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Abstract

We study a model of two interdependent contests, each of which includes two heterogeneous players with commonly known types. The winners of both contests have a common winning value that depends on their types and, therefore, endogenous win probabilities in each match depend on the other contests' outcomes through the identity of the winner. The designer seeds players according to their ranks, and we assume that he wishes either to maximize or minimize the total effort. For such interdependent contests we consider two different types of a winning value function in order to demonstrate how its type plays a crucial effect on the structure of the optimal seeding.

Keywords: Seedings, Tullock contest, interdependent contests.

JEL classification: D44, J31, D72, D82

1 Introduction

Two contests are interdependent if the result of each contest depends and affects the result of the other. An example of such a contest is the elections to the United States House of Representatives

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whose members are elected for a two-year term in single-seat constituencies. There are 435 House districts that cover the United States and in each district there is a first-past-the-post election, namely, voters cast their vote for a candidate of their choice and the candidate who receives the most votes wins (irrespective of the vote share). The elections in each district are not independent since the value of winning for each candidate depends on the identity of the winners in the other districts whether or not his party will have the majority of members in the House. Another example is a software company that develops a product. The members of this company are divided into several teams, and in each team they compete with each other for the position of team leader. When the product finally hits the market, the team leaders are usually rewarded commensurably. The higher the leaders' abilities are, the better is the product, so that a more valuable product gives the team leaders a more generous reward. As such, the competitions for the leader positions are interdependent since the higher the ability of the team leaders, the higher is the reward. Last example of interdependent contests is the elimination tournament where in each stage some of the contestants are removed while the others advance to the next stage until the final stage in which some of the contestants (usually one) win prizes. In this kind of a tournament the contests in each stage are interdependent since the players' expected payoffs in any stage depend on the winners of the other contests at the same stage.

We study a model of two interdependent contests in which there are two sets of agents, each of which has two heterogeneous players with commonly known types. In both sets the players compete simultaneously such that in each set the two players compete against each other in a Tullock contest (see Tullock 1980).[1] The winners of both contests have winning value functions which are later derived from the interaction of the winners, namely, after the end of the competition. It is assumed that these winning value functions are monotonically increasing in both types of winners. Therefore,

[1]On the existence of the equilibrium in Tullock contests see Szidarovszky and Okuguchi (1997) and Einy et al. (2015).

endogenous win probabilities in each contest depend on the other contests' outcomes through the identity of the winner. This mutual influence explains why the equilibrium analysis of such a model might be very complex, and therefore we focus on the case of only two players in each set. We explicitly calculate the players' equilibrium efforts with either multiplicative or additive winning value functions of the winners' types. In both cases, these winning value functions include two parameters, α and β, which indicate the relative impact of each set on the players' winning values.

In a model with multiple contests there are several ways to seed the heterogeneous players. If the contests are interdependent, the seeding may have a crucial effect on the results (see, for example, Groh et al. 2012) since the type of each player has an effect not only on his own value of winning but also on all the players' values of winning in both sets. There are numerous real-life interdependent contests for which the seeding of the players is important. To illustrate, consider our previous example of a software-development company where the manager has to determine how to seed the software developers among the teams. Is it optimal to divide the talented software developers among the teams equally, or, alternatively, to seed all of them in the most dominant team?

It is important to note that the nature of the interdependency of the contests may not be identical among all kinds of contests and therefore the optimal seedings is not identical for all interdependent contests. In this paper, we focus on multiplicative and additive winning value functions where each function is suitable for a different application. We first study the issue of players' seeding when the winning value functions are multiplicative. We show that when the relative impact of each set on the players' winning values is the same, the optimal seeding for a designer who wishes to maximize the players' total effort in both contests is to place the players with the highest and the lowest types in the same contest. On the other hand, the optimal seeding for a designer who wishes to minimize the total effort is to place the two players with the highest types in the same contest. We also show that when the two sets have different impacts on the players' winning values and this asymmetry is sufficiently large, the same optimal seedings hold.

Then, we show that when the winning value functions are additive, if the relative impact of both sets on the players' winning value is the same, the players' total effort is independent of the players' seeding. However, when the two sets have different impacts on the players' winning values, the optimal seeding for a designer who wishes to maximize the total effort is to place the players with the highest types in the set with the higher impact, while to minimize the total effort it is optimal to place the two players with the lowest types in the set with the higher impact.

