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Abstract

We consider two-stage parallel contests with a finite set of agents and a finite set of heterogeneous contests. In the first stage, each agent simultaneously chooses in which contest he wants to compete, and in the second stage, the agents who chose the same contest compete against each other in a contest for a single prize. We first assume general contest success functions with nonlinear cost functions, and then we provide an algorithm that converges to equilibrium; namely, the algorithm organizes the allocation of agents among the contests until no agent wants to change his current contest. Later, we assume the Tullock contest success function with linear cost functions, and for homogeneous or heterogeneous agents, we show that our model and congestion games are equivalent, establishing the existence of a subgame perfect equilibrium with pure strategies in our two-stage parallel contests model.

JEL classification: D44, D72, D82, J31

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1 Introduction

In our world, agents frequently encounter several parallel contests such that they can take part in only one of them and then they have to choose which contest to enter. In that case, each agent who needs to choose one contest may confront the famous dilemma that was attributed to Julius Caesar "I had rather be the first in this village than the second in Rome" (see Damiano et al. 2010). For example, researchers from the same country or even from several countries compete to win one of several competitive grants with the constraint that each researcher cannot submit his proposal to more than one competitive grant. If a researcher submits his proposal to one of the highly competitive grants, he will have to compete against other excellent researchers, and therefore his chance to win the grant will not be very high, but if he wins the value of the prize is high particularly because of the reputation of winning that grant. On the other hand, if he chooses to submit his proposal to an ordinary competitive grant, the reputation of winning this grant will be lower, but the competition with his opponents will be easier such that his chance to win will be higher. This is a classic problem of parallel contests where the prizes (the reputations of the grants) are different among the contests, and the heterogeneous agents (researchers) have to choose in which one they want to compete.

There are many other real-life situations that can be modeled as parallel contests. For example, consider a professional tennis player who has to decide in which tournament to compete where some of them take part at the same time. A tennis player who is ranked in the top 50 can choose to compete in almost every tournament. However, if he has to choose between an ATP-1000 tournament in which the prizes are relatively large and an ATP-250 tournament in which the prizes are smaller, he knows that in the ATP-1000 tournament he is going to match stronger tennis players than in the ATP-250 tournament such that his chance to win the ATP-1000 tournament is smaller than to win the ATP-250 tournament. Thus, in parallel contests an agent should consider his expected payoff in each of them, and since it is not possible to compete in more than one contest, he should choose the one in which he has the highest expected payoff. Since an agent's expected payoff is based on the reward in each contest as well as his chance to win it, the choice may not be that simple. Thus, the relevant question is if there exists an equilibrium in parallel contests at all, and if so, how to find it? In this paper, we develop a model of parallel contests to provide answers to these questions.

Specifically, we consider a two-stage model of parallel contests, each of which has a single prize. In the first stage, the agents simultaneously choose the contest in which they want to compete with the condition that each agent can compete in at most one contest. In the second stage, after the agents choose their contests, the agents in each contest compete against each other, and the winner of each contest wins a prize where all the other agents bear the costs of their efforts.

We first consider parallel contests each of which has a generalized Tullock contest success function (see Yamazaki 2008). There are n heterogeneous agents who have two possible types, each with a different cost of effort. Furthermore, the costs of effort are non-linear, and are not explicitly given. All we know about these costs of effort is that they are increasing and weakly convex in effort. We also assume some ranking including the agents' costs of effort that clarify the relative strength of the agents of different types. In that case, it is impossible to characterize the subgame perfect equilibrium since the costs of effort are not explicitly given, so that the agents' expected payoffs over the parallel contests cannot be calculated. To overcome this obstacle, we use revealed preference techniques to demonstrate the existence of a subgame perfect equilibrium with pure strategies and provide an algorithm that enables us to calculate this equilibrium, namely, the allocation of agents across the contests. The algorithm is based on finite sequential steps each of which includes an instruction that enables some types of agents to move across the contests. One might think that these steps can repeat in an infinite loop. However, the algorithm reaches a point according to which no agent wants to change his position, and then we say that the algorithm converges to equilibrium and our parallel contests have a subgame perfect equilibrium with pure strategies. We first show that every model of parallel contests with two contests and two types of agents has a subgame perfect equilibrium with pure strategies, and then we generalize this result to parallel contests with any finite number of contests.

Later, we consider parallel contests, each with a Tullock contest success function (Tullock 1980). First, we assume that the parallel contests include heterogeneous agents with different linear costs of effort. When the agents are homogeneous and have linear effort costs, we show that our parallel contest model is a potential game, or more specifically, a congestion game, which is well known to have a pure-strategy equilibrium (Rosenthal 1973, Monderer and Shapley 1996).[1] When the agents are heterogeneous, it is unclear whether our model is comparable to a congestion game. For this purpose, we introduce the "exchange ratio," which states that the expected payoff of all agents remains constant when the correct number of two types of agents from different contests is exchanged. The "exchange ratio" allows us to "convert" the case of heterogeneous agents to a case of homogeneous agents, demonstrating that our heterogeneous agent model is also equivalent to a congestion game, implying the existence of pure-strategy equilibrium. This "exchange ratio" also implies that an equilibrium is not always unique, because each agent's expected payoff remains constant when the appropriate number of agents are switched between contests.

[1] A game is considered a potential game if the incentive for all players to change their strategy can be expressed using a single global function known as the potential function.

The main findings of this paper are that in our two-stage parallel contests with heterogeneous agents of two types (cost of effort), whether players have linear or nonlinear cost functions, and regardless of the number of contests or the prize values in these contests, there is always subgame perfect equilibrium in pure strategies. Furthermore, we provide algorithms for calculating these equilibrium points, specifically the player distribution across contests, so that no player wants to switch to another contest.

