risk do not exert any effort or they exert a very low effort.

8.7 Proof of Proposition 8

According to (9), the derivative of the highest effort in d_1 is

$$\begin{aligned}\frac{dx}{dd_1} &= \frac{d}{dd_1}\left(\frac{d_1 d_2}{\left(nd_2+md_1\right)^2}\left(md_1-(m-1)d_2\right)(m+n-1)\right) \\ &= \frac{d_2^2}{\left(md_1+nd_2\right)^3}(m+n-1)\left(nd_2-md_1+m^2d_1+2mnd_1-mnd_2\right) \\ &= \frac{d_2^2}{\left(md_1+nd_2\right)^3}(m+n-1)\left(d_1(m^2-m+2mn)-d_2(m-1)n\right).\end{aligned}$$

Because that $d_1 > d_2$ we obtain that $\frac{dx}{dd_1} > 0$, that is, the highest effort increases in d_1 and therefore the prize contract with the higher risk (w_1, l_1) should be the one with the highest risk $(v, 0)$. On the other hand, by (9), the derivative of the highest effort in d_2 is

$$\frac{dx}{dd_2} = m\frac{d_1^2}{\left(md_1+nd_2\right)^3}(m+n-1)\left(2d_2+md_1-2md_2-nd_2\right).$$

This implies that the optimal value of d_2 is

$$d_2^* = \frac{md_1}{(n+2m-2)}.$$

Proposition 7 states that the optimal number of players receiving the prize contract with the highest risk is either $n=1$ or $n=2$. If $n=1$, then $d_2^* = \frac{md_1}{2m-1}$, and by (11), the highest effort is

$$\begin{aligned}x &= \frac{d_1 d_2}{\left(nd_2+md_1\right)^2}\left(md_1-(m-1)d_2\right)(m+n-1) \\ &= \frac{d_1\frac{md_1}{(2m-1)}}{\left(\frac{md_1}{(2m-1)}+md_1\right)^2}\left(md_1-(m-1)\frac{md_1}{(2m-1)}\right)(m+1-1) \\ &= \frac{1}{4}d_1.\end{aligned}$$

If, on the other hand, $n=2$, $d_2^* = \frac{1}{2}d_1$ which means that only the two players with the highest risk prize contract will compete, and the highest effort is

$$x = \frac{1}{4}d_1.$$

Hence, the optimal value for the highest effort is $\frac{1}{4}d_1$ which is obtained when one player receives the prize contract with the highest risk.

8.8 Proof of Proposition 9

According to Proposition 6, if all players are active, no more than one player will receive the prize contract with the higher risk (w_1, l_1). The relevant question is if when the designer chooses $d_2 = d_1 \frac{m}{2m-1} > \frac{1}{2} d_1$ which is optimal for maximizing the highest effort according to Proposition 8, there is one player who wants to receive the prize contract with the highest risk.

If a player receives the prize contract with the highest risk, by (12), his expected payoff is

$$u_1 = \frac{d_1}{(d_2 + md_1)^2} (md_1 - (m-1)d_2)^2 + l_1.$$

On the other hand, if this player receives the prize contract with the lower risk, all the players then receive the prize contract with the lower risk, and by (13), his expected payoff is

$$u_2 = \frac{d_2}{m^2} + l_2.$$

Then we have

$$u_1 - u_2 = \frac{d_1}{(d_2 + md_1)^2} (md_1 - (m-1)d_2)^2 + l_1 - \frac{d_2}{m^2} - l_2$$

Because that $(d_1 - d_2) = 2(l_2 - l_1)$, we obtain that

$$\begin{aligned} u_1 - u_2 &= \frac{d_1}{(d_2 + md_1)^2} (md_1 - (m-1)d_2)^2 - \frac{d_2}{m^2} - \frac{d_1 - d_2}{2} \\ &= \frac{1}{2m^2 (d_2 + md_1)^2} \begin{pmatrix} m^4 d_1^3 - 3m^4 d_1^2 d_2 + 2m^4 d_1 d_2^2 + 2m^3 d_1^2 d_2 - 2m^3 d_1 d_2^2 \\ -2m^2 d_1^2 d_2 + m^2 d_1 d_2^2 + m^2 d_2^3 - 4md_1 d_2^2 - 2d_2^3 \end{pmatrix} \end{aligned}$$

Because that $d_2 = d_1 \frac{m}{2m-1}$ we have

$$\begin{aligned} &u_1 - u_2 = \\ &= \frac{1}{2m^2 \left(d_1 \frac{m}{2m-1} + md_1\right)^2} \begin{pmatrix} m^4 d_1^3 - 3m^4 d_1^2 d_1 \frac{m}{2m-1} + 2m^4 d_1 (d_1 \frac{m}{2m-1})^2 + 2m^3 d_1^2 d_1 \frac{m}{2m-1} \\ -2m^3 d_1 (d_1 \frac{m}{2m-1})^2 - 2m^2 d_1^2 (d_1 \frac{m}{2m-1}) + m^2 d_1 (d_1 \frac{m}{2m-1})^2 \\ +m^2 (d_1 \frac{m}{2m-1})^3 - 4md_1 (d_1 \frac{m}{2m-1})^2 - 2(d_1 \frac{m}{2m-1})^3 \end{pmatrix} \\ &= \frac{1}{2m^2 \left(d_1 \frac{m}{2m-1} + md_1\right)^2} \left(\frac{d_1}{8m^2 - 4m} (m-4)\right). \end{aligned}$$

Thus, If $m > 4$, then $u_1 - u_2 > 0$, resulting in an equilibrium with the highest effort.

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