These results indicate that the type of the players' winning value function has a significant effect on the optimal seeding for a designer whether he wishes to maximize or to minimize the players' total effort. When we assume a significant asymmetry of the impact of the two sets on the players' values of winning, we obtain that for a designer who wishes to maximize the total effort, if the winning value function is multiplicative the worst seeding is to place the players with the highest types in the set with the higher impact. On the other hand, if the winning value function is additive, placing the players with the highest types in the same set is best. The results are completely different for multiplicative and additive value functions since these forms of functions represent completely different families of winning value functions. For the multiplicative value function, the mixed partial derivatives are positive which implies that there exists a strong mutual effect between the matched types. But for the additive value function, the mixed partial derivatives are equal to zero which implies that there exists a weak mutual effect between the matched types.

1.1 Related literature

A competition among a finite group of contestants can be designed in several ways. The contestants can compete in a grand contest or can be split into several sub contests (see Moldovanu and Sela 2006 and Fu and Lu 2009). Furthermore, if there are several contests they can be played sequentially or simultaneously (see Fu and Lu 2012, Fu et al. 2015, Jian et al. 2017, Mago and Sheremeta 2019, and Juang et al. 2020), and when they are played simultaneously the contests can be independent or interdependent.

The most common example of interdependent contests is the elimination tournament where in each stage some of the contestants are removed while the others advance to the next stage until the final stage in which some of the contestants (usually one) win prizes. In this kind of a tournament the contests in each stage are interdependent since the players' expected payoffs in any stage depend on the winners of the other contests in the same stage. The elimination tournament was first studied in the statistical literature. The pioneering paper of David (1959) considered the winning probability of the top player in a four-player tournament with a random seeding (see also Glenn (1960) and Searles (1963) for early contributions). Several papers (see for example, Hwang 1982, Horen and Reizman 1985 and Schwenk (2000)) consider various optimality criteria for choosing seedings. In particular, the optimal seeding for a given criterion may depend on the particular matrix of win probabilities. These works from the statistical literature assume that for each game among players i and j there is a fixed, exogenously given probability that i will beat j. This probability does not depend on the stage of the tournament in which the particular game takes place nor on the identity of the expected opponent at the next stage. As opposed to the statistical literature, in the economic literature the winning probabilities in each game become endogenous in that they result from equilibrium strategies and are dependent on continuation values of winning. Moreover, the win probabilities depend on the stage of the tournament in which the game takes place as well as on the identity of the future expected opponents.

One of the main issues regarding elimination tournaments is the optimal seeding of players. Rosen (1986) studied an elimination tournament in which the probability of winning a match is a stochastic function of the players' efforts. He found numerically that a random seeding yields a higher total effort than a seeding where strong players meet weak players in the semifinals. Gradstein and Konrad (1999) studied elimination tournaments in which homogenous players compete against each other in the Tullock contest, while Groh et al. (2012) studied a two-stage elimination tournaments with four players who are ranked in decreasing order of strength (1,2,3,4) and compete against each other in the all-pay auction. They showed that the seeding of $1-4, 2-3$ in the first stage maximizes the win probability of the strongest player, but the seeding of $1-3, 2-4$ in the first stage maximizes both the total effort across the tournament and the probability of a final among the two top players.[2]

The issue of seeding has been studied also when the contests are not simultaneous but sequential. Some examples include Linster (1993) who studied a sequential two-stage Tullock contest and showed that if the stronger player is the first (second) mover, the players' total effort is larger (smaller) than in the simultaneous contest. More recently, Levi-Tsedek and Sela (2018) studied a model in which a defender competes sequentially against n heterogeneous attackers in n different contests. They showed that if the players compete in all-pay contests, the order of the attackers does not affect the defender's expected payoff, but if the players compete in Tullock contests, the defender maximizes his expected payoff if he competes first against the strongest attacker (the attacker with the highest value of winning), next against the second strongest attacker, and so on until the last stage in which he competes against the weakest attacker.

Similarly to players' seedings in our model, in round-robin tournaments the designer has to decide the scheduling of the pair-wise games. Krumer et al. (2017) showed that in round-robin tournaments with three symmetric players, each player's expected payoff and probability of winning is maximized when he competes in the first and the last rounds, and with four players, they showed that a player who plays in the first game of each of the first two rounds has a first-mover advantage as reflected by a significantly higher winning probability as well as by a significantly higher expected payoff than his opponents. Later, Krumer et al. (2020) analyzed the optimal allocations of players for a designer who wishes to maximize the players' expected total effort in tournaments with one and two prizes.