1.1 Related literature

We are not the first to deal with a model of parallel contests, although thus far they have received quite limited attention despite their widespread presence in our lives. Among these, we have the work of Azmat and Möller (2009) who study two parallel Tullock contests with homogeneous agents that have linear costs of effort where each agent chooses a contest and in each contest, there are several heterogenous prizes. Konrad and Kovenock (2012) study a model of parallel all-pay contests with homogeneous agents who have linear costs of effort where each agent first chooses a contest and then in each contest the agents compete in an all-pay contest for several identical prizes. Juang et al. (2020) also study the same model with two parallel all-pay contests with homogenous agents who have linear costs of effort. However, in contrast to Konrad and Kovenock they assume that the number of identical prizes in each contest is not the same. They also assume that each agent chooses his contest when he does not necessarily know the number of his opponents in that contest. Azmat and Möller (2018) study parallel equilibrium contests with continuum heterogeneous agents who first choose the contest and then compete in all-pay contests with several identical prizes. Morgan et al. (2018) analyze two parallel contests with a continuum of heterogenous agents where each agent's performance is deterministic but noisily measured. In all the above models, the authors usually assume that agents are homogenous with a linear cost of effort, and those who consider heterogeneous agents also assume a continuum of agents. Our present work differs in that we are the first to deal with a finite number of heterogeneous agents. When there is a continuum of agents, the movement of a single agent between two contests has no effect on the expected payoff of the other agents, but the opposite is true when there are a finite number of agents. Moreover, while in the above papers it is assumed that the agents have linear costs of effort, we also assume that the agents' costs of effort are nonlinear and, moreover, are not even explicitly given such that the agents' payoffs cannot be calculated.

In our model of two-stage parallel contests, the agents face several prizes, each of which is in a different contest, and they have to choose the prize they want to compete for. In most models of contests with several prizes either under complete or incomplete information (see, among others, Clark and Riis 1996, 1998, Moldovanu and Sela 2001, 2006, Moldovanu et al. 2012, Akerlof and Holden 2012, Gonzalez-Diaz and Siegel 2013, Fu et al. 2014, Xiao 2016 and Sela 2020) it is assumed that the prizes are identical to all the agents, and if they are not identical, then the ratio of the values for every pair of prizes is the same for all the agents who differ from each other by their ability or, alternatively, their marginal cost of effort. In contrast, in our model when we assume non-linear costs of effort, the agents of the different types also have different ratios of prizes over different contests. In other words, the heterogeneity of the prizes between the two types of agents in our model with non-linear costs of effort is unrestricted.

In the second stage of our model, in each contest, the agents compete in a Tullock contest (see, among others, Baye and Hoppe 2003, Cornes and Hartley 2005, Corchon and Dahm 2010, Fu and Lu 2012, and Ewerhart 2015). The convergence of our algorithm to the equilibrium in the case of non-linear costs of effort is based on the uniqueness of the equilibrium in one-stage Tullock contests (for the existence and uniqueness of equilibrium in contests with complete information, see Szidarovszky and Okuguchi 1997 and Yamazaki 2008; for contests with incomplete information, see Einy et al. 2015, 2020 and Ewerhart and Quartieri 2020). The reason is that we assume that for any change in the allocation of agents that is solved by the algorithm, there is a unique equilibrium in every contest in the second stage. Nonetheless, in the first stage, we demonstrate that the equilibrium allocation of agents is not necessarily unique; consequently, the equilibrium in our model of parallel contests with homogeneous agents and linear effort costs is not necessarily unique.

When players are homogeneous and have linear cost functions, we show that the first stage of our two-stage parallel contests corresponds to a potential game (see Monderer and Shapley 1996) and, more specifically, a congestion game (see, for example, Milchtaich 1996; Holzman and Law-Yone 1997). In this case, we explicitly define the potential function in our model. When the players are heterogeneous and have linear cost functions, we do not find the potential function; however, we demonstrate that our model corresponds to a model with homogeneous players, implying that it also corresponds to a potential game. When the players are diverse and have nonlinear cost functions, our model of two-stage parallel contests is unlikely to be representative of a potential game. However, using algorithms, we can demonstrate that our model has a subgame perfect equilibrium in pure strategies.

The rest of the paper is organized as follows: In Section 2, we describe the model of parallel contests. In Section 3, we analyze the subgame perfect equilibrium when the agents have non-linear effort cost functions. In subsection 3.1, we assume that there are two contests while in subsection 3.2 we generalize the analysis for any finite number of contests. In Section 4, we analyze the subgame perfect equilibrium in this model where agents have linear costs of effort. In subsection

4.1, we assume that the agents are homogeneous while in subsection 4.2 we assume that the agents are heterogeneous. Section 5 concludes. The proofs appear in the Appendix.

2 Two-stage parallel contests

Consider a set $N = \{1, 2, 3, ..., n\}$ of n heterogeneous agents, and a set $M = \{1, 2, 3, ..., m\}$ of m, $m < n$, heterogeneous contests. In the first stage each agent chooses to compete in one of the m contests where each contest $k \in M$ has a single prize v_k, and, without loss of generality, we assume that $v_k \geq v_{k+1}$, $k = 1, ..., m-1$. The set of agents that choose to compete in contest $k \in M$ will be defined as N_k, where n_k is the number of agents in that contest. In the second stage, after the agents were allocated across the contests, the agents observe the other agents who chose the same contest, and then the agents in each contest compete against each other to win the prize.

The agents in contest $k \in M$ simultaneously exert their efforts. Let x_i denote the effort of agent $i \in N_k$. Then, each agent i's probability to win in contest k is $p_i = \frac{f_i(x_i)}{\sum_{j \in N_k} f_j(x_j)}$ where $f_i(x_i)$, which be called the production function, is assumed to be zero at $x_i = 0$, continuous, strictly increasing, and weakly concave in x_i for all $i \in N$. Each agent $i \in N$ bears the cost of his effort whether he wins or not, where his effort cost function $c_i(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and weakly convex in the effort x. The expected payoff of agent $i \in N_k$ is:

$$v_k \frac{f_i(x_i)}{\sum_{j \in N_k} f_j(x_j)} - c_i(x_i) \tag{1}$$

Let $y_j = f_j(x_j)$ for all $j \in N_k$. Then by (1), we obtain that

$$v_k \frac{y_i}{\sum_{j \in N_k} y_j} - \hat{c}_i(y_i) \tag{2}$$

where $\widehat{c}_i = f_i^{-1} \circ c_i$.[2] The assumptions on f_i and c_i imply that $\widehat{c}_i$ is a well-defined function with

[2]This transformation was proposed by Szidarovszky and Okuguchi (1997), and Yamazaki (2008).

