ALL-PAY AUCTIONS WITH HETEROGENEOUS PRIZES AND PARTIALLY ASYMMETRIC PLAYERS

Chen Cohen, David Lagziel, Ofer Levi, and Aner Sela

Discussion Paper No. 20-10

July 2020

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

All-Pay Auctions with Heterogenous Prizes and Partially Asymmetric Players

Chen Cohen[*], David Lagziel[†], Ofer Levi[‡], Aner Sela[§]

May 21, 2020

Abstract

We study complete information all-pay contests with n players and two heterogeneous prizes with distinct values. Among the players, $n-1$ are symmetric (i.e., they evaluate the prizes in a similar manner), whereas the remaining player has different valuations than his opponents for each of the prizes. Our analysis focuses on the equilibrium profiles and expected payoffs for the case of three players, and we partially extend our analysis for cases with additional players. Our results show that in all-pay auctions with heterogeneous prizes, the ordering of the players according to their expected payoffs in equilibrium might vary significantly, depending on both prizes. In particular, a relatively high first prize does not necessarily entail a high (or even positive) expected payoff compared to a relatively high second prize.

JEL Classification numbers: D44, D82, J31, J41.

Keywords: All-pay contests, multiple prizes, complete information.

[*]Cohen: Department of Public Policy and Administration, Ben-Gurion University of the Negev, Beer-Sheva, Israel.

[†]Lagziel (corresponding author): Economics Department, Ben-Gurion University of the Negev, Beer-Sheva, Israel.

[‡]Levi: Department of Mathematics and Computer Science, The Open University of Israel, Raanana, Israel.

[§]Sela: Economics Department, Ben-Gurion University of the Negev, Beer-Sheva, Israel.

1 Introduction

In our society, contests in which multiple prizes are awarded are quite ubiquitous. Examples include employees who exert effort for the purpose of promotions in organizational hierarchies, students who compete over grades (and the adjacent ranking), political competitions for ranked places in parliamentary systems, and obviously sports events where athletes compete over medals or various monetary prizes. Such contests with multiple prizes can be modeled in several ways, one of the most well-known being the all-pay auction.[1] In this contest form, the players with the highest bids receive the prizes, but all the players, including those who do not win anything, bear the costs of their bids.

Thus far, most of the contest literature has focused on single-prize all-pay auctions where the highest bidder is awarded the prize (known as the winner-take-all contest), whereas studies concerning all-pay auctions with multiple prizes, especially heterogenous ones, are rather neglected. The reason for this is quite straightforward - there is a substantial difference, in terms of complexity, between the analysis of a single-prize all-pay auction or even one with several identical prizes, and that of an all-pay auction with heterogenous prizes. For example, in a complete information single-prize contest, the player with the highest valuation wins the prize with the highest probability and has the highest expected payoff. Moreover, if one player has a strictly higher valuation for winning compared to all the other players, then only he has a positive expected payoff, while all others have an expected payoff of zero (see Baye et al. 1996). Likewise, in a complete information all-pay auction with $k \geqslant 2$ identical prizes, the players with the k highest values gain positive expected payoffs, where a higher private value entails a higher expected payoff.

In contrast, when there are at least two heterogeneous prizes and the ordering of the players' valuations vary across prizes, the identity of the winners for each of the prizes, as well as the order of the players' expected payoffs, are ambiguous. A priori, it is unclear how one should evaluate the winning probability and expected payoff of a player with a high value for the first prize and a low value for the second one to those of a player with a lower value for the first prize and a higher value for the second one.

[1] See, among others, Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993), Amman and Leininger (1996), Krishna and Morgan (1997), Che and Gale (1998), Lizzeri and Persico (2000), Siegel (2009), Sela (2012), Hart (2016), Einy et al. (2017), and Lu and Parreiras (2017).

To illustrate the above argument, consider an all-pay auction with two heterogeneous prizes and four players, all with different values for both prizes, while the sum of their values is the same. In addition, for each of the players, the value for the first prize is higher than the value for the second one. In such a case, it can be shown that the player who has the highest value for the first prize and the lowest value for the second prize gains the highest expected payoff. Moreover, the player who has the highest value for the second prize and the lowest value for the first prize has the second highest expected payoff. This result is somewhat puzzling since each of the four players can potentially win each of the prizes.

In this paper, we try to shed light on the players' behavior in all-pay auctions with heterogenous prizes (i.e., the players' valuations vary across prizes). We assume that each player has a higher value for the first prize than for the second one, but the player with the highest value for each prize is not necessarily the same. In order to deal with the players' behavior in such complex contests, we assume that each of the n players is one of two types such that there are $n-1$ symmetric players all of whom have the same valuations for each of the prizes (to be clear, all have different values for the first and second prize), while the remaining player has different values for both prizes compared to his opponents (note that all valuations are public). The contest evolves as follows. First, each player chooses a bid. Next, the player with the highest bid wins the first prize, and the player with the second highest bid wins the second prize. Finally, all players bear the cost of their bids, independently of their winning status.

It turns out that the most complex scenario in our model is when there are only three players, namely, two symmetric players and a single asymmetric one. The rationale is that if there are more than two symmetric players, in any equilibrium their expected payoff will necessarily be zero since the number of prizes is smaller than the number of the symmetric players. On the other hand, if there are only two symmetric players, they might have positive expected payoffs. Thus, most of the present paper focuses on three players, while providing some generalizations for $n > 3$ players.

In contrast to the equilibrium profiles in the all-pay auction with a single prize in which the players' efforts (or bids) are derived from a common support, in the all-pay auction with two heterogenous prizes, they are not necessarily derived from the same support. Moreover, the supports of the players' strategies are not necessarily convex, namely, they include gaps such that the players' mixed strategies (distributions over bids) are not strictly increasing. Thus, we divide our analysis into five cases according to the relationship between the players' values for the prizes. For each case, we provide sufficient conditions ensuring that the players' distributions of bids are strictly increasing, and then analyze the players' equilibrium bids. Since we provide an explicit solution of the players' equilibrium strategies, we are able to calculate the players' expected payoffs as well.

A player type with the higher (lower) value for the first prize will be referred to as an S-type player (W-type player, respectively). Our equilibrium analysis shows that, depending on the players' values for the prizes, either the W-type player(s) or the S-type player(s) has a positive expected payoff, but both types never have positive expected payoffs at the same time. Furthermore, if the S-type player is the asymmetric player, he is the only one who has a positive expected payoff. On the other hand, if the S-type players are the symmetric players, the asymmetric W-type player does not necessarily have an expected payoff of zero. In that case, depending on his value for the second prize, he might be the only player with a positive expected payoff although he is allegedly considered the weaker player. Hence, we conclude that although the values for the first (larger) prize have the greatest effect on the identity of the players with positive expected payoffs, the value of the second prize might have a non-negligible effect. In other words, the order of the players according to their expected payoffs depends on the valuations of all the prizes.

We then consider the all-pay auction with $n > 3$ players. Although we do not provide a complete analysis of this case, we do show how our results for three players can be generalized. We prove that the asymmetric player may have a positive expected payoff, whether or not he has the higher value for the first prize. On the other hand, the $n-1$ symmetric players will always have an expected payoff of zero. This is due to the fact that even if these players have higher values for either the first prize or for both prizes, the competition among them yields an expected payoff of zero.

As mentioned earlier, we are not the first to deal with the all-pay auction with heterogeneous prizes. Incomplete information auctions where only the common distribution of private values is commonly known has been studied, among others, by Moldovanu and Sela (2001, 2006), Moldovanu et al. (2012), and Liu and Lu (2017). Complete information auctions with identical prizes and linear costs in which the players' values are common knowledge has been studied by Barut and Kovenock (1998), and Clark and Riis (1998). Siegel

(2010) analyzed such contests with nonlinear costs. Bulow and Levin (2006) studied all-pay auctions with heterogenous prizes and linear costs in which the first-order differences in successive prizes are constants, and Gonzalez-Diaz and Siegel (2013) extended this work by allowing nonlinear costs. Later, Xiao (2016) investigated another version of the all-pay auction with heterogenous prizes in which either the ratio of successive prizes is constant or the second-order differences are a positive constant.

The model most similar to ours, namely, with two symmetric players and one asymmetric player who compete over two prizes, was studied by Dahm (2018). However, this work places several restrictions on the prizes' values so that the value of the second prize is zero for the asymmetric player. Thus, Dahm is mainly interested in one prize, and considered the symmetric players' values for the first prize to be larger than the respective asymmetric player's value. Xiao (2018) also studied all-pay auctions with two nonidentical prizes, but he assumed that the sequence of prizes is either convex or concave, that is, the second-order differences (among prizes) are either a positive or a negative constant. Therefore, in these studies the heterogeneity among the prizes is limited due to some special properties imposed on the sequence of the prizes' valuations.[2] Furthermore, it is assumed that the ratio of the values for every pair of prizes is the same for all the players who differ from each other by their ability or, alternatively, their bid cost. In other words, the players technically have the same value for each prize, but due to the heterogenous cost functions, they differ in their expected payoff for winning. Nevertheless, the ratio between the values of each pair of prizes is identical among all the players. In contrast, in our model the players differ in their prize valuations and in the ratios among these valuations. In other words, the heterogeneity of the prizes between the two types of players in our model is unrestricted.

The rest of the paper proceeds as follows. In Section 2, we introduce the model, and in Section 3, we present general properties of the equilibria. In Section 4, we analyze the equilibrium strategies with three players and two heterogenous prizes where the players' supports are convex. In Section 5, we illustrate an equilibrium with non-convex supports, and generalize our equilibrium analysis to contests with more than three players. Section 6 concludes. Most of the proofs appear in the Appendix.

[2] In Olzewski and Siegel (2016) the heterogeneity of the prizes is not limited, but they assume that the numbers of prizes and players go to infinity.

2 The model

We first consider a two-prize all-pay auction with three players. There are two types of players who differ in their prize valuations: the 'strong' type, denoted by S, has valuations s_1 and s_2 for the first and second prize, and the 'weak' type, denoted by W, has valuations w_1 and w_2 for the first and second prizes. Note that $s_1 > s_2 \geqslant 0$ and $w_1 > w_2 \geqslant 0$. We refer to the types as strong and weak since $s_1 > w_1$ is the basic assumption that affects the type which has a positive expected payoff in equilibrium. Unless stated otherwise, we assume that among the three players, there are two S-type players and one W-type player.

The bid set of each player is $\mathbb{R}_+$ and, without loss of generality, we can assume that the bids of S-type and W-type players are bounded on $[0, s_1]$ and $[0, w_1]$, respectively. A strategy of a player is a distribution over the set of feasible bids (i.e., the CDF) which is denoted by F_T for every $T \in \{S, W\}$. We denote the random bids of the S-type and W-type players by $X_S \in I_S$ and $X_W \in I_W$, where I_S and I_W are the relevant supports. For clarity of exposition, the analysis is confined to symmetric equilibria with respect to the players' types.

Under the mentioned assumptions and given a strategy profile (F_S, F_W), the expected payoffs of both types under a bid of $x \in \mathbb{R}$ are

$$\begin{aligned} U_S(x|F_S, F_W) &= s_1 F_S(x) F_W(x) + s_2[F_W(x)(1 - F_S(x)) + F_S(x)(1 - F_W(x))] - x \\ &= \left[(s_1 - 2s_2)F_S(x) + s_2\right] F_W(x) + s_2 F_S(x) - x \end{aligned} \tag{1}$$

and

$$\begin{aligned} U_W(x|F_S, F_W) &= w_1 F_S^2(x) + 2w_2 F_S(x)(1 - F_S(x)) - x \\ &= (w_1 - 2w_2)F_S^2(x) + 2w_2 F_S(x) - x \end{aligned} \tag{2}$$

Note that the expected payoffs do not account for possible ties since ties occur with 0-probability in equilibrium.

3 General properties of equilibria

We first introduce some general properties of the equilibrium profile (F_S, F_W) when there are several S-type players and one W-type player.

Lemma 1 *In a symmetric equilibrium, F_S has no atoms in $[0, s_1)$ and F_W has no atoms in $(0, w_1)$.*

Proof. We begin with the CDF F_S. Assume, by contradiction, that there exists a symmetric equilibrium where all S-type players support some atom $a \in [0, s_1)$. There are at least two S-type players, so a tie occurs with positive probability, and a symmetric tie-breaking rule dictates a final allocation. Now consider an infinitesimal and unilateral upward-deviation of an S-type player, from a to $a + \epsilon < s_1$. On the one hand, this deviation increases the player's cost by an infinitesimal amount, but on the other, the expected prize increases by a strictly positive and relatively high amount due to the increased probability of winning without the need to split the prize according to some tie-breaking rule.[3] Thus, in a symmetric equilibrium, the bids' distributions of S-type players have no atoms in $[0, s_1)$.