The rest of the paper is organized as follows: In Section 2, we present our model of two interdependent contests. In Section 3, we analyze the equilibrium of this model for multiplicative

[2] Other papers that deal with elimination tournaments include Krakel (2014) where players are matched in the rank-order tournament, and Stracke et al. (2014) where players are matched in the Tullock contest.

and additive winning value functions. In Sections 4 and 5, we study the optimal seeding in our model with these two forms of winning value functions. Section 6 concludes. Some of the proofs appear in the Appendix.

2 The model

We consider two sets each of which includes two players. The players' types in set M are m_i, $i = h, l$ where $m_h \geq m_l$, and the players' types in set W are w_j, $j = h, l$ where $w_h \geq w_l$. The types, m_h and w_h, are the high-type players, and the other types, m_l and w_l, are the low-type players. These types are commonly known. Each player i in set M exerts an effort x_i, and each player j in set W exerts an effort y_j, $j = h, l$. The players compete in a Tullock contest in each set, namely, player i in set M wins with probability $\frac{x_i}{x_h+x_l}$, and player j in set W wins with probability $\frac{y_j}{y_h+y_l}$. The contests in both sets are interdependent since if player i from set M and player j from set W are the winners, the players' utilities are $f(m_i, w_j) - x_i$ and $g(m_i, w_j) - y_j$, respectively, where $f : R^2 \to R^1$ and $g : R^2 \to R^1$ are the winning value functions which are monotonically increasing in the players' types.[3] This model will be referred to as a model of two interdependent contests, and we say that this model has an equilibrium if and only if each player maximizes his payoff given the efforts of the other players in both contests.

3 Equilibrium analysis

We begin with the contest in set M. The maximization problem of the high-type player in set M is

$$\max_{x_h} \frac{x_h}{x_h + x_l} \left(f(m_h, w_h) \frac{y_h}{y_h + y_l} + f(m_h, w_l) \frac{y_l}{y_h + y_l} \right) - x_h, \tag{1}$$

and that of the low-type player is

$$\max_{x_l} \frac{x_l}{x_h + x_l} \left(f(m_l, w_h) \frac{y_h}{y_h + y_l} + f(m_l, w_l) \frac{y_l}{y_h + y_l} \right) - x_l. \tag{2}$$

[3] The players' winning value function is derived from the ex-post interaction of the winners in both contests.

Similarly, in set W, the maximization problem of the high-type player is

$$\max_{y_h} \frac{y_h}{y_h + y_l} \left(g(m_h, w_h) \frac{x_h}{x_h + x_l} + g(m_l, w_h) \frac{x_l}{x_h + x_l} \right) - y_h, \tag{3}$$

and that of the low-type player is

$$\max_{y_l} \frac{y_l}{y_h + y_l} \left(g(m_h, w_l) \frac{x_h}{x_h + x_l} + g(m_l, w_l) \frac{x_l}{x_h + x_l} \right) - y_l. \tag{4}$$

The first-order conditions (FOC) of the maximization problems (1), (2), (3), and (4) are

$$\begin{aligned} \frac{x_l}{(x_h + x_l)^2} \left(f(m_h, w_h) \frac{y_h}{y_h + y_l} + f(m_h, w_l) \frac{y_l}{y_h + y_l} \right) &\leq 1 \\ \frac{x_h}{(x_h + x_l)^2} \left(f(m_l, w_h) \frac{y_h}{y_h + y_l} + f(m_l, w_l) \frac{y_l}{y_h + y_l} \right) &\leq 1 \\ \frac{y_l}{(y_h + y_l)^2} \left(g(m_h, w_h) \frac{x_h}{x_h + x_l} + g(m_l, w_h) \frac{x_l}{x_h + x_l} \right) &\leq 1 \\ \frac{y_h}{(y_h + y_l)^2} \left(g(m_h, w_l) \frac{x_h}{x_h + x_l} + g(m_l, w_l) \frac{x_l}{x_h + x_l} \right) &\leq 1. \end{aligned} \tag{5}$$

These FOC are similar to those of two independent Tullock contests where each player's value of winning depends on his own type. In our model, however, each player's value of winning depends on his own type and the type of the winner in the other set. It is well known that the Tullock contest with two players has only an interior equilibrium, namely, there is no equilibrium in which players do not exert positive efforts, and by the same argument, there is no such equilibrium in our model. Thus, the only solution that is possible is when there is an equality between the LHS and the RHS of (5) which yields

Proposition 1 *In our model of two interdependent contests the equilibrium efforts are obtained by the solution of the equations given by (5).*

Proof. *See Appendix.* ■

In the following, we explicitly provide the players' equilibrium efforts for two families of winning value functions.