$\widehat{c}_i(0) = 0$ and that $\widehat{c}$ is strictly increasing and weakly convex with respect to y_i. The initial maximization problem (1) with respect to x_i is equivalent to the maximization of (2) with respect to y_i.

We assume that the agents' costs of effort can be ranked such that for all $x \geq 0$, $\widehat{c}_i(x) \leq \widehat{c}_{i+1}(x)$, $i = 1, ..., n-1$. This competition will be referred to as the model of two-stage parallel contests.[3]

3 An algorithmic analysis of parallel contests

For this section, we assume that there are two types of agents, strong (h) and weak (l), such that the costs of effort satisfy $\widehat{c}_h(x) \leq \widehat{c}_l(x)$ for all $x \geq 0$. By this assumption, we know that in any competition in the second stage with strong and weak agents there is a unique equilibrium (see Szidarovszky and Okuguchi 1997 and Yamazaki 2008) in which the strong agents exert higher efforts and have higher expected payoffs than the weak agents. We will use the property that the equilibrium in every contest is unique since this implies that if any agent joins another contest then the expected payoffs of all the other agents in that contest decrease while the expected payoffs of all the other agents in the contest that he just left increase. Likewise, the uniqueness of the equilibrium in every contest implies that if an agent leaves and then returns to the same contest and no other agent left or joined that contest, his expected payoff will not change. In the subgame perfect equilibrium of the model of two-stage parallel contests agents do not want to move to another contest. However, because we allow a general form of non-linear costs of effort, we cannot prove that the equilibrium exists by comparing agents' expected payoffs. Instead, we use revealed preference

[3]We can assume here a much more general contest success function than the one given by (1). However, we also need the uniqueness of the equilibrium in the second stage in each contest, and for this reason, we consider the contest success function given by (1).

techniques to demonstrate unequivocally that our model has a subgame perfect equilibrium where in the first stage the agents choose their contests by pure strategies. In the next section, we will begin by analyzing the case of two parallel contests, and then we will generalize our result to any finite number of parallel contests. The results for the case of two parallel contests are included in the general case of any finite number of contests. However, because the equilibrium analysis in the case of two parallel contests is slightly simpler than the equilibrium analysis in the general case, we find it easier to explain the equilibrium analysis of the two parallel contests first, followed by the equilibrium analysis of any number of parallel contests.

3.1 Two parallel contests

Consider a model of two-stage parallel contests with two contests 1 and 2, and two prizes v_1, v_2 respectively, where $v_1 \geq v_2$. For any allocation of agents between the two contests, each agent's expected payoff is well defined by the uniqueness of the equilibrium of the competition in each contest. Thus, we can assume that if all the agents stay in their places instead one agent who moves from contest i to contest j, then his expected payoff in contest j is larger than his expected payoff in contest i. The first result describes the response of h-type agents to a move of l-type agents between the two contests.

Lemma 1 *Consider a model of two-stage parallel contests with two contests and with two types of agents h and l where $\widehat{c}_h(x) \leq \widehat{c}_l(x)$ for all $x \geq 0$. If for some allocation of agents between the two contests there is no h-type agent who wants to move between the two contests, but there is one l-type agent who moves from contest i to contest $j, i \neq j, i, j \in \{1, 2\}$. Then, in response to the l-type move, no more than one h-type agent will want to move in the opposite direction from contest j to contest i.*

Lemma 1 implies that the number of moves between the two contests is not increasing in time such that in response to each move of one l-type agent there will be at most a move of one h-type agent. Accordingly, we construct an algorithm based on finite sequential steps each of which includes an instruction that enables a type of agents to move across the contests. In this algorithm, we allow one l-type agent to move and then we allow all the h-type agent to move. By Lemma 1, only one h-type agent will move in the opposite direction of the l-type agents. One might think that these steps can repeat in an infinite loop and never reach a stable state. However, the algorithm does reach a point according to which the agents do not react, namely, they do not want to change their positions and the algorithm stops. Then, we say that the algorithm converges to equilibrium and our model of two-stage parallel contests has a subgame perfect equilibrium with pure strategies in both stages. To find the equilibrium, consider the following algorithm:

Algorithm 1 *Algorithm has the following four steps:*

Step 1 *Allocate the h-type agents in contest 1 and the l-type agents in contest 2.*

Step 2 *H-type agents have the option to move between the contests.*[4]

Step 3 *A single l-type agent has the option to move between the contests.*

Step 4 *Repeat steps 2-3 until in step 3 no agent wants to move.*

The following result shows that the above algorithm always converges to equilibrium.

Proposition 1 *Algorithm 1 always converges to equilibrium and therefore a two-stage parallel contests with two contests and two types of agents h and l where $\widehat{c}_h(x) \leq \widehat{c}_l(x)$ for all $x \geq 0$ has a subgame perfect equilibrium in which agents choose pure strategies in both stages.*

[4]In this step, the h-type agents may move in any order, either simultaneously or sequentially, and there are no limits on the number of possible movements for each agent. This step is complete when no one desires to move.

The outcome of Proposition 1 is not intuitive since strong agents (h-type agents) move from contest 1 with the larger prize to contest 2 with the smaller prize and they never go back to contest 1.

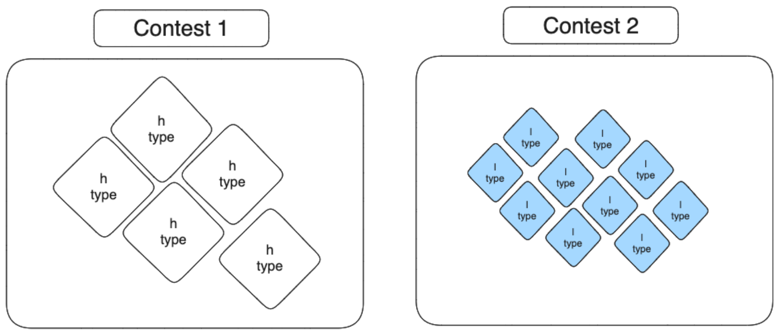

Figure 1: An illustration of the two contests at the end of Step 1 of Algorithm 1

Furthermore, the expected payoffs of the strong agents in contest 2 will be higher than the strong agents' expected payoffs in contest 1. The opposite holds for the weak agents (l-type agents) who move from contest 2 to contest 1 and never go back to contest 2, namely, the weak agents in contest 1 will have a higher expected payoff than the weak agents in contest 2.