For the CDF F_W, we assume, by contradiction, that there exists a symmetric equilibrium in which the W-type player supports some atom $a \in (0, w_1)$. Since a cannot be an atom of X_S, either there exists some small $\epsilon > 0$ such that $\Pr(X_S \in (a - \epsilon, a)) > 0$, or there exists $\epsilon^* > 0$ such that $\Pr(X_S \in (a - \epsilon, a)) = 0$ for every $\epsilon \in (0, \epsilon^*)$. If the latter is the case, then the W-type player has a profitable deviation downwards. Specifically, for some $\epsilon > 0$, bids in $(a - \epsilon, a)$ are not supported by the S-type players, so the W-type player can shift his atom from a to $a - \frac{\epsilon}{2}$ such that the probability of getting a prize is not affected while the cost decreases. If, however, there exists some small $\epsilon > 0$ such that $\Pr(X_S \in (a - \epsilon, a)) > 0$, then any of the S-type players can shift bids from this small interval upwards to $a + \epsilon'$, for some small $\epsilon' > 0$. Such a deviation increases the expected payoff by strictly increasing the probability of winning the first prize, while the increased cost is infinitesimal. Thus, we can conclude that this cannot be an equilibrium, and F_W has no interior atoms in equilibrium. ■

Corollary 1 *In a symmetric equilibrium, if $\Pr(X_W \in [0, \epsilon)) > 0$ for any $\epsilon > 0$, $U_W(x|F_S, F_W) = 0$ for any $x \in I_W$.*

[3] The tie-breaking rule does not have to be symmetric, and any rule would motivate at least one player to shift the private bid upwards.

The proof is omitted since it is a straightforward conclusion from the fact that F_S is non-atomic at 0. Namely, since the payoffs are right-side continuous and without an atom at 0 of an S-type player, then the point-wise expected payoff of the W-type player converges to zero when a bid x approaches 0. Therefore, by the indifference principle, the expected payoff must be zero.

Lemma 2 *In a symmetric equilibrium, for every open interval $I \subseteq \mathbb{R}_{++}$ such that $\Pr(X_W \in I) > 0$, it follows that $\Pr(X_S \in I) > 0$.*

Proof. Fix a symmetric equilibrium (F_S, F_W). Assume, by contradiction, that there is an open interval I such that $\Pr(X_W \in I) > 0 = \Pr(X_S \in I)$. If the W-type player has an atom $a \in I$, then there exists a strictly profitable deviation downwards from a to $a' \in (\inf I, a) \subset I$ since the probability of winning a prize does not change while the realized cost strictly decreases. Moreover, even if the W-type player has no atoms in I, then the player can shift a positive-probability set of values (from I) downwards in a similar manner to the atom shift, while remaining within I. Again, this would not change the probability of winning a prize, whereas the realized cost strictly decreases. Therefore, we conclude that this cannot be an equilibrium since the W-type player always has a strictly profitable deviation. ∎

Remark 1 *The last lemma suggests that for every symmetric equilibrium in which the random bids X_S of the S-type players and the random bid X_W of the W-type player are supported on I_S and I_W, respectively, then $I_W \subseteq I_S$ up to a zero-measure deviation of the S-type players.*

Lemma 3 *In a symmetric equilibrium, I_S is a connected set.*

Proof. Assume, by contradiction, that I_S is not a connected set. By the lack of interior atoms, there exists an open interval $I \subset \mathbb{R}_{++}$ such that $\Pr(X_S \geqslant \sup I) \cdot \Pr(X_S \leqslant \inf I) > 0 = \Pr(X_S \in I)$. By Lemma 2, it follows that $\Pr(X_W \in I) = 0$. Without loss of generality, take I to be the largest possible interval, which suggests that $\Pr(X_S \in [\sup I, \sup I + \epsilon)) > 0$ for any $\epsilon > 0$.

Now consider two scenarios: either the W-type player has an atom at $\sup I$ or he does not have one. If an atom exists, then the W-type player has a profitable deviation downwards, for example from $\sup I$ to $\frac{\inf I + \sup I}{2}$. This follows from the fact that the probability of winning the prize does not change by this shift, while the realized price strictly decreases.

If, however, the W-type player does not have an atom at $\sup I$, then the S-type players have a profitable deviation from bids $x \in [\sup I, \sup I + \epsilon)$ downwards, for example, to $\frac{\inf I + \sup I}{2}$. Again, by the indifference principle, all bids produce the same expected payoff and a shift from $\sup I$ to $\frac{\inf I + \sup I}{2}$ does not entail any decrease in the winning probability, while the price strictly decreases. Thus, we conclude that this cannot be an equilibrium, and I_S is indeed a connected set. ■

4 Three-player contests with one weak and two strong players

We next analyze the equilibrium in the all-pay auction with three players who compete for two heterogeneous prizes. We assume that there are two S-type players and one W-type player. Below, we divide our analysis to four cases A-D, depending on the players' valuations of the prizes.

Remark 2 *Unless stated otherwise, all subsequent proofs are differed to the Appendix.*

4.1 Case A: The weak player stays out of the contest.

The first case depicts an equilibrium where the W-type player stays out of the contest, and the two S-type players compete against each other, so that each wins one of the prizes.

Proposition 1 *In the all-pay auction with two S-type players and one W-type player, if $[s_1 - s_2] \geqslant \max\{w_1, 2w_2\}$, there exists an equilibrium in which both S-type players randomize on the interval $[0, s_1 - s_2]$ according to their cumulative distribution bid function $F_S(x)$ which is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{s_1 - s_2}, & \text{for } 0 \leqslant x \leqslant s_1 - s_2, \\ 1, & \text{for } x \geqslant s_1 - s_2, \end{cases} \tag{3}$$

while the W-type player bids 0 with probability 1. Under this equilibrium, the expected payoff of both S-type players is s_2, while the expected payoff of the W-type player is 0.

The following example illustrates an equilibrium under the conditions of Proposition 1.

Example 1 *Assume that there are two S-type players whose prize valuations are $s_1 = 10$, $s_2 = 5$, and a W-type player whose prizes' valuations are $w_1 = 4$, $w_2 = 2$, so that the conditions of* Proposition 1 *hold. Then, the mixed-strategy equilibrium described in* Proposition 1 (*see* Figure 1) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{5}, & \text{for } 0 \leqslant x \leqslant 5, \\ 1, & \text{for } x \geqslant 5, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x \geqslant 0. \end{cases}$$

The expected payoff of each S-type player is 5, *and that of the W-type player is* 0.

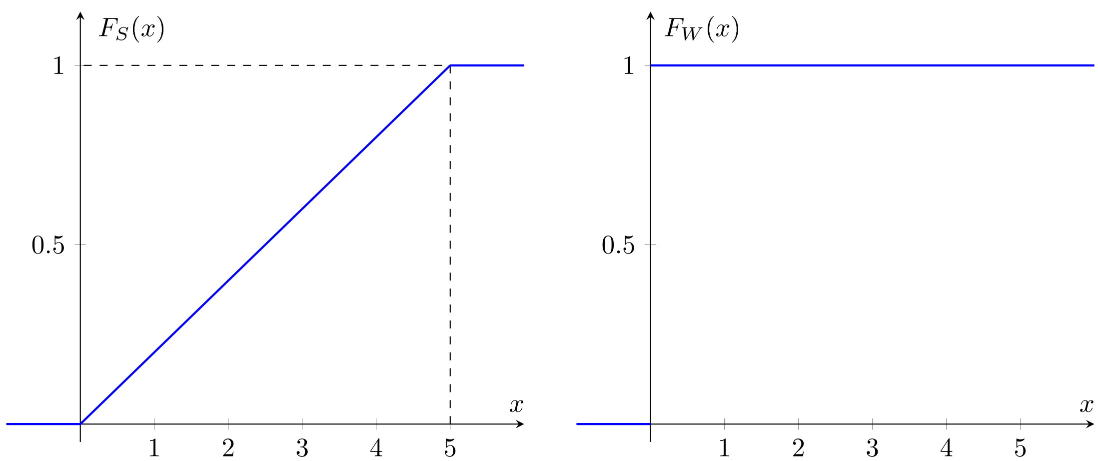

Figure 1: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 10, s_2 = 5, w_1 = 4$, and $w_2 = 2$ (values sustain the condition of Proposition 1).

If the conditions of Proposition 1 are violated, the W-type player may actually compete. Then, we would need to verify that the W-type player's distribution F_W, in equilibrium, is well-defined and specifically, non-decreasing.[4] Thus, below we now provide several sufficient conditions so that the relevant distribution

[4]As we will later show, this assumption is not trivial, since the function F_W, described in Equation (1), might be partially decreasing in some intervals under various parametric assumptions.

is non-decreasing:

$$
\begin{array}{lrcl}
[\mathrm{A}_1] & w_1 & > & 2w_2; \\
[\mathrm{A}_2] & s_1 & > & 2s_2; \\
[\mathrm{A}_3] & 2(w_1 - w_2) & > & (s_1 - s_2); \\
[\mathrm{A}_4] & 2w_2 s_2 & > & s_2^2 + (s_1 - 2s_2)(s_1 - w_1).
\end{array}
$$

4.2 Case B: All the players have symmetric supports

We continue our analysis by describing an equilibrium where all the players compete against each other (i.e., support a strictly positive bid with probability 1), and both types, S and W, use mixed strategies with a common support.

Proposition 2 *In the all-pay auction with two S-type players and one W-type player, if $w_1 \geqslant s_1 - s_2$ and the monotonicity conditions, either $(A_1, \neg A_2, A_3, A_4)$, or $(\neg A_1, A_2, A_3, A_4)$ hold, then there exists an equilibrium in which the players randomize on the interval $[0, w_1]$ according to their non-decreasing cumulative distribution bid functions (F_S, F_W) which are*

$$
F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1}, & \text{for } 0 \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x - s_2 F_S(x) + s_1 - w_1}{(s_1 - 2s_2) F_S(x) + s_2}, & \text{for } 0 \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1. \end{cases} \tag{4}
$$

Under the given equilibrium, the expected payoffs of both S-type players is $s_1 - w_1$, whereas the expected payoff of the W-type player is 0.

The following example shows that the conditions of Proposition 2 are feasible, and that there are parameters which simultaneously support all the required constraints.

Example 2 *Assume that there are two S-type players whose prize valuations are $s_1 = 10, s_2 = 6$, and a W-type player whose prize valuations are $w_1 = 8$, and $w_2 = 3$, so that the conditions of* Proposition 2 *hold.*

Then, a mixed-strategy equilibrium (*see* Figure 2) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{\sqrt{9+2x}-3}{2}, & \text{for } 0 \leqslant x \leqslant 8, \\ 1, & \text{for } x > 8, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x-3\sqrt{9+2x}+11}{9-\sqrt{9+2x}}, & \text{for } 0 \leqslant x \leqslant 8, \\ 1, & \text{for } x > 8. \end{cases}$$

The expected payoff of each S-type player is 4, *while the expected payoff of the W-type player is* 0.

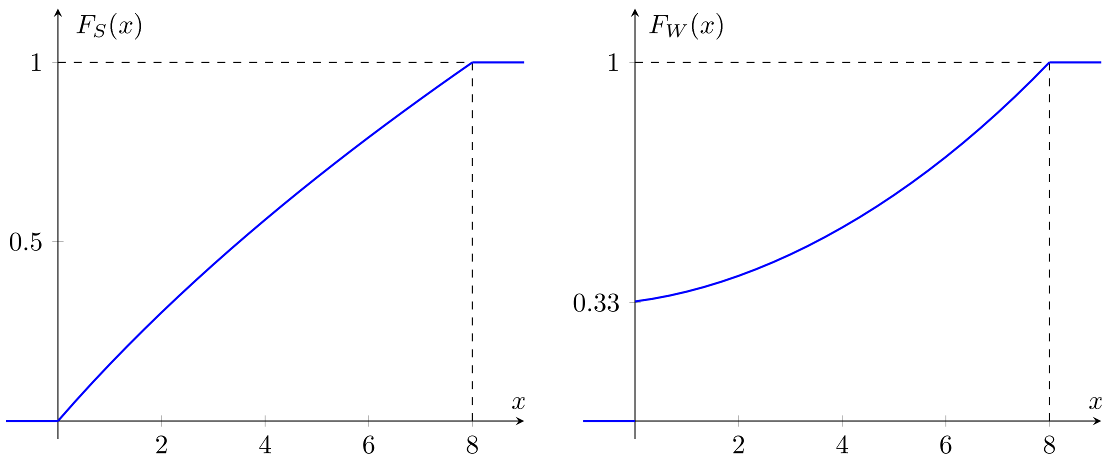

Figure 2: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 10, s_2 = 6, w_1 = 8$, and $w_2 = 3$. Note that these parameters meet the conditions of Proposition 2.