3.1 Equilibrium for multiplicative winning value functions

We assume that the players' winning value function belongs to the family of multiplicative functions $f(m_i, w_j) = g(m_i, w_j) = m_i^{\alpha} w_j^{\beta}$ where m_i is the type of the winner in set M, w_j is the type of winner in set W, and $\alpha, \beta \in R^1$ are the parameters denoting the relative impact of each set on the players' winning value. If $\alpha > \beta$, we say that the impact of set M on the players' winning value is larger than that of set W and vice versa. Note that the multiplicative value function satisfies $\frac{d^2 f}{dm_i dw_j} = \alpha\beta m_i^{\alpha-1} w_j^{\beta-1} > 0$ which shows a strong mutual effect between the matched types. By (5), the players' equilibrium efforts satisfy

$$\begin{aligned}
\frac{x_l}{(x_h+x_l)^2}\left((m_h^{\alpha} w_h^{\beta})\frac{y_h}{y_h+y_l} + (m_h^{\alpha} w_l^{\beta})\frac{y_l}{y_h+y_l}\right) &= 1 \\
\frac{x_h}{(x_h+x_l)^2}\left((m_l^{\alpha} w_h^{\beta})\frac{y_h}{y_h+y_l} + (m_l^{\alpha} w_l^{\beta})\frac{y_l}{y_h+y_l}\right) &= 1 \\
\frac{y_l}{(y_h+y_l)^2}\left((m_h^{\alpha} w_h^{\beta})\frac{x_h}{x_h+x_l} + (m_l^{\alpha} w_h^{\beta})\frac{x_l}{x_h+x_l}\right) &= 1 \\
\frac{y_h}{(y_h+y_l)^2}\left((m_h^{\alpha} w_l^{\beta})\frac{x_h}{x_h+x_l} + (m_l^{\alpha} w_l^{\beta})\frac{x_l}{x_h+x_l}\right) &= 1.
\end{aligned} \tag{6}$$

Dividing the LHS and RHS of the first two FOC by each other yields the following relation between the players' efforts in set M:

$$\frac{x_h}{x_l} = \frac{m_h^{\alpha}}{m_l^{\alpha}}. \tag{7}$$

And dividing the LHS and RHS of the last two FOC by each other yields the following relation between the players' efforts in set W:

$$\frac{y_h}{y_l} = \frac{w_h^{\beta}}{w_l^{\beta}}. \tag{8}$$

Inserting (7) and (8) into (6) gives us

Proposition 2 *In our model of two interdependent contests with a multiplicative winning value* *function, the players' equilibrium efforts are*

$$
\begin{aligned}
x_h &= \frac{m_h^{2\alpha} m_l^{\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)^2\left(w_h^{\beta}+w_l^{\beta}\right)} \\
x_l &= \frac{m_h^{\alpha} m_l^{2\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)^2\left(w_h^{\beta}+w_l^{\beta}\right)} \\
y_h &= \frac{w_h^{2\beta} w_l^{\beta}\left(m_h^{2\alpha}+m_l^{2\alpha}\right)}{\left(w_h^{\beta}+w_l^{\beta}\right)^2\left(m_h^{\alpha}+m_l^{\alpha}\right)} \\
y_l &= \frac{w_h^{\beta} w_l^{2\beta}\left(m_h^{2\alpha}+m_l^{2\alpha}\right)}{\left(w_h^{\beta}+w_l^{\beta}\right)^2\left(m_h^{\alpha}+m_l^{\alpha}\right)}.
\end{aligned} \tag{9}
$$