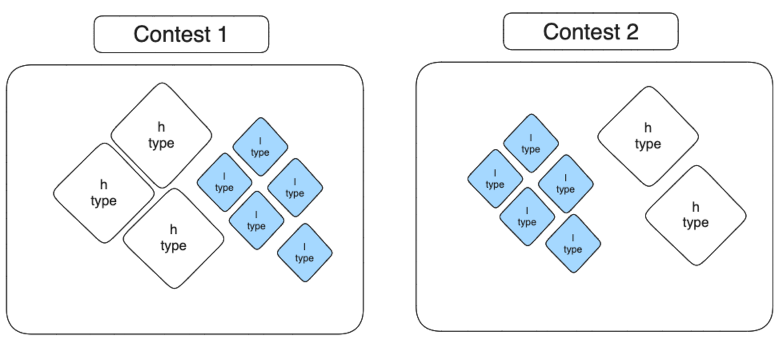

Figure 2: An illustration of the two contests at the end of Step 4 of Algorithm 1

3.2 More than two parallel contests

Consider a set $M = \{1, 2, 3, ..., m\}$ of m contests. Each contest $j \in M$ has a prize v_j where $v_j \geq v_{j+1}$, $j = 1, ..., m-1$. Similarly to the previous subsection, regardless of the allocation of the agents among the m contests, by the uniqueness of the equilibrium in each of the m contests, we can assume that if an agent moves from contest i to contest j, then his expected payoff in contest j is larger than his expected payoff in contest i. We can generalize Lemma 1 as follows:

Lemma 2 *Consider a model of two-stage parallel contests with m contests and two types of agents h and l where $\widehat{c}_h(x) \leq \widehat{c}_l(x)$ for all $x \geq 0$. If for some allocation of agents among the m contests, there is not a h-type agent who wants to move across the contests but there is an l-type agent who wants to move from contest i to contest j, then no more than $(m-1)$ h-type agents will want to move across the contests where at most one of them will join contest i.*

By Lemma 2, a move of one l-type agent implies at most $m-1$ moves of h-type agents. But, in fact, the move of one l-type agent implies at most one move of an h-type agent since except for one move of a h-type agent to the initial place of the l-type agent, the other moves are of h-type agents to contests that other h-type agent just left. By using this argument, we construct the following algorithm:

Algorithm 2 *The algorithm has the following four steps:*

Step 1 *Allocate all the h-type agents in contest 1 and all the l-type agents in contest 2 where all the other contests are without agents.*

Step 2 *H-type agents have the option to move across the contests.*[5]

Step 3 *A single l-type agent has the option to move across the contests.*

Step 4 *Repeat steps 2-3 until in step 3 no agent wants to move.*

The next result shows that this algorithm has to be stopped at some point since agents do not want to move across the contests, and then we say that the algorithm converges to equilibrium.

Proposition 2 *Algorithm 2 always converges to equilibrium and therefore every model of two-stage parallel contests with m contests and two types of agents h and l where $\widehat{c}_h(x) \leq \widehat{c}_l(x)$ for all $x \geq 0$, has a subgame perfect equilibrium in which agents choose pure strategies in both stages.*

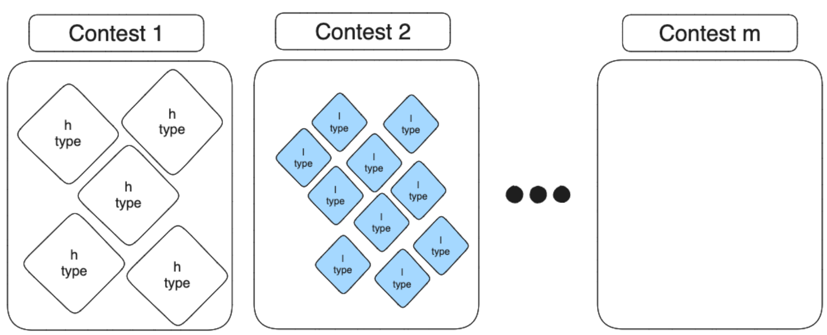

Figure 3: An illustration of the m contests at the end of Step 1 of Algorithm 2

Proposition 2 demonstrates that in a model of two-stage parallel contests with non-linear contests, independent of the number of contests, there is a subgame perfect equilibrium. In the initial

[5] Similar to step 2 of the preceding algorithm, neither the order nor the number of possible movements for each agent is constrained. When no one wishes to proceed, this step concludes.

point of our algorithm, all the h-type agents are in contest 1 and all the l-type agents are in contest 2. The initial point might affect the agents' equilibrium strategies in the first stage such that we might have equilibrium points that depend on the initial point of algorithm 2.

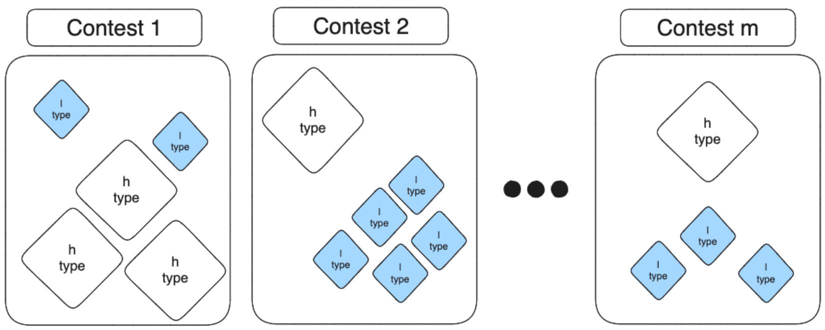

Figure 4: An illustration of the m contests at the end of Step 4 of Algorithm 2

So far, we have assumed that the contests in the second stage have a generalized Tullock contest success function (CSF) and non-linear costs of effort. In the following section, we study the model of two-stage parallel contests with a Tullock CSF (Tullock 1980) and assume linear costs of effort for the agents. We will show that, given those assumptions, we can explicitly characterize the equilibrium and generalize our results to all numbers of types of agents and contests.