4.3 Case C: The weak player has a one-sided short support

In Case C, both types of players participate in the contest and none of them stays out with a positive probability. However, the W-type player has a shorter support relative to the S-type players, namely, the W-type player's maximal bid is smaller than the maximal bids of the S-type players.

Proposition 3 *In the all-pay auction with two S-type players and one W-type player, if $2w_2 > s_1 - s_2 \geqslant w_1$, $K_1 = s_2 - \frac{[2w_2-(s_1-s_2)]^2}{4(2w_2-w_1)} \geqslant 0$ and the monotonicity condition A_2 holds* (*i.e., if $s_1 > 2s_2$*), *then there exists an equilibrium in which the W-type player randomizes on the interval* $[0, \alpha]$, *where $\alpha = \frac{(2w_2)^2-(s_1-s_2)^2}{4(2w_2-w_1)}$, and the S-type players randomize on the interval* $[0, s_1 - K_1]$ *according to their non-decreasing cumulative* *distribution bid functions* (F_S, F_W) *which are*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1}, & \text{for } 0 \leqslant x \leqslant \alpha, \\ 1 + \frac{x + K_1 - s_1}{s_1 - s_2}, & \text{for } \alpha \leqslant x \leqslant s_1 - K_1, \\ 1, & \text{for } x > s_1 - K_1, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x - s_2 F_S(x) + K_1}{(s_1 - 2s_2) F_S(x) + s_2}, & \text{for } 0 \leqslant x \leqslant \alpha, \\ 1, & \text{for } x > \alpha. \end{cases} \quad (5)$$

Then, the expected payoffs of both S*-type players is* K_1*, and that of the* W*-type player is* 0.

The following example shows that the conditions of Proposition 3 are feasible, and that there are parameters which simultaneously support all needed constraints.

Example 3 *Assume that there are two S-type players whose prize valuations are* $s_1 = 10, s_2 = 4$*, and a* W*-type player whose prize valuations are* $w_1 = 5$*, and* $w_2 = 4$*, so that the conditions of* Proposition 3 *hold. Then, the mixed-strategy equilibrium described in* Proposition 3 (*see* Figure 3) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{4 - \sqrt{16 - 3x}}{3}, & \text{for } 0 \leqslant x \leqslant 7/3, \\ \frac{3x - 1}{18}, & \text{for } 7/3 \leqslant x \leqslant 19/3, \\ 1, & \text{for } x > 19/3, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{3x + 4\sqrt{16 - 3x} - 5}{20 - 2\sqrt{16 - 3x}}, & \text{for } 0 \leqslant x \leqslant 7/3, \\ 1, & \text{for } x > 7/3. \end{cases}$$

The expected payoff of each S*-type player is* $6\frac{1}{3}$*, while the expected payoff of the* W*-type player is* 0.

4.4 Case D: The weaker player has a two-sided short support

In this case, both types support a positive bid with a probability of 1, but the W-type player has a shorter support relative to the S-type players. Specifically, the W-type player's maximal bid is smaller than that of the S-type players, and the W-type player's minimal bid is larger than that of the S-type players.

Proposition 4 *In the all-pay auction with two* S*-type players and one* W*-type player, if* $2w_2 > s_1 - s_2 \geqslant w_1$, $K_2 = \frac{[2w_2 - (s_1 - s_2)]^2}{(4w_2 - w_1)} - s_2 > 0$*, and the monotonicity condition* A_2 *holds (i.e., if* $s_1 > 2s_2$*), then there exists*

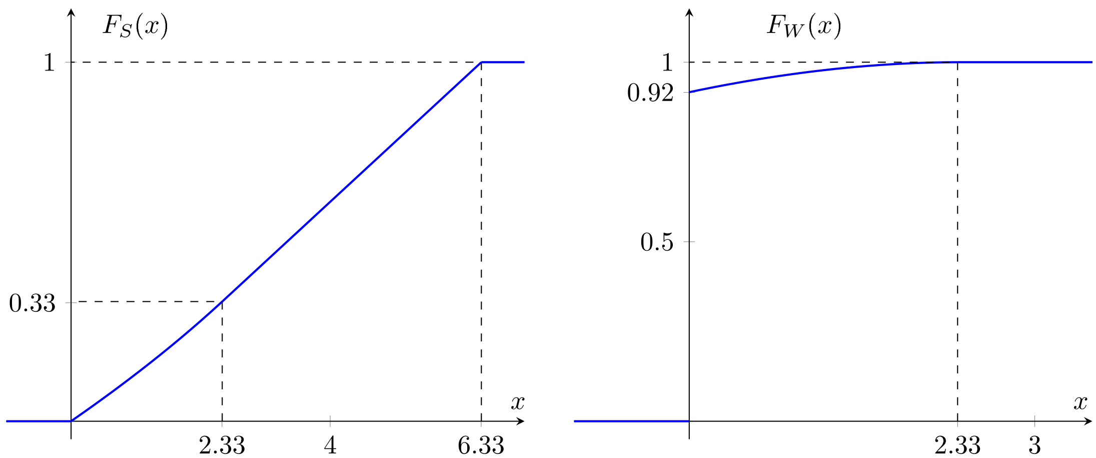

Figure 3: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 10, s_2 = 4, w_1 = 5$, and $w_2 = 4$. These values sustain the conditions of Proposition 3.

an equilibrium where the W-type player randomizes on the interval $[\alpha_1, \alpha_2]$*, where*

$$\alpha_1 = s_2\frac{2w_2-s_2-\sqrt{(2w_2-s_2)^2-4K_2(2w_2-w_1)}}{2(2w_2-w_1)} \quad \text{and} \quad \alpha_2 = s_2 + (s_1 - s_2)\frac{2w_2 - (s_1 - s_2)}{2(2w_2 - w_1)},$$

and the S-type players randomize on the interval $[0, s_1]$ *according to the following non-decreasing bid-distributions*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{s_2}, & \text{for } 0 \leqslant x \leqslant \alpha_1, \\ \frac{w_2-\sqrt{w_2^2-(x+K_2)(2w_2-w_1)}}{2w_2-w_1}, & \text{for } \alpha_1 \leqslant x \leqslant \alpha_2, \\ \frac{x-s_2}{s_1-s_2}, & \text{for } \alpha_2 \leqslant x \leqslant s_1, \\ 1, & \text{for } x > s_1, \end{cases} \quad F_W(x) = \begin{cases} 0, & \text{for } x < \alpha_1, \\ \frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}, & \text{for } \alpha_1 \leqslant x \leqslant \alpha_2, \\ 1, & \text{for } x > \alpha_2. \end{cases} \tag{6}$$

Under this equilibrium, the expected payoffs of the S-type players is 0*, while the expected payoff of the W-type player is* K_2*.*

The following example illustrates that the conditions of Proposition 4 are feasible, and that there are parameters that simultaneously support all the required constraints.

Example 4 *Assume that there are two S-type players whose prize valuations are* $s_1 = 30, s_2 = 1$*, and a*

W-type player whose prize valuations are $w_1 = 25$, and $w_2 = 20$, so that the conditions of Proposition 4 *hold. Then, the players' mixed-strategy equilibrium-strategies* (*see* Figure 4) *are*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ x, & \text{for } 0 \leqslant x \leqslant \frac{39-\sqrt{1460}}{30} \approxeq 0.026, \\ \frac{40-\sqrt{1539-60x}}{30}, & \text{for } 0.026 \approxeq \frac{39-\sqrt{1460}}{30} \leqslant x \leqslant \frac{349}{30} \approxeq 11.633, \\ \frac{x-1}{29}, & \text{for } 11.633 \approxeq \frac{349}{30} \leqslant x \leqslant 30, \\ 1, & \text{for } x > 30, \end{cases}$$

$$F_W(x) = \begin{cases} 0, & \text{for } x < \frac{39-\sqrt{1460}}{30} \approxeq 0.026, \\ \frac{30x-40+\sqrt{1539-60x}}{1150-28\sqrt{1539-60x}}, & \text{for } 0.026 \approxeq \frac{39-\sqrt{1460}}{30} \leqslant x \leqslant \frac{349}{30} \approxeq 11.633, \\ 1, & \text{for } x > \frac{349}{30} \approxeq 11.633. \end{cases}$$

The expected payoff of each S-type player is 0, and that of the W-type player is 1.2.

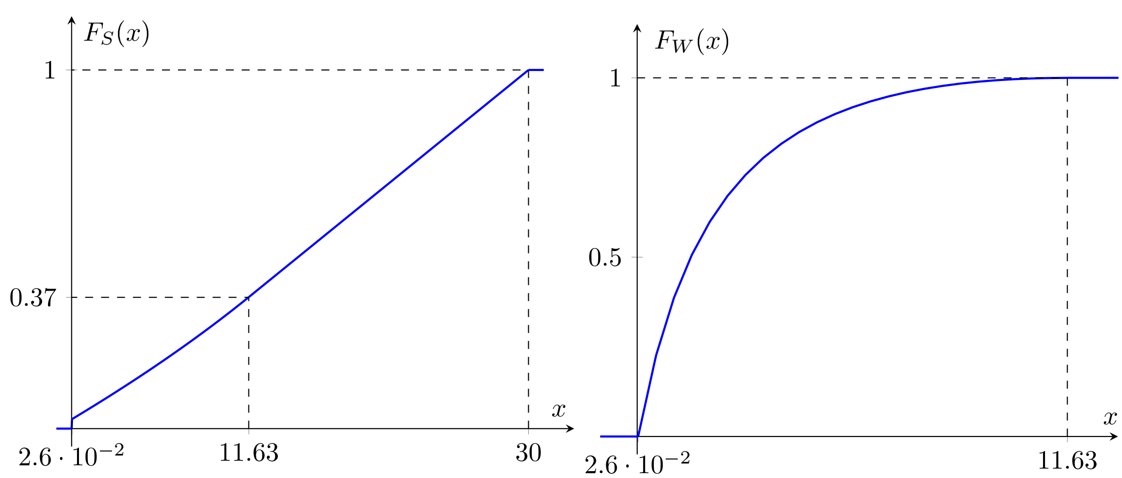

Figure 4: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 30, s_2 = 1, w_1 = 25$, and $w_2 = 20$. Note that these parameters sustain the conditions of Proposition 4.

5 Three-player contests with one strong and two weak players

In this section, we assume that there are two W-type players and a single S-type player. Thus, given a strategy profile (F_S, F_W), the expected payoffs of all types under a bid of $x \in \mathbb{R}$ are

$$\begin{aligned} U_W(x|F_S, F_W) &= w_1 F_W(x) F_S(x) + w_2[F_S(x)(1 - F_W(x)) + F_W(x)(1 - F_S(x))] - x \\ &= [(w_1 - 2w_2)F_W(x) + w_2]\, F_S(x) + w_2 F_W(x) - x, \end{aligned}$$

and

$$\begin{aligned} U_S(x|F_S, F_W) &= s_1 F_W^2(x) + 2s_2 F_W(x)(1 - F_W(x)) - x \\ &= (s_1 - 2s_2)F_W^2(x) + 2s_2 F_W(x) - x. \end{aligned}$$

We construct our analysis as follows. First, we define a profile (F_S, F_W) of strategies. Then, we prove that the function F_S is a well-defined CDF (specifically, a non-decreasing function), and later we explicitly use this result to establish an equilibrium.

5.1 Case E: The strong player has a one-sided short support

In this set-up of two W-type players and a single S-type player, our equilibrium analysis shows that both types participate with a probability of 1, but the S-type player has a shorter support relative to the W-type players. Specifically, the S-type player's minimal bid is larger than that of the W-type players.