The players' probabilities of winning in set M are

$$
\begin{aligned}
p_{M-h} &= \frac{x_h}{x_h+x_l} = \frac{m_h^{\alpha}}{m_h^{\alpha}+m_l^{\alpha}} \\
p_{M-l} &= \frac{x_l}{x_h+x_l} = \frac{m_l^{\alpha}}{m_h^{\alpha}+m_l^{\alpha}},
\end{aligned}
$$

and that of the players in set W are

$$
\begin{aligned}
p_{W-h} &= \frac{y_h}{y_h+y_l} = \frac{w_h^{\beta}}{w_h^{\beta}+w_l^{\beta}} \\
p_{W-l} &= \frac{y_l}{y_h+y_l} = \frac{w_l^{\beta}}{w_h^{\beta}+w_l^{\beta}}.
\end{aligned}
$$

By (9), we have

Proposition 3 *In our model of two interdependent contests with a multiplicative winning value function, the players' total effort is*

$$
\begin{aligned}
TE &= TE_M + TE_W = (x_h + x_l) + (y_h + y_l) \\
&= \frac{(m_h^{\alpha} w_h^{\beta} + m_l^{\alpha} w_l^{\beta})(m_h^{\alpha} w_l^{\beta} + m_l^{\alpha} w_h^{\beta})}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)}.
\end{aligned} \tag{10}
$$

Proof. See Appendix. ■

3.2 Equilibrium for additive winning value functions

We assume now that the players' winning value function belongs to the family of additive functions $f(m_i, w_j) = g(m_i, w_j) = \alpha m_i + \beta w_j$ where m_i is the type of the winner in set M, w_j is the type of the winner in set W, and $\alpha, \beta \in R^1$ are the parameters denoting the relative impact of each set on the players' winning values. If $\alpha > \beta$, we say that the impact of set M on the players' winning values is larger than that of set W. Note that the additive value function satisfies $\frac{d^2 f}{dm_i dw_j} = 0$ which shows a weak mutual effect between the matched types. By (5), the players' equilibrium efforts satisfy:

$$\begin{aligned}
\frac{x_l}{(x_h + x_l)^2}\left((\alpha m_h + \beta w_h)\frac{y_h}{y_h + y_l} + (\alpha m_h + \beta w_l)\frac{y_l}{y_h + y_l}\right) &= 1 \\
\frac{x_h}{(x_h + x_l)^2}\left((\alpha m_l + \beta w_h)\frac{y_h}{y_h + y_l} + (\alpha m_l + \beta w_l)\frac{y_l}{y_h + y_l}\right) &= 1 \\
\frac{y_l}{(y_h + y_l)^2}\left((\alpha m_h + \beta w_h)\frac{x_h}{x_h + x_l} + (\alpha m_l + \beta w_h)\frac{x_l}{x_h + x_l}\right) &= 1 \\
\frac{y_h}{(y_h + y_l)^2}\left((\alpha m_h + \beta w_l)\frac{x_h}{x_h + x_l} + (\alpha m_l + \beta w_l)\frac{x_l}{x_h + x_l}\right) &= 1.
\end{aligned} \tag{11}$$

Denote

$$\begin{aligned}
\tilde{m} &= \left(\frac{x_h}{x_h + x_l} m_h + \frac{x_l}{x_h + x_l} m_l\right) \\
\tilde{w} &= \left(\frac{y_h}{y_h + y_l} w_h + \frac{y_l}{y_h + y_l} w_l\right).
\end{aligned}$$

Then, we can rewrite (11) as

$$\begin{aligned}
\frac{x_l}{(x_h + x_l)^2}(\alpha m_h + \beta \tilde{w}) &= 1 \\
\frac{x_h}{(x_h + x_l)^2}(\alpha m_l + \beta \tilde{w}) &= 1 \\
\frac{y_l}{(y_h + y_l)^2}(\alpha \tilde{m} + \beta w_h) &= 1 \\
\frac{y_h}{(y_h + y_l)^2}(\alpha \tilde{m} + \beta w_l) &= 1.
\end{aligned} \tag{12}$$

Dividing the LHS and RHS of the first two FOC by each other yields the following relation between the players' efforts in contest M:

$$\frac{x_h}{x_l} = \frac{\alpha m_h + \beta \tilde{w}}{\alpha m_l + \beta \tilde{w}}. \tag{13}$$

Similarly, dividing the LHS and RHS of the last two FOC by each other yields the following relation between the players' efforts in contest W:

$$\frac{y_h}{y_l} = \frac{\alpha \tilde{m} + \beta w_h}{\alpha \tilde{m} + \beta w_l}. \tag{14}$$