4 Parallel contests with linear cost functions

We assume here that agent i's production function is $f_i(x_i) = x_i$ and that the effort cost function is linear, that is, $c_i(x) = c_i x$ where $c_i \in \mathbb{R}_+$ and $c_i \leq c_{i+1}, i \in \{1, ..., n-1\}$. We refer to all agents with a marginal effort cost c_i as i-type agents. In order to analyze the subgame perfect equilibrium in this case, we begin with the analysis of the second stage and go backwards to the first one (the necessary conditions for existence of a unique equilibrium in the second stage is satisfied according to Szidarovszky and Okuguchi 1997 and Yamazaki 2008).

4.1 The second stage

In the second stage, the agents in contest $k, k = 1, 2, ..., m$ compete against each other. The maximization problem of agent $i \in N_k$ is

$$\max_{x_i} v_k \frac{x_i}{\sum_{j \in N_k} x_j} - c_i x_i, \tag{3}$$

When agents are heterogeneous, some of the agents with high marginal effort costs may stay out of the competition in equilibrium. Without loss of generality, we assume that all the agents are active in contest k, otherwise, only the active agents will be considered. Then, the solution of (3) yields the following well-known result:

Proposition 3 *In the second stage of the model of two-stage parallel contests, if $i \in N_k$, then agent i's equilibrium effort is*

$$x_i = p_i \sum_{j \in N_k} x_j, \tag{4}$$

where p_i is agent i's probability of winning and is given by

$$p_i = 1 - \frac{(n_k - 1)c_i}{\sum_{j \in N_k} c_j}, \tag{5}$$

and agent i's expected payoff is given by

$$u_i = v_k (p_i)^2. \tag{6}$$

So far, we have analyzed the agents' equilibrium efforts in the second stage. Next, we analyze the first stage where each agent chooses the contest in which he wants to compete

4.2 The first stage

First, we assume that the agents are homogeneous, which means that their marginal costs of effort are the same, i.e., $c_i = c_j, \forall i, j \in N$. By the symmetry of the agents, since the equilibrium in each contest is unique (see Szidarovszky and Okuguchi 1997 and Yamazaki 2008), in the second stage, the agents in each contest choose the same effort, and then each of them has the same probability to win the contest. When agents are homogeneous, the first stage of our parallel contest model corresponds to a potential game, specifically a congestion game. Potential games, particularly congestion games, are well known to have a pure-strategy equilibrium (Rosenthal 1973, Monderer and Shapley 1996). Consider the set $M = \{1, 2\}$ of two contests such that the n homogeneous agents have to choose to compete either in contest 1 or contest 2 . Then, the potential function is

$$P(n_1, n_2) = \sum_{i=1}^{n_1} \frac{v_1}{(i)^2} + \sum_{i=1}^{n_2} \frac{v_2}{(i)^2},$$

when n_j agents compete for a prize of v_j in a contest $j, j = 1, 2$. If there are n_j agents in contest j and an agent moves from contest 1 to contest 2, there exists

$$P(n_1, n_2) - P(n_1 - 1, n_2 + 1) = u_i(n_1, n_2) - u_i(n_1 - 1, n_2 + 1)$$

where agent i is the one who switched between the contests, and u_i is his payoff function, which is given by (6).

Thus, in our model of two-stage parallel contests with two contests and n homogeneous agents, there is a subgame perfect equilibrium in which, in the first stage, n_k^* agents choose to compete in contest $k = 1, 2$. We will now characterize this pure-strategy equilibrium. Assume that $v_1 \geq v_2$ and $\frac{v_1}{(n-1)^2} < \frac{v_2}{4}$ such that both contests 1 and 2 are active. Otherwise, only contest 1 is active, namely, in the first stage, all the agents choose to compete in contest 1. If the agents' expected payoffs in both contests would be the same, then by Proposition 3 we have

$$u_i^1 = v_1(p_i^1)^2 = \frac{v_1}{n_1^2} = \frac{v_2}{n_2^2} = v_2(p_i^2)^2 = u_i^2.$$

In that case, the number of agents in the both contests will be

$$\begin{aligned} \tilde{n_1} &= \frac{n\sqrt{v_1}}{\sqrt{v_1}+\sqrt{v_2}} \\ \tilde{n_2} &= \frac{n\sqrt{v_2}}{\sqrt{v_1}+\sqrt{v_2}}. \end{aligned} \tag{7}$$

Thus, $u_i^1 = u_i^2$ when exactly $\widetilde{n}_1$agents choose to compete in contest 1 and $\widetilde{n}_2$choose to compete in contest 2, but $\widetilde{n}_i, i = 1,2$ are not necessarily integers. Nevertheless, in that case there is a subgame perfect equilibrium in which in the first stage, the agents use pure strategies. To see that, assume that in equilibrium $\tilde{n_1} - 2$ agents choose to compete in contest 1 and $\tilde{n_2} + 2$ agents choose to compete in contest 2 . Then, there exists that $u^1 > u^2$, namely, the agents' expected payoff in contest 1 is larger than in contest 2. But, then, an agent from contest 2 could move to contest 1 and he will have a larger expected payoff than an agent in contest 2. This is because if the allocation of agents is $\tilde{n_1}-1$ in contest 1 and $\tilde{n_2}+1$ agents in contest 2 , there exists that $u^1 > u^2$. This argument reduces the number of possible equilibrium allocations to only two combinations of $\lfloor \tilde{n_i} \rfloor$-the largest integer that smaller than $\tilde{n_i}$, and $\lceil \tilde{n_i} \rceil$ the smallest integer that larger than $\tilde{n_i}$ $i = 1,2$. Hence, in the model of two-stage parallel contests with two contests and n homogeneous agents, there is a subgame perfect equilibrium in which in the first stage n_i^* agents choose to compete in contest $k = 1,2$ where $(n_1^*, n_2^*) \in \{(\lfloor \tilde{n_1} \rfloor, \lceil \tilde{n_2} \rceil), (\lceil \tilde{n_1} \rceil, \lfloor \tilde{n_2} \rfloor)\}$ and $\tilde{n_1}, \tilde{n_2}$ are given by (7).