Proposition 5 *In the all-pay auction with two W-type players and one S-type player, if*

$$0 < \alpha = \tfrac{w_2}{2\Delta(s)} \left[-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}\right] < w_1 \quad , \quad \Delta(s) = s_1 - 2s_2 > 0,$$

and the monotonicity condition $4s_2 > 2w_2 > w_1$ hold, then there exists an equilibrium in which the W-type players randomize on the interval $[0, w_1]$, and the S-type player randomizes on the interval $[\alpha, w_1]$ according *to their non-decreasing cumulative distribution bid functions* (F_S, F_W) *which are*

$$F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{w_2}, & \text{for } 0 \leqslant x \leqslant \alpha, \\ \frac{-s_2+\sqrt{s_2^2+\Delta(s)(s_1-w_1+x)}}{\Delta(s)}, & \text{for } \alpha \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1, \end{cases} \qquad F_S(x) = \begin{cases} 0, & \text{for } x < \alpha, \\ \frac{x-w_2F_W(x)}{(w_1-2w_2)F_W(x)+w_2}, & \text{for } \alpha \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1. \end{cases} \tag{7}$$

Then, the expected payoff of both W*-type players is* 0*, and that of the* S*-type player is* $s_1 - w_1$*.*

The following example shows that the conditions of Proposition 5 are feasible, and that there are parameters which simultaneously support all the required constraints.

Example 5 *Assume that there is a single* S*-type player whose prize valuations are* $s_1 = 5, s_2 = 2$*, and two* W*-type players whose prize valuations are* $w_1 = 3$*, and* $w_2 = 2$*, so that the conditions of* Proposition 5 *hold, where*

$$\alpha = \tfrac{w_2}{2\Delta(s)}\left[-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}\right] = \sqrt{12} - 2.$$

Then, the players' mixed-strategy equilibrium-strategies described in Proposition 5 (*see* Figure 5) *are*

$$F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{2}, & \text{for } 0 \leqslant x \leqslant \sqrt{12} - 2, \\ -2 + \sqrt{6 + x}, & \text{for } \sqrt{12} - 2 \leqslant x \leqslant 3, \\ 1, & \text{for } x > 3, \end{cases} \qquad F_S(x) = \begin{cases} 0, & \text{for } x < \sqrt{12} - 2, \\ \frac{x+4-2\sqrt{6+x}}{4-\sqrt{6+x}}, & \text{for } \sqrt{12} - 2 \leqslant x \leqslant 3, \\ 1, & \text{for } x > 3. \end{cases}$$

The expected payoff of the S*-type player is* 2*, while that of each of the* W*-type players is* 0*.*

6 Extensions

6.1 A non-convex support for the 'weak' player

Thus far, we have provided sufficient conditions such that the W-type player's distribution over bids, F_W, is monotonically increasing. However, in some cases, these conditions do not hold and a different type of

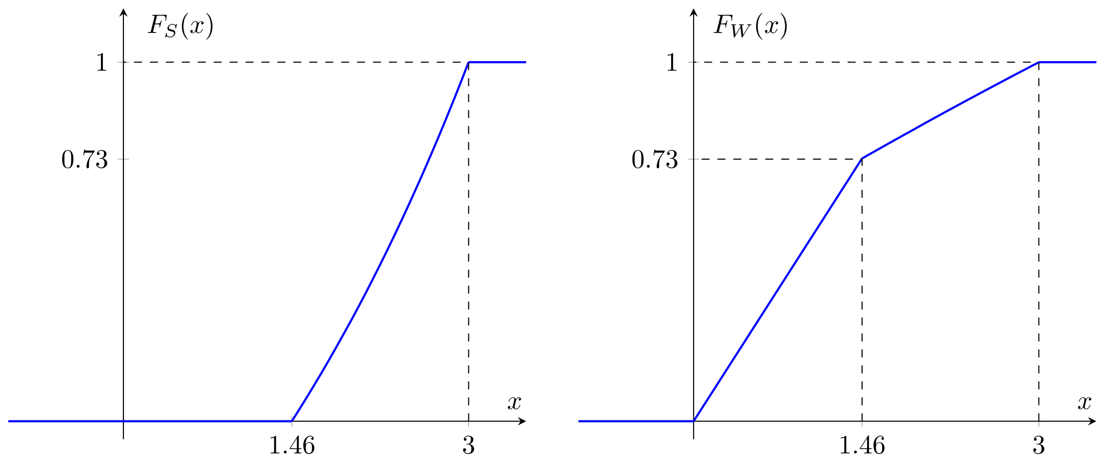

Figure 5: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 5, s_2 = 2, w_1 = 3$, and $w_2 = 2$. These values sustain the conditions of Proposition 5.

equilibrium arises. Specifically, according to Proposition 2, if A_4 is violated, then F_W may decrease close to zero. Thus, we need to depict new equilibrium strategies for which the support of the W-type player's strategy is non-convex.

Claim 1 *Assume that there are two S-type players whose values of the prizes are $s_1 = 8, s_2 = 6$, and a single W-type player whose values are $w_1 = 4$, and $w_2 = 0$. Then, a mixed-strategy equilibrium (see* Figure 6*) is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{3x}{10}, & \text{for } 0 \leqslant x \leqslant 25/9 \\ \frac{\sqrt{x}}{2}, & \text{for } 25/9 \leqslant x \leqslant 4 \\ 1, & \text{for } x > 4, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{2}{3}, & \text{for } 0 \leqslant x \leqslant 25/9, \\ \frac{4+x-3\sqrt{x}}{6-2\sqrt{x}}, & \text{for } 25/9 \leqslant x \leqslant 4 \\ 1, & \text{for } x > 4. \end{cases} \tag{8}$$

Note that $F_W(x)$ is not strictly increasing, and is fixed for all $0 \leqslant x \leqslant 25/9$. In that case, the expected payoffs of both S-type players is 4, and that of the W-type player is 0.

6.2 More than three players

We now proceed to study the case of $n > 3$ players, where there are at least three players of the same type and one player of a different type. This model is not only tractable, but even simpler to analyze than the

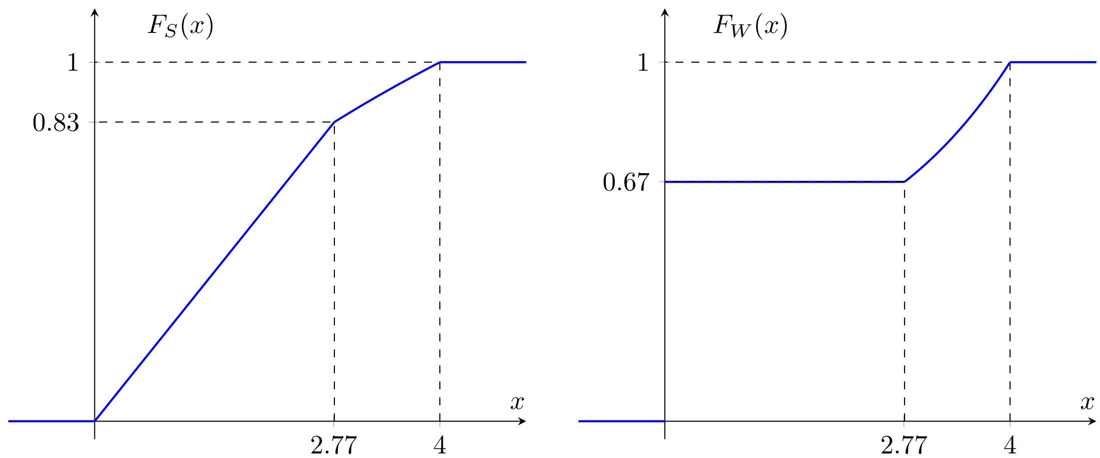

Figure 6: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 8, s_2 = 6, w_1 = 4$, and $w_2 = 0$ (as given in Claim 1).

three-player contest, since the competition among more than two players of the same type, regardless of whether their type is S or W, implies that their expected payoffs are zero. This is demonstrated in the following propositions, where in Proposition 6 there are multiple S-type players, and in Proposition 7 there are multiple W-type players.

Proposition 6 *In the all-pay auction with $n-1$ S-type players and one W-type player, where either $[s_1 - (n-2)s_2] \geqslant \max\{w_1, (n-1)w_2\}$ or $(n-2)s_2 \geqslant (n-1)w_2$ hold, there exists an equilibrium where the S-type players randomize on the interval $[0, s_1]$ according to their cumulative distribution bid function $F_S(x)$ which is given by*

$$s_1 F_S^{n-2}(x) + s_2(n-2)F_S^{n-3}(x)[1 - F_S(x)] - x = 0, \tag{9}$$

while the single W-type player bids 0 with a probability of 1. Then, the expected payoffs of all the players is 0.

Now, let us consider the case with more-than-two W-type players and a single S-type one.

Proposition 7 *Consider an all-pay auction with $n-1$ W-type players, one S-type player, and the functions*

$$F_W(x)=\begin{cases}0, & \text{for } x<0,\\ \left[\frac{x}{w_2}\right]^{\frac{1}{(n-2)}}, & \text{for } 0\leqslant x\leqslant \alpha_1,\\ G(x), & \text{for } \alpha_1\leqslant x\leqslant w_1,\\ 1, & \text{for } x>w_1,\end{cases}\qquad F_S(x)=\begin{cases}0, & \text{for } x<\alpha_1,\\ \frac{x-w_2F_W^{n-2}(x)}{F_W^{n-3}(x)[(w_1-(n-2)w_2)F_W(x)+w_2(n-3)]}, & \text{for } \alpha_1\leqslant x\leqslant w_1,\\ 1, & \text{for } x>w_1,\end{cases} \tag{10}$$

where α_1 and $G(x)$ are given by

$$\begin{aligned} s_1-w_1+\alpha_1 &= s_1\left[\frac{\alpha_1}{w_2}\right]^{(n-1)/(n-2)}+s_2(n-2)\frac{\alpha_1}{w_2}\left[1-\left[\frac{\alpha_1}{w_2}\right]^{1/(n-2)}\right],\\ s_1-w_1+x &= s_1G^{n-1}(x)+s_2(n-2)G^{n-2}(x)[1-G(x)]. \end{aligned}$$

If $F_S(\cdot)$ is non-decreasing on $[\alpha_1,w_1]$ and $s_1\geqslant s_2(n-2)$, then there exists an equilibrium in which the W-type players randomize on the interval $[0,w_1]$ and the S-type player randomizes on the interval $[a_1,w_1]$ according to the given strategies (F_S,F_W). Moreover, under this equilibrium, the expected payoffs of all W-type players are 0, while the expected payoff of the single S-type player is s_1-w_1.

7 Conclusion

Most of the contest literature has focused on the all-pay auction with a single prize or several identical prizes. In the current work, we study all-pay auctions with heterogeneous prizes and demonstrate that the equilibrium strategies might be rather complex. In particular, we show that the players' distributions over bids are not necessarily strictly increasing. When the players' distributions are strictly increasing, we analyze the equilibrium strategies and show that the results may significantly differ from the standard all-pay auctions, with either identical prizes or heterogeneous prizes where the ratio of each pair of prizes is the same for all the players. We demonstrate that the identity of the dominant player, namely, the player with the highest expected payoff changes (in a non-trivial manner) depending on the heterogeneity of the prizes. Due to the complexity of the analysis of our model with heterogeneous prizes, we assume a partial asymmetry among the players. Obviously a sharper asymmetry among the players will produce less predictable and plausibly, even more interesting results.

8 Appendix

8.1 Proof of Proposition 1

Proof. Consider the strategy profile (F_S, F_W) in which $F_W(x) = 0$ and $F_S(x)$ is given by (3), and under which, the expected payoffs of all the players for a bid of $x \in [0, s_1 - s_2]$ are

$$\begin{aligned} U_S(x|F_S, F_W) &= s_1 F_S(x) + s_2[1 - F_S(x)] - x \\ &= (s_1 - s_2) \cdot \frac{x}{s_1 - s_2} + s_2 - x = s_2, \\ U_W(x|F_S, F_W) &= w_1 F_S^2(x) + 2w_2 F_S(x)[1 - F_S(x)] - x \\ &= x^2 \frac{w_1 - 2w_2}{(s_1 - s_2)^2} + x \frac{2w_2 - s_1 + s_2}{s_1 - s_2}. \end{aligned}$$

Clearly, the S-type players have no profitable deviations upwards which would induce a higher cost while the probability of winning then is identical when the bid is equal to $x = s_1 - s_2$.