Inserting (13) and (14) into (12) gives us

Proposition 4 *In our model of two interdependent contests with an additive winning value function, the players' equilibrium efforts are*

$$\begin{aligned} x_h &= \frac{(\alpha m_h + \beta \tilde{w})^2 (\alpha m_l + \beta \tilde{w})}{(\alpha (m_h + m_l) + 2\beta \tilde{w})^2} \\ x_l &= \frac{(\alpha m_h + \beta \tilde{w}) (\alpha m_l + \beta \tilde{w})^2}{(\alpha (m_h + m_l) + 2\beta \tilde{w})^2} \\ y_h &= \frac{(\alpha \tilde{m} + \beta w_h)^2 (\alpha \tilde{m} + \beta w_l)}{(2\alpha \tilde{m} + \beta (w_h + w_l))^2} \\ y_l &= \frac{(\alpha \tilde{m} + \beta w_h) (\alpha \tilde{m} + \beta w_l)^2}{(2\alpha \tilde{m} + \beta (w_h + w_l))^2}, \end{aligned} \tag{15}$$

where $\tilde{m}$ and $\tilde{w}$ are given by

$$\begin{aligned} \tilde{m} = {} & \frac{\sqrt{\left((\alpha m_h + \beta w_h)^2 + (\alpha m_l + \beta w_l)^2\right)\left((\alpha m_h + \beta w_l)^2 + (\alpha m_l + \beta w_h)^2\right)}}{2\alpha (\alpha (m_h + m_l) + \beta (w_h + w_l))} \\ & + \frac{\alpha^2 \left(m_h^2 + m_l^2\right) - \beta^2 \left(w_h^2 + w_l^2\right)}{2\alpha (\alpha (m_h + m_l) + \beta (w_h + w_l))}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \tilde{w} = {} & \frac{\sqrt{\left((\alpha m_h + \beta w_h)^2 + (\alpha m_l + \beta w_l)^2\right)\left((\alpha m_h + \beta w_l)^2 + (\alpha m_l + \beta w_h)^2\right)}}{2\beta (\alpha (m_h + m_l) + \beta (w_h + w_l))} \\ & + \frac{\beta^2 \left(w_h^2 + w_l^2\right) - \alpha^2 \left(m_h^2 + m_l^2\right)}{2\beta (\alpha (m_h + m_l) + \beta (w_h + w_l))}. \end{aligned} \tag{17}$$

The players' probabilities of winning in set M are

$$\begin{aligned} q_{M-h} &= \frac{x_h}{x_h+x_l} = \frac{(\alpha m_h + \beta \tilde{w})}{(\alpha m_h + \beta \tilde{w}) + (\alpha m_l + \beta \tilde{w})} \\ q_{M-l} &= \frac{x_l}{x_h+x_l} = \frac{(\alpha m_l + \beta \tilde{w})}{(\alpha m_h + \beta \tilde{w}) + (\alpha m_l + \beta \tilde{w})}, \end{aligned}$$

and that of the players in set W are

$$\begin{aligned} q_{W-h} &= \frac{y_h}{y_h+y_l} = \frac{(\alpha \tilde{m} + \beta w_h)}{(\alpha \tilde{m} + \beta w_h) + (\alpha \tilde{m} + \beta w_l)} \\ q_{W-l} &= \frac{y_l}{y_h+y_l} = \frac{(\alpha \tilde{m} + \beta w_l)}{(\alpha \tilde{m} + \beta w_h) + (\alpha \tilde{m} + \beta w_l)}. \end{aligned}$$

Proof. See Appendix. ■

By (9), we have

Proposition 5 *In our model of two interdependent contests with an additive winning value function, the players' total effort is*

$$TE = TE_M + TE_W = (x_h + x_l) + (y_h + y_l) = \frac{\alpha m_h + \alpha m_l}{2} + \frac{\beta w_h + \beta w_l}{2}. \tag{18}$$

Proof. See Appendix. ■

4 Optimal seedings

Consider four players $1, 2, 3, 4$ where each player i has type v_i, $i = 1, 2, 3, 4$ and $v_i \geq v_{i+1}$, $i = 1, 2, 3$. According to the players' types, the designer can determine the seeding of these players in both contests in order to either maximize or minimize their expected total effort by the following six ways:

$$\begin{aligned} 1.\ M &= \{1,2\}, W = \{3,4\} \\ 2.\ M &= \{1,3\}, W = \{2,4\} \\ 3.\ M &= \{1,4\}, W = \{2,3\} \\ 4.\ M &= \{3,4\}, W = \{1,2\} \\ 5.\ M &= \{2,4\}, W = \{1,3\} \\ 6.\ M &= \{2,3\}, W = \{1,4\}. \end{aligned} \tag{19}$$