When there are more than two sets, by the same argument, in the model of two-stage parallel contests with m contests and n homogeneous agents, there is a subgame perfect equilibrium in which n_k^* agents choose to compete in contest $k, k = 1, 2, ..., m$ where $n_k^* = \lfloor \tilde{n_k} \rfloor$ or $n_k^* = \lceil \tilde{n_k} \rceil$, $\sum_{k=1}^m n_k^* = n$ and $\tilde{n_k}$ are given by

$$\tilde{n_k} = \frac{n\sqrt{v_k}}{\sum_{i=1}^m \sqrt{v_i}}. \tag{8}$$

When agents are heterogeneous, the equivalence between our model of parallel contests and congestion games is not clear at all. Nonetheless, we will demonstrate that, even when the agents are heterogeneous, our model of parallel contests remains a congestion game, and thus has a subgame perfect equilibrium in which the agents choose pure strategies in the first stage. For this purpose, consider now a set $N = \{1, 2, 3, ..., n\}$ of n agents and a set $M = \{1, 2, 3, ..., m\}$ of m contests. The agents have heterogeneous marginal costs of effort where $c_i \leq c_{i+1}, i = 1, ..., n-1$. Agents with the same marginal costs of effort c_s will be referred to as s-type agents. Since the agents are heterogeneous there might be situations in which weak agents, namely agents with high marginal cost of effort will choose to stay out of the contest in equilibrium. In any subgame perfect equilibrium in which in the first stage agents use pure strategies, an agent in any contest has no incentive to move to another contest since by such a move he will not increase his expected payoff. However, the following result demonstrates that agents of the same type in different contests with different prizes never have the same expected payoff in any subgame perfect equilibrium.

Proposition 4 *In a model of two-stage parallel contests with $m \geq 2$ contests and n heterogenous agents, no subgame perfect equilibrium exists in pure strategies in which agents of the same type in different contests with different prizes have the same expected payoff.*

In the following we will show that the model of two-stage parallel contests always has a subgame perfect equilibrium in which in the first stage agents choose pure strategies. Note that such an equilibrium in two-stage parallel contests is based on two conditions. The first is that in the second stage no agent in any contest k wants to change his effort given the other efforts of the agents. The second condition is that in the first stage, no agent wants to move from his contest to another one given the agents' efforts in the second stage. The next result is crucial regarding the existence of a subgame perfect equilibrium with pure strategies in the first stage when the agents are heterogeneous.

Lemma 3 *In every contest $k \in M$, an i-type agent can be replaced by a number $n_{i-j}(k) \in \mathbb{R}_{++}$ of j-type agents such that the expected payoffs of all the other agents in contest k, independent of their types, will not be changed.*

The number $n_{i-j}(k)$ will be referred to as the "exchange ratio," which denotes how many j-type agents are equivalent to one i-type agent in contest k, namely, if we exclude one i-type agent from contest k and instead add $n_{i-j}(k)$ j-type agents, the expected payoff of all the other agents in contest k, independent of their types, are not changed. Furthermore, this equivalence relation is transitive. For example, assume that an h-type agent has the same impact on the other agents' expected payoffs as four m-type agents. This indicates that if one h-type agent is replaced by four m-type agents, the expected payoff of all the agents in the contest will be unchanged. Assume also that every m-type agent has the same impact on the other agents' expected payoffs as two l-type agents. Then, one h-type agent is equivalent to three m-type agents and two l-type agents, and so on. By this argument, a model of two-stage parallel contests with s-type agents is equivalent to a model with only s-type agents where all other type-agents are transferred to s-type agents according to the exchange ratios. Because each agent competes in a contest with homogeneous agents, and because a model of two-stage parallel contests with homogeneous agents is equivalent to a congestion game, a model of two-stage parallel contests with heterogeneous agents is also equivalent to a congestion game, resulting in a subgame perfect equilibrium with pure strategies.

Proposition 5 *In a model of two-stage parallel contests with m contests and n heterogeneous agents with linear costs of effort, there is always a subgame perfect equilibrium in which in the first stage the agents choose pure strategies.*

It is worth noting that even if the exchange ratios $n_{i-j}(k)$, $i, j = 1, ..., n$, $j \neq i$ are not integers, Proposition 5 proves that the proportion of agents in each contest is the same independent of the agent type. Thus, each move of agents across contests brings the game closer to its equilibrium proportions of agents, until no agent wants to switch his position. In addition, the subgame perfect equilibrium in our parallel two-stage model is typically not unique. The rationale is followed by the "exchange ratio," which states that it is possible to exchange $n_{i-j}(k)$ j-type agents by one i-type agent from another contest without affecting the expected payoffs of all agents participating in the two-stage parallel contests.

5 Conclusion

Parallel contests where each agent can participate in only one contest might be complicated especially when we assume that contests are asymmetric in size as well as in the value of the prize(s) and the agents are also asymmetric in their costs of effort. In this paper, we analyze the subgame perfect equilibrium of two-stage parallel contests with a finite number of agents under asymmetric assumptions on the contests as well on the agents.

We first assumed non-linear costs of effort and a generalized Tullock contest success function. Moreover, we assumed that these costs of efforts are given in the most general form. In that case, since we do not know the exact form of the costs of effort, we cannot explicitly calculate the equilibrium strategies. However, we do know the agents' preferences among different allocations, and accordingly, we are able to develop an algorithm that converges to equilibrium, namely, the algorithm allocates the agents across the contests such that no agent wants to move from his position. Thus, we do not just prove that there is a subgame perfect equilibrium with pure strategies in both stages, but also provide an efficient mechanism to find it.

Next, we consider two-stage parallel contests with linear effort costs and the Tullock contest success function. When the agents are homogeneous, our model is equivalent to a congestion game with a pure-strategy equilibrium, and we explicitly define the potential function. Thus, in that case, our model of parallel contest has a subgame perfect equilibrium in which agents in the first stage choose their contests using pure strategies. When the agents are heterogeneous, it is unclear whether our model is equivalent to a congestion game. However, we show that, from the perspective of each type of agent, our two-stage parallel contest model with heterogeneous agents is equivalent to another model with homogeneous agents, and thus to a congestion game. Thus, we can conclude that even when the players are heterogeneous, there exists a subgame perfect equilibrium in which they choose their contests using pure strategies in the first stage.

Our model of two-stage parallel contests can be extended in several directions. For example, we can assume similarly to Damiano et al. (2010, 2012), that the prize in each contest is not fixed but rather is determined by the average type (ability) of the agents in each contest. Then, if an agent joins another contest he might increase the expected payoffs of the agents in that contest. This is in contrast to our present model in which when an agent joins another contest he necessarily decreases the expected payoffs of all the agents there since the competition becomes more intensive.