Now, to see that the W-type player has no profitable deviation from $x = 0$, note that $U_W(x|F_S, F_W)$ is a quadratic function of x. For $x = s_1 - s_2$, we get $U_W(s_1 - s_2|F_S, F_W) = w_1 - s_1 + s_2 \leqslant 0$, where the inequality follows from the lemma's conditions. So, we now need to verify that the derivative of U_W at $x = 0$ is negative. Specifically, $U'_W(0|F_S, F_W) = \frac{2w_2 - s_1 + s_2}{s_1 - s_2} = \frac{2w_2}{s_1 - s_2} - 1 \leqslant 1 - 1 = 0$, and the W-type player has no profitable deviations as well, thus concluding the proof. ■

8.2 Proof of Proposition 2

Proof. Consider the strategy profile (F_S, F_W) given by (4). The proof is divided into two parts: First we establish that F_W is non-decreasing on $[0, w_1]$, and then we prove that the given profile (F_S, F_W) is an equilibrium.

Part I: F_W is non-decreasing on $[0, w_1]$.

It easy to verify that $F_S(x)$ is strictly increasing on $[0, w_1]$, and its derivative is

$$f_S(x) = \frac{1}{2} \left[w_2^2 - x(2w_2 - w_1) \right]^{-1/2}.$$

Note that A_1 implies that F_S is concave (i.e., $f'_S(x) < 0$ for every $x \in [0, w_1]$), and $\neg A_1$ suggests that F_S is convex (i.e., $f'_S(x) \geqslant 0$ for every $x \in [0, w_1]$).

Denote $\Delta(s) = (s_1 - 2s_2)$. To see that $F_W(x)$ is strictly increasing on $[0, w_1]$ as well, we differentiate both sides of the equation $U_S(x|F_S, F_W) = s_1 - w_1$, and then we get

$$\begin{aligned} 0 &= \Delta(s) f_S(x) F_W(x) + \Delta(s) F_S(x) f_W(x) + s_2 f_W(x) + s_2 f_S(x) - 1. \\ f_W(x) &= \frac{1 - [\Delta(s) F_W(x) + s_2] f_S(x)}{[\Delta(s) F_S(x) + s_2]} \end{aligned}$$

Note that $[\Delta(s) F_S(x) + s_2] > 0$ for every $x \in [0, w_1]$, since

$$[\Delta(s) F_S(0) + s_2] = s_2 > 0 \quad , \quad [\Delta(s) F_S(w_1) + s_2] = s_1 - s_2 > 0,$$

and $F_S(x)$ in increasing on $[0, w_1]$. Therefore, $F_W(x)$ is increasing if and only if

$$[\Delta(s) F_W(x) + s_2] f_S(x) < 1, \quad \text{for all } x \in [0, w_1].$$

If F_W is either convex or concave (which means that f_W is a monotone function), we only need to verify that F_W is increasing near the end points of its support, 0 and w_1. If that is indeed the case (namely, if $f_W(x) > 0$, for $x = 0, w_1$), then F_W is increasing on the interval $[0, w_1]$. Therefore, we can differentiate the previous equation once more, and get

$$\begin{aligned} 0 &= \Delta(s)[f_S'(x) F_W(x) + 2 f_S(x) f_W(x) + F_S(x) f_W'(x)] + s_2[f_W'(x) + f_S'(x)], \\ f_W'(x) &= \frac{-f_S'(x)[s_2 + \Delta(s) F_W(x)] - 2\Delta(s) f_S(x) f_W(x)}{s_2 + \Delta(s) F_S(x)}. \end{aligned}$$

The conditions $(A_1, \neg A_2)$ imply that $f_W'(x) \geqslant 0$ and F_W is convex, while the conditions $(\neg A_1, A_2)$ ensure that $f_W'(x) \leqslant 0$ and F_W is concave. In any case, f_W is monotone, and we need to verify that $f_W(x) > 0$ for $x = 0, w_1$. Specifically,

$$\begin{aligned} [\Delta(s) F_W(w_1) + s_2] f_S(w_1) = [\Delta(s) + s_2] \frac{1}{2(w_1 - w_2)} &< 1, \\ [\Delta(s) F_W(0) + s_2] f_S(0) = \left[\Delta(s) \frac{s_1 - w_1}{s_2} + s_2\right] \frac{1}{2w_2} = \frac{\Delta(s)(s_1 - w_1) + s_2^2}{2 w_2 s_2} &< 1, \end{aligned}$$

where the first inequality follows from A_3, and the second inequality follows from A_4, thus concluding the first part of the proof.

Part II: (F_S, F_W) is an equilibrium.

We begin by showing that both functions are well-defined CDFs given that F_W is non-decreasing. Note that $F_W(0) = \frac{s_1 - w_1}{s_2} \geqslant F_S(0) = 0$, where the inequality follows from the assumption that $w_1 \geqslant s_1 - s_2$.

Also, note that $F_W(w_1) = F_S(w_1) = 1$, and that one can easily verify that $F_S(x)$ is strictly increasing on $[0, w_1]$. Thus, the functions F_S and F_W are well-defined CDFs, and we can now evaluate the players' payoffs at every point x, to establish an equilibrium.

Under the given strategy profile, the expected payoff of all S-type players for a bid of $x \in [0, w_1]$ is

$$
\begin{aligned}
U_S(x|F_S, F_W) &= \Delta(s)F_S(x)F_W(x) + s_2\left[F_W(x) + F_S(x)\right] - x \\
&= \Delta(s)F_S(x)\frac{x - s_2F_S(x) + s_1 - w_1}{\Delta(s)F_S(x) + s_2} + s_2\left[\frac{x - s_2F_S(x) + s_1 - w_1}{\Delta(s)F_S(x) + s_2} + F_S(x)\right] - x \\
&= \frac{\Delta(s)\left[xF_S(x) - s_2F_S^2(x)\right]}{\Delta(s)F_S(x) + s_2} + \frac{xs_2 + s_2\Delta(s)F_S^2(x)}{\Delta(s)F_S(x) + s_2} + (s_1 - w_1)\frac{\Delta(s)F_S(x) + s_2}{\Delta(s)F_S(x) + s_2} - x \\
&= \frac{\Delta(s)xF_S(x) + xs_2}{\Delta(s)F_S(x) + s_2} + s_1 - w_1 - x = s_1 - w_1.
\end{aligned}
$$

Therefore, all the S-type players are indifferent between any bid $x \in [0, w_1]$, and no player has an incentive to deviate upwards above w_1. The expected payoff of the W-type player for a bid of $x \in [0, w_1]$ is

$$
\begin{aligned}
U_W(x|F_S, F_W) &= \left[w_1 - 2w_2\right]F_S^2(x) + 2w_2F_S(x) - x \\
&= \left[w_1 - 2w_2\right]\left[\frac{w_2 - \sqrt{w_2^2 - 2w_2x + w_1x}}{2w_2 - w_1}\right]^2 + 2w_2\frac{w_2 - \sqrt{w_2^2 - 2w_2x + w_1x}}{2w_2 - w_1} - x \\
&= -\left[\frac{w_2^2 - 2w_2\sqrt{w_2^2 - 2w_2x + w_1x} + w_2^2 - 2w_2x + w_1x}{2w_2 - w_1}\right] \\
&+ 2w_2\frac{w_2 - \sqrt{w_2^2 - 2w_2x + w_1x}}{2w_2 - w_1} - x \\
&= \frac{2w_2\sqrt{w_2^2 - 2w_2x + w_1x} + 2w_2x - w_1x - 2w_2\sqrt{w_2^2 - 2w_2x + w_1x}}{2w_2 - w_1} - x \\
&= \frac{2w_2x - w_1x}{2w_2 - w_1} - x = 0.
\end{aligned}
$$

Thus, the W-type player has no profitable deviation, as well, and the profile is indeed an equilibrium with expected payoffs $s_1 - w_1$ and 0, as stated. ■

8.3 Proof of Proposition 3

Proof. Consider the strategy profile (F_S, F_W) given by (5). The proof is divided into two parts: First we establish that F_W is non-decreasing on $[0, w_1]$, and second we prove that the given profile (F_S, F_W) is an equilibrium.

Part I: F_W is non-decreasing on $[0, w_1]$.

Note that $F_S(x)$ is strictly increasing on $[0, \alpha]$, and its derivative is

$$f_S(x) = \frac{1}{2}\left[w_2^2 - x(2w_2 - w_1)\right]^{-1/2}.$$

Note that $2w_2 > s_1 - s_2 \geqslant w_1$ implies that F_S is convex (i.e., $f_S'(x) \geqslant 0$ for every $x \in [0, w_1]$). Recall that $\Delta(s) = (s_1 - 2s_2)$. Similarly to the first part of the proof of Proposition 2, we differentiate both sides of the equation $U_S(x|F_S, F_W) = K_1$, and we get

$$f_W(x) = \frac{1 - [\Delta(s)F_W(x) + s_2]\, f_S(x)}{[\Delta(s)F_S(x) + s_2]},$$

Thus, we conclude that $F_W(x)$ is non-decreasing in $[0, \alpha]$ if and only if

$$[\Delta(s)F_W(x) + s_2]\, f_S(x) \leqslant 1, \quad \text{for all } x \in [0, \alpha].$$

Again, as in the proof of Proposition 2, the conditions $2w_2 > s_1 - s_2 \geqslant w_1$ and A_2 ensure that $F_W''(x) \leqslant 0$ and that F_W is concave. Thus, f_W is a monotone function and it remains to verify that $f_W(x) \geqslant 0$ for $x = 0$ and $x = \alpha$. Specifically, for $x = 0$ we get

$$\begin{aligned}
[\Delta(s)F_W(0) + s_2]\, f_S(0) &= \left[\Delta(s)\frac{K_1}{s_2} + s_2\right]\frac{1}{2w_2} \\
&= \left[\Delta(s)\left[1 - \frac{[2w_2 - (s_1 - s_2)]^2}{4s_2(2w_2 - w_1)}\right] + s_2\right]\frac{1}{2w_2} \\
&< \left[s_1 - s_2 - \frac{\Delta(s)[2w_2 - (s_1 - s_2)]^2}{4s_2(2w_2 - w_1)}\right]\frac{1}{s_1 - s_2} \\
&= 1 - \frac{\Delta(s)[2w_2 - (s_1 - s_2)]^2}{4s_2(s_1 - s_2)(2w_2 - w_1)} \leqslant 1,
\end{aligned}$$

where the first inequality follows from the condition $2w_2 > s_1 - s_2$, and the second inequality follows from the fact that $\frac{\Delta(s)[2w_2 - (s_1 - s_2)]^2}{4s_2(s_1 - s_2)(2w_2 - w_1)} \geqslant 0$. Moving on to $x = \alpha$, we get

$$\begin{aligned}
[\Delta(s)F_W(\alpha) + s_2]\, f_S(\alpha) &= [\Delta(s) \cdot 1 + s_2]\frac{1}{2\sqrt{w_2^2 - \alpha(2w_2 - w_1)}} \\
&= [s_1 - s_2]\frac{1}{\sqrt{4w_2^2 - 4(2w_2 - w_1)\frac{(2w_2)^2 - (s_1 - s_2)^2}{4(2w_2 - w_1)}}} = 1,
\end{aligned}$$

as needed. This concludes the first part of the proof.

Part II: (F_S, F_W) is an equilibrium.

We begin by showing that both functions are well-defined CDFs, given that F_W is non-decreasing. Note that $F_W(0) = \frac{K_1}{s_2} \geqslant F_S(0) = 0$, where the inequality follows from the assumption that $K_1 \geqslant 0$. Also note that $F_S(s_1 - K_1) = 1$, and that

$$F_S(\alpha) = \frac{w_2 - \sqrt{w_2^2 + \frac{(2w_2)^2-(s_1-s_2)^2}{4(2w_2-w_1)}(w_1 - 2w_2)}}{2w_2 - w_1} = \frac{w_2 - \frac{s_1-s_2}{2}}{2w_2 - w_1} = 1 + \frac{\alpha + K_1 - s_1}{s_1 - s_2}. \tag{11}$$

Therefore, $(s_1 - s_2)F_S(\alpha) = \alpha + K_1 - s_2$. Hence,

$$F_W(\alpha) = \frac{\alpha - s_2 F_S(\alpha) + K_1}{(s_1 - s_2)F_S(\alpha) - s_2 F_S(\alpha) + s_2} = \frac{\alpha - s_2 F_S(\alpha) + K_1}{\alpha + K_1 - s_2 - s_2 F_S(\alpha) + s_2} = 1.$$

Similarly to the proof of Proposition 2, it is straightforward to verify that $F_S(x)$ is strictly increasing on $[0, w_1]$. We thus conclude that the functions F_S and F_W are well-defined CDFs, and can now evaluate the players' point-wise payoffs in order to establish an equilibrium.

Under the given strategy profile, the expected payoffs of the W-type player for a bid of $x \in [0, \alpha]$ is

$$\begin{aligned}
U_W(x|F_S, F_W) &= [w_1 - 2w_2]\, F_S^2(x) + 2w_2 F_S(x) - x \\
&= [w_1 - 2w_2] \left[\frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1}\right]^2 + 2w_2 \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1} - x \\
&= -\left[\frac{w_2^2 - 2w_2\sqrt{w_2^2 - 2w_2 x + w_1 x} + w_2^2 - 2w_2 x + w_1 x}{2w_2 - w_1}\right] \\
&+ 2w_2 \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1} - x \\
&= \frac{2w_2 x - w_1 x}{2w_2 - w_1} - x = 0.
\end{aligned}$$

Therefore, the W-type player is indifferent between any bid $x \in [0, \alpha]$. In addition, a bid of $x \in (\alpha, s_1 - K_1]$ would produce a negative payoff for the W-type player as $[F_S(x) - 1](s_1 - s_2) + s_1 - K_1 = x$ and

$$\begin{aligned}
U_W(x|F_S, F_W) &= [w_1 - 2w_2]\, F_S^2(x) + 2w_2 F_S(x) - [F_S(x) - 1](s_1 - s_2) - s_1 + K_1 \\
&= \Delta(w)t^2 + (2w_2 - \Delta(s'))t - s_2 + K_1,
\end{aligned}$$

where $t = F_S^2(x)$, $\Delta(w) = w_1 - 2w_2$, and $\Delta(s') = s_1 - s_2$. Denote $H(t) = \Delta(w)t^2 + (2w_2 - \Delta(s'))t - s_2 + K_1$, which is a parabolic function with a unique maximum point (by the assumption that $\Delta(w) < 0$) and $H(F_S(\alpha)) = 0$. Moreover, $H'(t) = 2\Delta(w)t + (2w_2 - \Delta(s'))$ and

$$H'(F_S(\alpha)) = 2\Delta(w)F_S(\alpha) + (2w_2 - \Delta(s')) = -2\left[w_2 - \tfrac{s_1-s_2}{2}\right] + (2w_2 - \Delta(s')) = 0,$$

where the second equality follows from Equation (11). Thus, $H(t)$ is decreasing for every $t > F_S(\alpha)$, which implies that $U_W(x|F_S, F_W) < 0$ for every $x > \alpha$, as needed. Thus, the W-type player has an incentive to deviate upwards, above α.

We now consider the S-type players. Denote $\Delta(s) = s_1 - 2s_2$. The expected payoff of the S-type players for a bid of $x \in [0, \alpha]$ is

$$
\begin{aligned}
U_S(x|F_S, F_W) &= \Delta(s)F_S(x)F_W(x) + s_2\left[F_W(x) + F_S(x)\right] - x \\
&= \Delta(s)F_S(x)\frac{x - s_2F_S(x) + K_1}{\Delta(s)F_S(x) + s_2} + s_2\left[\frac{x - s_2F_S(x) + K_1}{\Delta(s)F_S(x) + s_2} + F_S(x)\right] - x \\
&= \frac{\Delta(s)\left[xF_S(x) - s_2F_S^2(x)\right]}{\Delta(s)F_S(x) + s_2} + \frac{xs_2 + s_2\Delta(s)F_S^2(x)}{\Delta(s)F_S(x) + s_2} + K_1\frac{\Delta(s)F_S(x) + s_2}{\Delta(s)F_S(x) + s_2} - x \\
&= \frac{\Delta(s)xF_S(x) + xs_2}{\Delta(s)F_S(x) + s_2} + K_1 - x = K_1,
\end{aligned}
$$

and for a bid of $x \in [\alpha, s_1 - K_1]$, the expected payoff is

$$
\begin{aligned}
U_S(x|F_S, F_W) &= \Delta(s)F_S(x) + s_2\left[1 + F_S(x)\right] - x \\
&= \Delta(s')F_S(x) + s_2 - x \\
&= \Delta(s')\left[1 + \frac{x + K_1 - s_1}{\Delta(s')}\right] + s_2 - x \\
&= \Delta(s') + x + K_1 - s_1 + s_2 - x = K_1.
\end{aligned}
$$

Thus, no player has a profitable deviation, and the stated profile is indeed an equilibrium, with expected payoffs of K_1 and 0, as needed. ■

8.4 Proof of Proposition 4

Proof. Consider the strategy profile (F_S, F_W) given by (6). The proof is divided into two parts: First we establish that F_W is non-decreasing on $[0, w_1]$, and then we prove that this profile is an equilibrium.

Part I: F_W is non-decreasing on $[0, w_1]$.

Note that $F_S(x)$ is strictly increasing on $[0, \alpha]$, and its derivative is

$$
f_S(x) = \frac{1}{2}\left[w_2^2 - (x + K_2)(2w_2 - w_1)\right]^{-1/2}.
$$

Note that $2w_2 > s_1 - s_2 \geqslant w_1$ implies that F_S is convex (i.e., $f_S'(x) \geqslant 0$ for every $x \in [0, w_1]$). Recall that $\Delta(s) = (s_1 - 2s_2)$. Similarly to the first part of the proof of Proposition 2, we differentiate both sides of the equation $U_S(x|F_S, F_W) = 0$, and we get

$$
f_W(x) = \frac{1 - \left[\Delta(s)F_W(x) + s_2\right]f_S(x)}{\left[\Delta(s)F_S(x) + s_2\right]},
$$

We can conclude that $F_W(x)$ is non-decreasing in $[\alpha_1, \alpha_2]$ if and only if

$$[\Delta(s)F_W(x) + s_2]\, f_S(x) \leqslant 1, \quad \text{for all } x \in [\alpha_1, \alpha_2].$$

As in the proof of Proposition 2, the conditions $2w_2 > s_1 - s_2 \geqslant w_1$ and A_2 ensure that $f'_W(x) \leqslant 0$ and F_W is concave. Thus, f_W is a monotone function and it remains to verify that $f_W(x) \geqslant 0$ for $x = \alpha_1, \alpha_2$. Note that

$$\begin{aligned}
\alpha_2 + K_2 &= s_2 + (s_1 - s_2)\frac{2w_2 - (s_1 - s_2)}{2(2w_2 - w_1)} + \frac{[2w_2 - (s_1 - s_2)]^2}{4(2w_2 - w_1)} - s_2 \\
&= (s_1 - s_2)\frac{2w_2 - (s_1 - s_2)}{2(2w_2 - w_1)} + \frac{4w_2^2 - 4w_2(s_1 - s_2) + (s_1 - s_2)^2}{4(2w_2 - w_1)} \\
&= \frac{-(s_1 - s_2)^2}{2(2w_2 - w_1)} + \frac{4w_2^2 + (s_1 - s_2)^2}{4(2w_2 - w_1)} = \frac{4w_2^2 - (s_1 - s_2)^2}{4(2w_2 - w_1)},
\end{aligned}$$

and

$$\begin{aligned}
f_S(\alpha_2) &= \frac{1}{2}\left[w_2^2 - (\alpha_2 + K_2)(2w_2 - w_1)\right]^{-1/2} \\
&= \frac{1}{2}\left[w_2^2 - \frac{4w_2^2 - (s_1 - s_2)^2}{4(2w_2 - w_1)}(2w_2 - w_1)\right]^{-1/2} = \frac{1}{s_1 - s_2}.
\end{aligned}$$

In addition, since $\alpha_1 \leqslant \alpha_2$, we get

$$f_S(\alpha_1) = \frac{1}{2}\left[w_2^2 - (\alpha_1 + K_2)(2w_2 - w_1)\right]^{-1/2} \leqslant \frac{1}{2}\left[w_2^2 - (\alpha_2 + K_2)(2w_2 - w_1)\right]^{-1/2} = \frac{1}{s_1 - s_2}.$$

Thus, using A_2 (i.e., $s_1 > 2s_2$), for each $i = 1, 2$, it follows that

$$\begin{aligned}
[\Delta(s)F_W(\alpha_i) + s_2]\, f_S(\alpha_i) &\leqslant [\Delta(s) \cdot 1 + s_2]\,\frac{1}{s_1 - s_2} \\
&= [(s_1 - 2s_2) + s_2]\,\frac{1}{s_1 - s_2} = 1,
\end{aligned}$$

as needed. Thus, we conclude the first part of the proof.

Part II: (F_S, F_W) is an equilibrium.

We begin by showing that both functions are well-defined CDFs, given that F_W is non-decreasing. Note that $F_S(0) = 0 < F_S(s_1) = 1$ and F_S is strictly increasing in $[0, s_1]$. Also note that

$$\begin{aligned}
\alpha_1 &= s_2 \frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 - 4K_2(2w_2 - w_1)}}{2(2w_2 - w_1)} \\
&= s_2 \frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 - 4\left(\frac{[2w_2-(s_1-s_2)]^2}{4(2w_2-w_1)} - s_2\right)(2w_2 - w_1)}}{2(2w_2 - w_1)} \\
&= s_2 \frac{2w_2 - s_2 - \sqrt{2s_1 s_2 - s_1^2 + 4s_1 w_2 - 4s_2 w_1}}{2(2w_2 - w_1)} \\
&\leqslant s_2 \frac{2w_2 - s_2 - \sqrt{2s_1 s_2 - s_1^2 + 2s_1(s_1 - s_2) - 4s_2(s_1 - s_2)}}{2(2w_2 - w_1)} \\
&= s_2 \frac{2w_2 - s_2 - \sqrt{s_1^2 - 4s_2 s_1 + 4s_2^2}}{2(2w_2 - w_1)} \\
&= s_2 \frac{2w_2 - s_2 - (s_1 - 2s_2)}{2(2w_2 - w_1)} \\
&= s_2 \frac{2w_2 - (s_1 - s_2)}{2(2w_2 - w_1)} \leqslant s_2 \cdot 1 < \alpha_2,
\end{aligned}$$

as needed. Moreover, we can show that the proposition's conditions imply that $\alpha_1 \geqslant 0$ (i.e., $2w_2 \geqslant s_2$), and it is a straightforward to verify that F_S is continuous, specifically at $x = \alpha_1, \alpha_2$. Therefore, we can conclude that both functions are well defined.

Let us now verify that the profile of strategies which consists of F_W and F_S constitutes an equilibrium. We begin with the W-type player. For $x \in [\alpha_1, \alpha_2]$ we get

$$\begin{aligned}
U_W(x|F_S, F_W) &= [w_1 - 2w_2] F_S^2(x) + 2w_2 F_S(x) - x \\
&= [w_1 - 2w_2] \left[\frac{w_2 - \sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} \right]^2 \\
&+ 2w_2 \frac{w_2 - \sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} - x \\
&= -\frac{2w_2^2 - (x + K_2)(2w_2 - w_1) - 2w_2\sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} \\
&+ 2w_2 \frac{w_2 - \sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} - x \\
&= \frac{(x + K_2)(2w_2 - w_1)}{2w_2 - w_1} - x = K_2.
\end{aligned}$$

Therefore, the W-type player is indifferent between all values of $x \in [\alpha_1, \alpha_2]$ that produce an expected payoff of K_2.

Now consider $x \in [0, \alpha_1)$,

$$\begin{aligned} U_W(x|F_S, F_W) &= [w_1 - 2w_2] F_S^2(x) + 2w_2 F_S(x) - x \\ &= [w_1 - 2w_2] \frac{x^2}{s_2^2} + 2w_2 \frac{x}{s_2} - x. \end{aligned}$$

Thus, for $x \in [0, \alpha_1)$, the function $U_W(x|F_S, F_W)$ is parabolic with $U_W(0|F_S, F_W) = 0$, $U_W'(0|F_S, F_W) \geqslant 0$ (which follows from $2w_2 > s_1 - s_2 \geqslant w_1$ and $K_2 > 0$) and

$$\begin{aligned} U_W'(\alpha_1|F_S, F_W) &= 2[w_1 - 2w_2] \frac{\alpha_1}{s_2^2} + \frac{2w_2}{s_2} - 1 \\ &= 2[w_1 - 2w_2] s_2 \frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 - 4K_2(2w_2 - w_1)}}{2(2w_2 - w_1)s_2^2} + \frac{2w_2}{s_2} - 1 \\ &= -\frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 - 4K_2(2w_2 - w_1)}}{s_2} + \frac{2w_2 - s_2}{s_2} \\ &= \frac{\sqrt{(2w_2 - s_2)^2 - 4K_2(2w_2 - w_1)}}{s_2} \geqslant 0. \end{aligned}$$

Since $U_W(\alpha_1|F_S, F_W) = K_2$ and $U_W(x|F_S, F_W)$ is increasing for $x \in [0, \alpha_1)$, we conclude that $U_W(x|F_S, F_W) \leqslant K_2$ for every $x \in [0, \alpha_1)$, and that there exists no profitable deviation downwards for the W-type player.