6 Appendix

6.1 Proof of Lemma 1

If an l-type agent wants to move from contest i to contest j, his expected payoff in contest j has to be greater than his current expected payoff in contest i. In response, an h-type agent who did not want to move at the beginning, might now want to move from contest j to contest i. In that case, at most one h-type agent will move to contest i. The reason is that if an additional h-type agent wants to move to contest i, he would do it before the move of the l-type agent, since after that one h-type agent joined contest i instead of the l-type agent who left contest i, this contest is now less attractive for every agent. Therefore, any move of an l-type agent will be followed by at most one move of an h-type agent.

6.2 Proof of Proposition 1

We first show that when we apply Algorithm 1 agents who leave a contest will never return to it. In Step 3, a single l-type agent gets the option to move from contest 2 to contest 1. He decides to move between the contests if his expected payoff in contest 1 will be higher than his current expected payoff in contest 2. By Lemma 1, this move of an l-type agent yields at most a move of a single (or non-) h-type agent in the opposite direction from contest 1 to contest 2. Then if an h-type agent moves to contest 2, the l-type agent who just moved to contest 1 (or any other agent of the same type) will not want to return to contest 2 since if he will return to contest 2 his expected payoff will be smaller than his expected payoff when he left. By this argument, the l-type agents will stop moving at some stage. Thus, if the l-type agents will stop moving between contests 1 and 2, the h-type agents will stop moving between the contests as well. Therefore, the situation in which none of the l-type agents wants to move between the contests always occurs since l-type agents never want to return to their initial position, and then the algorithm terminates. In such a case we say that Algorithm 1 converges to equilibrium.

6.3 Proof of Lemma 2

The move of an l-type agent from contest i to contest j affects only those contests in which contest i becomes more attractive to other agents, and contest j becomes less attractive since an additional agent joined. Furthermore, by Lemma 1, following the move of an l-type agent from contest i to contest j, at most one h-type agent who did not want to move earlier, might want to move to contest i, but does not want to move to contest j. Now, at most one h-type agent might want to move to the initial contest of the h-type agent who joined contest i and so on. Thus, we can conclude that at most $m-1$ h-type agent will follow the move of an l-type agent.

6.4 Proof of Proposition 2

We first claim that by Algorithm 2, if an l-type agent leaves contest i at some point, he will not return to this contest throughout the remaining steps of the algorithm. The reason is that when an l-type agent wants to leave contest i, this indicates that there is another contest in which his expected payoff will be larger than his current expected payoff in contest i. By Lemma 2, we know that no more than one h-type agent will join contest i if the l-type agent leaves this contest. If an h-type agent moves to contest i, the expected payoff of an l-type agent will be smaller than if he would return to contest i. This argument holds independent of the stage of the algorithm, and therefore an l-type agent will never return to a contest that he left.

According to Lemma 2 and the symmetry of the h-type agents, in step 2, after that an l-type agent moves, only one h-type agent will move. Furthermore, this h-type agent will move to the previous contest of the l-type agent. As a result of the above argument, l-type agents will never return to the contests that they left, and therefore, h-type agents as well will never return to the contests that they left. This implies that, at most, all l-type agents will move to all other contests until each agent has visited all the contests. Thus, the algorithm converges to equilibrium, and in the first stage the model of the parallel contests has a subgame perfect equilibrium with pure strategies.

6.5 Proof of Proposition 3

The FOC of agent i's maximization problem (3) is

$$v_k \frac{\sum_{j=1}^{n_k} x_j - x_i}{\left(\sum_{j=1}^{n_k} x_j\right)^2} \leq c_i. \tag{9}$$

Without loss of generality, assume that for all agents in N_k the FOC is non-negative, and then we obtain:

$$\sum_{i \in N_k} \frac{v_k(\sum_{j \in N_k} x_j - x_i)}{(\sum_{j \in N_k} x_j)^2} = v_k \left(\sum_{i \in N_k} \frac{\sum_{j \in N_k} x_j}{(\sum_{j \in N_k} x_j)^2} - \frac{x_i}{(\sum_{j \in N_k} x_j)^2} \right) =$$

$$v_k \left(\frac{n_k}{\sum_{j \in N_k} x_j} - \frac{\sum_{j \in N_k} x_j}{(\sum_{j \in N_k} x_j)^2} \right) = v_k \left(\frac{n_k}{\sum_{j \in N_k} x_j} - \frac{1}{\sum_{j \in N_k} x_j} \right) = \sum_{i \in N_k} c_i.$$

Thus, we have

$$v_k \left(\frac{n_k - 1}{\sum_{j \in N_k} x_j} \right) = \sum_{i \in N_k} c_i.$$

which implies that the total effort in contest k is

$$\sum_{j \in N_k} x_j = \frac{v_k(n_k - 1)}{\sum_{i \in N_k} c_i}. \tag{10}$$

Inserting (10) into (9) yields

$$\frac{v_k\left(\frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j} - x_i\right)}{\left(\frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j}\right)^2} = c_i,$$

or, alternatively,

$$\frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j} - x_i = v_k c_i \left(\frac{n_k-1}{\sum_{j\in N_k} c_j}\right)^2.$$

Thus, agent i's equilibrium effort in contest k is

$$x_i = \frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j}\left(1 - \frac{(n_k-1)c_i}{\sum_{j\in N_k} c_j}\right). \tag{11}$$

By (10) and (11) agent i's probability of winning in contest k is

$$p_i = \frac{x_i}{\sum_{j\in N_k} x_j} = \frac{\frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j}\left(1 - \frac{(n_k-1)c_i}{\sum_{j\in N_k} c_j}\right)}{\frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j}} = 1 - \frac{(n_k-1)c_i}{\sum_{j\in N_k} c_j}. \tag{12}$$

Then, by (11) and (12), agent i's expected payoff is

$$\begin{aligned}
u_i &= v_k p_i - x_i \\
&= v_k\left(1 - \frac{(n_k-1)c_i}{\sum_{j\in N_k} c_j}\right) - \frac{v_k(n_k-1)}{\sum_{j\in N_k} c_j}\left(1 - \frac{(n_k-1)c_i}{\sum_{j\in N_k} c_j}\right) \\
&= v_k\left(1 - \frac{(n_k-1)c_i}{\sum_{j\in N_k} c_j}\right)^2 = v_k(p_i)^2.
\end{aligned}$$

6.6 Proof of Proposition 4

In order to prove that agents with the same type from different contest do not have the same expected payoff in equilibrium, without loss of generality, we consider two contests and two types of agents. In the case of more than two contests and two types of agents, the following argument holds for any choice of two contests from the set of contests.