We now consider $x \in (\alpha_2, s_1]$.

$$\begin{aligned} U_W(x|F_S, F_W) &= [w_1 - 2w_2] F_S^2(x) + 2w_2 F_S(x) - x \\ &= [w_1 - 2w_2] \frac{(x - s_2)^2}{(s_1 - s_2)^2} + 2w_2 \frac{x - s_2}{s_1 - s_2} - x. \end{aligned}$$

So,

$$\begin{aligned} U_W'(\alpha_2|F_S, F_W) &= 2[w_1 - 2w_2] \frac{\left(s_2 + (s_1 - s_2)\frac{2w_2 - (s_1 - s_2)}{2(2w_2 - w_1)} - s_2\right)}{(s_1 - s_2)^2} + \frac{2w_2}{s_1 - s_2} - 1 \\ &= -\frac{2w_2 - (s_1 - s_2)}{s_1 - s_2} + \frac{2w_2}{s_1 - s_2} - 1 = 0, \end{aligned}$$

while $U_W(\alpha_2|F_S, F_W) = K_2$, and $U_W(s_1|F_S, F_W) = w_1 - s_1 < 0$. Therefore, we can conclude that $U_W(x|F_S, F_W) \leqslant K_2$ for every $x \in (\alpha_2, s_1]$, as needed. Therefore, we have established that the W-type player has no profitable deviations.

We now consider the S-type players. For $x \in [0, \alpha_1)$, we get

$$\begin{aligned} U_S(x|F_S, F_W) &= (s_1 - 2s_2) F_S(x) F_W(x) + s_2 [F_W(x) + F_S(x)] - x \\ &= (s_1 - 2s_2) F_S(x) \cdot 0 + s_2 \left[0 + \frac{x}{s_2}\right] - x = 0, \end{aligned}$$

whereas, for $x \in (\alpha_2, s_1]$, we get

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)F_W(x)+s_2\left[F_W(x)+F_S(x)\right]-x \\ &= (s_1-2s_2)\frac{x-s_2}{s_1-s_2}\cdot 1+s_2\left[1+\frac{x-s_2}{s_1-s_2}\right]-x \\ &= (s_1-2s_2)\frac{x-s_2}{s_1-s_2}+s_2\frac{x+s_1-2s_2}{s_1-s_2}-x=0, \end{aligned}$$

Therefore, in these intervals, the S-type players get an expected payoff of 0 for every bid. In addition, for $x \in [\alpha_1, \alpha_2]$,

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)\frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}+s_2\left[\frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}+F_S(x)\right]-x \\ &= \left[(s_1-2s_2)F_S(x)+s_2\right]\frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}+s_2F_S(x)-x=0. \end{aligned}$$

Hence, we can conclude that the S-type players have an expected payoff of 0 for every $x \in [0, s_1]$, and that there are no profitable deviations for any of the players, thus establishing an equilibrium. ∎

8.5 Proof of Proposition 5

Proof. Consider the strategy profile (F_S, F_W) given by (7). The proof is divided into two parts: First we establish that F_S is non-decreasing on $[\alpha, w_1]$, then we prove that the given profile (F_S, F_W) is an equilibrium.

Part I: F_S is non-decreasing on $[\alpha, w_1]$.

Note that $F_W(x)$ is strictly increasing and continuous in $[0, w_1]$ (i.e., α is fixed specifically so that F_W is continuous), and its derivatives in $[\alpha, w_1]$ are

$$\begin{aligned} f_W(x) &= \frac{1}{2}\left[s_2^2+\Delta(s)(s_1-w_1+x)\right]^{-1/2}, \\ f'_W(x) &= -\frac{\Delta(s)}{4}\left[s_2^2+\Delta(s)(s_1-w_1+x)\right]^{-3/2}. \end{aligned}$$

Since $\Delta(s) > 0$, we deduce that F_W is concave (namely, $f'_W(x) \leqslant 0$ for every $x \in [\alpha, w_1]$). Now, we can differentiate (twice) both sides of the following equation

$$U_W(x|F_S,F_W)=\left[(w_1-2w_2)F_W(x)+w_2\right]F_S(x)+w_2F_W(x)-x=0,$$

and get

$$\begin{aligned} f_S(x) &= \frac{1-\left[(w_1-2w_2)F_S(x)+w_2\right]f_W(x)}{\left[(w_1-2w_2)F_W(x)+w_2\right]}, \\ f'_S(x) &= \frac{f'_W(x)\left[(2w_2-w_1)F_S(x)-w_2\right]+2(2w_2-w_1)f_W(x)f_S(x)}{\left[(w_1-2w_2)F_W(x)+w_2\right]}. \end{aligned}$$

Thus, we conclude that $F_S(x)$ is non-decreasing in $[\alpha, w_1]$ if and only if

$$[(w_1 - 2w_2)F_S(x) + w_2]\, f_W(x) \leqslant 1, \quad \text{for all } x \in [\alpha, w_1].$$

Combining the fact that $f'_W(x) \leqslant 0$ and $2w_2 > w_1$ (by assumption), we get that $f'_S(x) > 0$ for every $x \in [\alpha, w_1]$. This means that f_S is a monotone function, F_S is convex, and it remains to verify that $f_S(x) \geqslant 0$ for $x \in \{\alpha, w_1\}$. Specifically,

$$\begin{aligned} f_S(\alpha) &= \frac{1 - [(w_1 - 2w_2)F_S(\alpha) + w_2]\, f_W(\alpha)}{[(w_1 - 2w_2)F_W(\alpha) + w_2]} \\ &= \frac{1 - w_2 f_W(\alpha)}{[(w_1 - 2w_2)F_W(\alpha) + w_2]} \\ &= \frac{1 - w_2[2\sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + \alpha)}]^{-1}}{[(w_1 - 2w_2)F_W(\alpha) + w_2]} > 0, \end{aligned}$$

where the last inequality follows from the assumptions that $2s_2 > w_2$ and $\Delta(s) > 0$. In addition,

$$\begin{aligned} f_S(w_1) &= \frac{1 - [(w_1 - 2w_2)F_S(w_1) + w_2]\, f_W(w_1)}{[(w_1 - 2w_2)F_W(w_1) + w_2]} \\ &= \frac{1 - (w_1 - w_2)\frac{1}{2}\left[s_2^2 + \Delta(s)(s_1 - w_1 + w_1)\right]^{-1/2}}{w_1 - w_2} \\ &= \frac{1 - \frac{w_1 - w_2}{2(s_1 - s_2)}}{w_1 - w_2} \\ &> \frac{1 - \frac{2w_2 - w_2}{2(2s_2 - s_2)}}{w_1 - w_2} = \frac{2s_2 - w_2}{2s_2(w_1 - w_2)} > 0, \end{aligned}$$

where the first inequality follows from $w_1 < 2w_2$ and $s_1 > 2s_2$, and the second inequality follows from $2s_2 > w_2$. Thus, F_S is increasing in $[\alpha, w_1]$, and we conclude the first part of the proof.

Part II: (F_S, F_W) is an equilibrium.

Note that both functions are well-defined CDFs, given that F_S is non-decreasing. Specifically, $F_W(0) = F_S(\alpha) = 0 < F_W(w_1) = F_S(w_1) = 1$, and F_W is strictly increasing and continuous (by the choice of α) in $[0, w_1]$.

We now verify that the profile of strategies (F_W, F_S) constitutes an equilibrium. We begin with the W-type players. For $x \in [0, \alpha]$, we get

$$\begin{aligned} U_W(x|F_S, F_W) &= [(w_1 - 2w_2)F_W(x) + w_2]\, F_S(x) + w_2 F_W(x) - x \\ &= \left[(w_1 - 2w_2)\tfrac{x}{w_2} + w_2\right] \cdot 0 + w_2 \tfrac{x}{w_2} - x = 0, \end{aligned}$$

and for $x \in [\alpha, w_1]$, we get

$$\begin{aligned} U_W(x|F_S, F_W) &= \left[(w_1 - 2w_2)F_W(x) + w_2\right] F_S(x) + w_2 F_W(x) - x \\ &= \left[(w_1 - 2w_2)F_W(x) + w_2\right] \frac{x - w_2 F_W(x)}{(w_1 - 2w_2)F_W(x) + w_2} + w_2 F_W(x) - x = 0. \end{aligned}$$

Hence, the W-type players are indifferent between all values of $x \in [0, w_1]$ which produce an expected payoff of 0.

We now consider the S-type player. For $x \in [\alpha, w_1]$, we get

$$\begin{aligned} U_S(x|F_S, F_W) &= \Delta(s)F_W^2(x) + 2s_2 F_W(x) - x \\ &= \Delta(s)\left[\frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)}\right]^2 \\ &+ 2s_2 \frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)} - x \\ &= \frac{s_2^2 - 2s_2\sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)} + s_2^2 + \Delta(s)(s_1 - w_1 + x)}{\Delta(s)} \\ &+ 2s_2 \frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)} - x \\ &= \frac{\Delta(s)(s_1 - w_1 + x)}{\Delta(s)} - x = s_1 - w_1, \end{aligned}$$

therefore, the expected payoff of the S-type player is $s_1 - w_1$ for every $x \in [\alpha, w_1]$. In addition, we consider $x \in [0, \alpha]$, and note that $U_S(x|F_S, F_W)$ constitutes the following parabolic function,

$$\begin{aligned} U_S(x|F_S, F_W) &= (s_1 - 2s_2)F_W^2(x) + 2s_2 F_W(x) - x \\ &= (s_1 - 2s_2)\frac{x^2}{w_2^2} + 2s_2 \frac{x}{w_2} - x. \end{aligned}$$

By differentiating and inserting in $x = \alpha$, we get

$$\begin{aligned} U_S'(\alpha|F_S, F_W) &= \Delta(s)\frac{2\alpha}{w_2^2} + \frac{2s_2}{w_2} - 1 \\ &= \Delta(s)\frac{2\frac{w_2}{2\Delta(s)}\left[-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}\right]}{w_2^2} + \frac{2s_2}{w_2} - 1 \\ &= \frac{-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}}{w_2} + \frac{2s_2}{w_2} - 1 > 0, \end{aligned}$$

As such, the function is increasing for x below and sufficiently close to α. Combining this result with the fact that $U_S(0|F_S, F_W) = 0$, we conclude that $U_S(x|F_S, F_W) < U_S(\alpha|F_S, F_W) = s_1 - w_1$ for $x \in [0, \alpha]$, and that the S-type player does not have a profitable deviation downwards. To conclude, we have shown that there are no profitable deviations for any of the players, thus establishing an equilibrium. ■

8.6 Proof of Claim 1

Proof. Consider the strategy profile (F_S, F_W) given by (8). It is straightforward to verify that both CDFs are well defined. Clearly, no player can deviate to $x < 0$, nor has an incentive to bid above 4, so we consider $x \in [0, 25/9]$. For the S-type players, we get

$$\begin{aligned} U_S(x|F_S, F_W) &= (s_1 - 2s_2)F_S(x)F_W(x) + s_2\left[F_W(x) + F_S(x)\right] - x \\ &= -4 \cdot \frac{2}{3} \cdot \frac{3x}{10} + 6\left[\frac{2}{3} + \frac{3x}{10}\right] - x = 4, \end{aligned}$$

while for the W-type player we get

$$\begin{aligned} U_W(x|F_S, F_W) &= (w_1 - 2w_2)F_S^2(x) + 2w_2F_S(x) - x \\ &= 4\frac{9x^2}{100} - x \leqslant 0. \end{aligned}$$

Now, we consider $x \in [25/9, 4]$, and get

$$\begin{aligned} U_S(x|F_S, F_W) &= (s_1 - 2s_2)F_S(x)F_W(x) + s_2\left[F_W(x) + F_S(x)\right] - x \\ &= -4 \cdot \frac{4 + x - 3\sqrt{x}}{6 - 2\sqrt{x}} \cdot \frac{\sqrt{x}}{2} + 6\left[\frac{4 + x - 3\sqrt{x}}{6 - 2\sqrt{x}} + \frac{\sqrt{x}}{2}\right] - x \\ &= (6 - 2\sqrt{x})\frac{4 + x - 3\sqrt{x}}{6 - 2\sqrt{x}} + 3\sqrt{x} - x = 4. \end{aligned}$$

Thus, both S-type players are indifferent between all values of $x \in [0, 4]$. For the W-type player we get

$$\begin{aligned} U_W(x|F_S, F_W) &= (w_1 - 2w_2)F_S^2(x) + 2w_2F_S(x) - x \\ &= 4\frac{x}{4} - x = 0. \end{aligned}$$

Hence, no player has an incentive to deviate, and the given profile is indeed an equilibrium. ■

8.7 Proof of Proposition 6

Proof. Consider the strategy profile (F_S, F_W) where $F_W(x) = 0$ and $F_S(x)$ is given by (9). Fix $F_W = \mathbf{1}_{\{x \geqslant 0\}}$ so that the W-type player always bids $x = 0$. Given some CDF F_S with no atoms in $[0, s_1)$, the W-type player has an expected payoff of 0, whereas an S-type player who bids x has an expected payoff of

$$U_S(x|F_S, F_W) = s_1 F_S^{n-2}(x) + s_2(n-2)F_S^{n-3}(x)(1 - F_S(x)) - x.$$

Now, we fix F_S such that $U_S(x|F_S, F_W) = 0$ for every $x \in [0, s_1]$. Note that this CDF is well defined since $F_S(x) = 0$ for every $x \leqslant 0$, $F_S(x) = 1$ for every $x \geqslant s_1$, and the function is strictly increasing in the given interval.

To show that (F_S, F_W) is an equilibrium, we consider a unilateral deviation of some player, either of type W or type S. An S-type player has no profitable deviation for a bid $x \in [0, s_1]$ since all bids generate a payoff of zero. In addition, any deviation upwards to $x > s_1$ entails a negative expected payoff. Thus, we can focus on a deviation of an W-type player.

Assume that the W-type player bids $x > 0$, and that $[s_1 - (n-2)s_2] \geqslant \max\{w_1, (n-1)w_2\}$. According the Eq. (2), the player's expected payoff would be

$$\begin{aligned}
U_W(x|F_S, F_W) &= w_1 F_S^{n-1}(x) + w_2(n-1)(1 - F_S(x))F_S^{n-2}(x) - x \\
&\leqslant [s_1 - (n-2)s_2]F_S^{n-1}(x) + [s_1 - (n-2)s_2][1 - F_S(x)]F_S^{n-2}(x) - x \\
&= (s_1 - (n-2)s_2)F_S^{n-2}(x) - x \\
&< s_1 F_S^{n-2}(x) + s_2(n-2)F_S^{n-3}(x)(1 - F_S(x)) - x \\
&= U_S(x|F_S, F_W) = 0,
\end{aligned}$$

where the first inequality follows from the condition $[s_1 - (n-2)s_2] \geqslant \max\{w_1, (n-1)w_2\}$, and the second inequality follows from the fact that $s_2(n-2)F^{n-3}(x) > 0$ for $x > 0$. Otherwise, assume that $(n-2)s_2 \geqslant (n-1)w_2$ and recall that $s_1 > w_1$. Then,

$$\begin{aligned}
U_W(x|F_S, F_W) &= w_1 F_S^{n-1}(x) + w_2(n-1)(1 - F_S(x))F_S^{n-2}(x) - x \\
&< s_1 F_S^{n-2}(x) + s_2(n-2)(1 - F_S(x))F_S^{n-3}(x) - x \\
&= U_S(x|F_S, F_W) = 0,
\end{aligned}$$

where the inequality follows from our preliminary assumptions, $(n-2)s_2 \geqslant (n-1)w_2$ and $s_1 > w_1$, along with the fact that $F_S(x) \leqslant 1$. We conclude that the W-type player has no profitable deviation upwards, and (F_S, F_W) is indeed an equilibrium. ∎

8.8 Proof of Proposition 7

Proof. Consider the strategy profile (F_S, F_W) given by (10). We begin by showing that the functions F_W and F_S are well-defined CDFs, given that F_S is non decreasing in $[\alpha_1, w_1]$. For that purpose, we first need to prove that α_1 and $G(x)$ are well-defined. Consider the equation

$$s_1 - w_1 + \alpha_1 = s_1 \left[\tfrac{\alpha_1}{w_2}\right]^{(n-1)/(n-2)} + s_2(n-2)\tfrac{\alpha_1}{w_2}\left[1 - \left[\tfrac{\alpha_1}{w_2}\right]^{1/(n-2)}\right].$$

If we substitute α_1 with 0, then the LHS is strictly greater than the RHS. However, for $\alpha_1 = w_2$, we obtain the reverse inequality. Thus, by the Mean-Value Theorem (MVT), there exists a solution $\alpha_1 \in [0, w_1]$. Similarly, for every $x \in (\alpha_1, w_1)$, we can take the equation

$$s_1 - w_1 + x = s_1 G^{n-1}(x) + s_2(n-2)G^{n-2}(x)[1 - G(x)],$$

and substitute $G(x)$ with 0 and 1. Again, we get reverse inequalities (between the two cases), and the MVT ensures that a solution $G(x)$ exists. Note that for $x = w_1$ we get $G(w_1) = 1$, and for $x = \alpha_1$ both equations coincide so that $G(\alpha_1) = \left[\tfrac{\alpha_1}{w_2}\right]^{\frac{1}{n-2}}$. Thus, α_1 and $G(x)$ are well-defined, and F_W is continuous, thus implying that F_S is continuous, as well. By differentiating both sides of the second equation, we get

$$G'(x) = \frac{1}{G^{n-3}(x)\left[G(x)[s_1(n-1) - s_2(n-2)(n-1)] + s_2(n-2)^2\right]} \geqslant 0, \quad \forall G(x) \in (0,1].$$

Therefore, $G(x)$ is non-decreasing. We conclude that both functions, F_W and F_S, are well-defined CDFs, as needed.

We next establish an equilibrium, beginning with the single S-type player. Taking the expected payoff of the single S-type player and inserting in F_W for $x \in [\alpha_1, w_1]$, we get

$$U_S(x|F_S, F_W) = s_1 G^{n-1}(x) + s_2(n-2)G^{n-2}(x)[1 - G(x)] - x = s_1 - w_1,$$

where the equality follows from the definition of $G(x)$. To evaluate a possible deviation of the S-type player downwards to $x \in [0, \alpha_1)$, consider the functions

$$\begin{aligned}
U_S(x|F_S, F_W) &= [s_1 - s_2(n-2)]F_W^{n-1}(x) + s_2(n-2)F_W^{n-2}(x) - x \\
&= [s_1 - s_2(n-2)] \cdot \left[\frac{x}{w_2}\right]^{\frac{n-1}{n-2}} + \frac{s_2(n-2)x}{w_2} - x \\
\frac{dU_S(x|F_S, F_W)}{dx} &= [s_1 - s_2(n-2)] \cdot \frac{n-1}{(n-2)x} \cdot \left[\frac{x}{w_2}\right]^{\frac{n-1}{n-2}} + \frac{s_2(n-2)}{w_2} - 1.
\end{aligned}$$

Since $s_1 \geqslant s_2(n-2)$, it follows that U'_S is non-decreasing for $x \in [0, \alpha_1)$. In other words, the monotonicity of U'_S implies that U_S is convex with no interior maxima in $x \in [0, \alpha_1)$. Since $U_S(0|F_S, F_W) = 0 < s_1 - w_1 = U_S(\alpha_1|F_S, F_W)$, we conclude that $U_S(x|F_S, F_W) < U_S(\alpha_1|F_S, F_W)$ for every $x \in [0, \alpha_1)$, which implies that the S-type player has no profitable deviations downwards.

For the W-type players, the expected payoff is given by

$$U_W(x|F_S, F_W) = w_1 F_W^{n-2}(x) F_S(x) + w_2 \left[(1 - F_S(x)) F_W^{n-2}(x) + (n-3) F_W^{n-3}(x) F_S(x)(1 - F_W(x))\right] - x.$$

For $x \in [0, \alpha_1]$ we get

$$\begin{aligned} U_W(x|F_S, F_W) &= w_1 F_W^{n-2}(x) \cdot 0 + w_2 \left[(1-0) F_W^{n-2}(x) + (n-3) F_W^{n-3}(x) \cdot 0 \cdot (1 - F_W(x))\right] - x \\ &= w_2 F_W^{n-2}(x) - x \\ &= w_2 \frac{x}{w_2} - x = 0. \end{aligned}$$

For $x \in [\alpha_1, w_1]$, we can see that F_S is specifically defined under the condition that $U_W = (x|F_S, F_W) = 0$. Therefore, again, no player has a profitable deviation, and (F_S, F_W) is an equilibrium as stated. ■

References

[1] Amman, E., Leininger, W.: Asymmetric all-pay auctions with incomplete information: The two-player case. *Games and Economic Behavior* 14, 1-18 (1996)

[2] Barut, Y., Kovenock, D.: The symmetric multiple prize all-pay auction with complete information. *European Journal of Political Economy* 14, 627-644 (1998)

[3] Baye, M.R., Kovenock, D., de Vries, C.: Rigging the lobbying process: an application of the all-pay auction. *American Economic Review* 83, 289-294 (1993)

[4] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[5] Bulow, J., Levin, J.: Matching and price competition. *American Economic Review* 96, 652-668 (2006)

[6] Che, Y-K., Gale, I.: Caps on political lobbying. *American Economic Review* 88, 643-651 (1998)

[7] Clark, D.J., Riis, C.: Competition over more than one prize. *American Economic Review* 88(1), 276-289 (1998)

[8] Dahm, M.: Semi-targeted all-pay auctions: A partial exclusion principle. *International Journal of Industrial Organization* 61, 256-282 (2018)

[9] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Common-value all-pay auctions with asymmetric information. *International Journal of Game Theory* 46, 79-102 (2017)

[10] Gonzalez-Diaz, J., Siegel, R.: Matching and price competition: beyond symmetric linear costs. *International Journal of Game Theory* 42, 835-844 (2013)

[11] Hart, S.: Allocation games with caps: from captain Lotto to all-pay auctions. *Games and Economic Behavior* 45, 37-61 (2016)

[12] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-39 (1989)

[13] Hillman, A., Samet, D.: Dissipation of contestable rents by small numbers of contenders. *Public Choice* 54(1), 63-82 (1987)

[14] Krishna, V., Morgan, J.: An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory* 72(2), 343-362 (1997)

[15] Liu, X., Lu, J.: Optimal prize-rationing strategy in all-pay contests with incomplete information. *International Journal of Industrial Organization* 50, 57-90 (2017)

[16] Lizzeri, A., Persico, N.: Uniqueness and existence of equilibrium in auctions with a reverse price. *Games and Economic Behavior* 30, 83-114 (2000)

[17] Lu, J., Parreiras, S.O.: Monotone equilibrium of two-bidder all-pay auctions Redux. *Games and Economic Behavior* 104, 78-91 (2017)

[18] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91, 542-558 (2001)

[19] Moldovanu, B., Sela, A.: Contest architecture. *Journal of Economic Theory* 126(1), 70-97 (2006)

[20] Moldovanu, B., Sela, A., Shi, X.: Carrots and sticks: prizes and punishments in contests. *Economic Inquiry* 50(2), 453-462 (2012)

[21] Olsewski, W., Siegel, R.: Large contests. *Econometrica* 84, 835-854 (2016)

[22] Sela, A.: Sequential two-prize contests. *Economic Theory* 51, 383-395 (2012)

[23] Siegel, R.: All-pay contests. *Econometrica* 77(1), 71-92 (2009)

[24] Siegel, R.: Asymmetric contests with conditional investments. *American Economic Review* 100(5), 2230-2260 (2010)

[25] Xiao, J.: Asymmetric all-pay contests with heterogeneous prizes. *Journal of Economic Theory* 163, 178-221 (2016)

[26] Xiao J.: Equilibrium analysis of the all-pay contest with two nonidentical prizes: complete results. *Journal of Mathematical Ecnomics* 74, 21-34 (2018)