The agents have marginal costs of effort either c_h or c_l. Assume that h_i agents with a marginal effort cost of c_h chooses to participate in contest $i, i = 1, 2$ where $h_1 + h_2 = h$. Similarly, l_i agents with a marginal effort cost of c_l chooses to participate in contest $i, i = 1, 2$ where $l_1 + l_2 = l$. We assume that the prizes in both contests are different $v_1 \neq v_2$ and the we will show that agents of the same type could not have the same expected payoff in both contests. For this purpose, suppose that agents of the same type have the same expected payoff in two different contests. Then, by Proposition 3 the agents' expected payoffs in contest 1, $u_j^1, j = l, h$ and in contest 2, $u_j^1, j = l, h$ satisfy:

$$u_h^1 = v_1 \left(1 - \frac{(h_1 + l_1 - 1)c_h}{h_1 c_h + l_1 c_l}\right)^2 = u_h^2 = v_2 \left(1 - \frac{(h_2 + l_2 - 1)c_h}{h_2 c_h + l_2 c_l}\right)^2, \tag{13}$$

and

$$u_l^1 = v_1 \left(1 - \frac{(h_1 + l_1 - 1)c_l}{h_1 c_h + l_1 c_l}\right)^2 = u_l^2 = v_2 \left(1 - \frac{(h_2 + l_2 - 1)c_l}{h_2 c_h + l_2 c_l}\right)^2. \tag{14}$$

We can divide both equations (13) and (14) by each other and obtain that,

$$\frac{c_h - (c_h - c_l)l_1}{c_l + (c_h - c_l)h_1} = \frac{c_h - (c_h - c_l)l_2}{c_l + (c_h - c_l)h_2}.$$

From the last equation we get

$$h_2 = \frac{c_h h_1 + c_l l_1 - c_l l_2 - c_h h_1 l_2 + c_l h_1 l_2}{c_h - c_h l_1 + c_l l_1}.$$

Since $h_1 + h_2 = h$, we have

$$h_2 = \frac{c_h h + c_l l_1 - c_l l_2 - c_h h l_2 + c_l h l_2}{2c_h - c_h l_1 - c_h l_2 + c_l l_1 + c_l l_2}, \tag{15}$$

and

$$h_1 = \frac{c_h h - c_l l_1 + c_l l_2 - c_h h l_1 + c_l h l_1}{2c_h - c_h l_1 - c_h l_2 + c_l l_1 + c_l l_2}. \tag{16}$$

Inserting (15) and (16) into (13), yields

$$v_1 \left(\frac{(l_2+l_1)c_l - (l_2+l_1-2)c_h}{c_h h + (l_2+l_1)c_l} \right)^2 = v_2 \left(\frac{(l_2+l_1)c_l - (l_2+l_1-2)c_h}{c_h h + (l_2+l_1)c_l} \right)^2 .$$

The last equation implies that $v_1 = v_2$, and this contradicts our assumption that $v_1 \neq v_2$.

6.7 Proof of Lemma 3

Consider contest $i \in M$ from which an agent of type h $(c = c_h)$ is excluded and is replaced by $x \in \mathbb{R}_{++}$ agents of type l $(c = c_l)$. We will find the value of x such that the expected payoffs of all the agents in contest i will not be changed by this replacement.

By Proposition 3, the expected payoff of an agent of type k $(c = c_k)$ in contest i is

$$v_i \left(1 - \frac{(n_i - 1)c_k}{\sum_{i \in N_i} c_i} \right)^2 \tag{17}$$

Suppose that one h-type agent leaves contest i and $x \in R_{++}$ l-type agents join it. Then, the expected payoff of a k-type agent is

$$v_i \left(1 - \frac{(n_i - 1 - 1 + x)c_k}{\sum_{i \in N_i} c_i - c_h + x c_l} \right)^2 \tag{18}$$

A comparison of (17) and (18) yields

$$v_i \left(1 - \frac{(n_i - 1)c_k}{\sum_{i \in N_i} c_i} \right)^2 = v_i \left(1 - \frac{(n_i - 1 - 1 + x)c_k}{\sum_{i \in N_i} c_i - c_h + x c_l} \right)^2$$

This implies that in order that the expected payoff of a k-type agent will be not changed, the number of l-type agents that should join contest i instead of the h-type agent that left this contest is

$$x = \frac{(n_i - 1)c_h - \sum_{i \in N_i} c_i}{(n_i - 1)c_l - \sum_{i \in N_i} c_i} \tag{19}$$

We can see that the number $x = n_{h-l}$ does not depend on the value of c_k. This yields that if n_{h-l} l-type agents replace one h-type agent, then the expected payoffs of all the agents in contest i, independent of their types, are not changed.

6.8 Proof of Proposition 5

As we showed the model of two-stage parallel contests is equivalent to other model of two-stage parallel contests in which all the agents have the same type of this agent's type. In such a model of two-stage parallel contests with homogeneous agents, we already proved that in the first stage there is a subgame perfect equilibrium with pure strategies. In the following, we show that independent of the choice of one agent's type, the proportion of agents in each contest is the same. Note that by (8), if we consider only h-type agents, then the proportion of agents in contest $k, k = 1, ..., m$ is

$$\frac{\frac{n_h \sqrt{v_k}}{\sum_{i=1}^{m} \sqrt{v_i}}}{n_h} = \frac{\sqrt{v_k}}{\sum_{i=1}^{m} \sqrt{v_i}} \tag{20}$$

where n_h is the number of h-type agents in the model of parallel contests with only agents of this type. We can see that the proportion of agents in each contest is the same, independent of the choice of the agent's type. Thus, any allocation of agents that after the transfer of the agents to a single type by the exchange ratios will result in the same proportion of agents as in (20) will be in equilibrium, namely, no agent will want to change his position.

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