4.1 Seedings for multiplicative winning value functions

We now assume that the players' winning value function belongs to the following family of multiplicative functions $f(v_i, v_j) = v_i^{\alpha} v_j^{\beta}$, $i,\, j \in \{1,2,3,4\}$. By (10), we obtain that the total efforts for the seedings $1-6$ given by (19) are

$$\begin{aligned} TE_1 &= \frac{\left(v_1^{\alpha} v_4^{\beta} + v_2^{\alpha} v_3^{\beta}\right)\left(v_1^{\alpha} v_3^{\beta} + v_2^{\alpha} v_4^{\beta}\right)}{\left(v_1^{\alpha} + v_2^{\alpha}\right)\left(v_3^{\beta} + v_4^{\beta}\right)} \\ TE_2 &= \frac{\left(v_1^{\alpha} v_4^{\beta} + v_3^{\alpha} v_2^{\beta}\right)\left(v_1^{\alpha} v_2^{\beta} + v_3^{\alpha} v_4^{\beta}\right)}{\left(v_1^{\alpha} + v_3^{\alpha}\right)\left(v_2^{\beta} + v_4^{\beta}\right)} \\ TE_3 &= \frac{\left(v_1^{\alpha} v_3^{\beta} + v_4^{\alpha} v_2^{\beta}\right)\left(v_1^{\alpha} v_2^{\beta} + v_4^{\alpha} v_3^{\beta}\right)}{\left(v_1^{\alpha} + v_4^{\alpha}\right)\left(v_2^{\beta} + w_3^{\beta}\right)} \\ TE_4 &= \frac{\left(v_3^{\alpha} v_1^{\beta} + v_4^{\alpha} v_2^{\beta}\right)\left(v_4^{\alpha} v_1^{\beta} + v_3^{\alpha} v_2^{\beta}\right)}{\left(v_3^{\alpha} + v_4^{\alpha}\right)\left(v_1^{\beta} + v_2^{\beta}\right)} \\ TE_5 &= \frac{\left(v_2^{\alpha} v_1^{\beta} + v_4^{\alpha} v_3^{\beta}\right)\left(v_4^{\alpha} v_1^{\beta} + v_2^{\alpha} v_3^{\beta}\right)}{\left(v_2^{\alpha} + v_4^{\alpha}\right)\left(v_1^{\beta} + v_3^{\beta}\right)} \\ TE_6 &= \frac{\left(v_2^{\alpha} v_1^{\beta} + v_3^{\alpha} v_4^{\beta}\right)\left(v_3^{\alpha} v_1^{\beta} + v_2^{\alpha} v_4^{\beta}\right)}{\left(v_2^{\alpha} + w_3^{\alpha}\right)\left(v_1^{\beta} + v_4^{\beta}\right)}. \end{aligned} \tag{20}$$

We next assume that the parameters denoting the relative impact of each set on the winners' winning value are the same, namely, $\alpha = \beta$. Then, we obtain that there are only three different seedings since $TE_i = TE_{i+3}$, $i = 1, 2, 3$. A comparison of the total effort for these three seedings yields:

1)

$$\begin{aligned} TE_3 - TE_1 &= \frac{(v_1^\alpha v_3^\alpha + v_4^\alpha v_2^\alpha)(v_1^\alpha v_2^\alpha + v_4^\alpha v_3^\alpha)}{(v_1^\alpha + v_4^\alpha)(v_2^\alpha + w_3^\alpha)} - \frac{(v_1^\alpha v_4^\alpha + v_2^\alpha v_3^\alpha)(v_1^\alpha v_3^\alpha + v_2^\alpha v_4^\alpha)}{(v_1^\alpha + v_2^\alpha)(v_3^\alpha + v_4^\alpha)} \\ &= \frac{(v_3^\alpha - v_1^\alpha)(v_4^\alpha - v_2^\alpha)(v_1^\alpha v_3^\alpha + v_2^\alpha v_4^\alpha)}{(v_1^\alpha + v_2^\alpha)(v_3^\alpha + v_4^\alpha)(v_1^\alpha + v_4^\alpha)(v_2^\alpha + w_3^\alpha)}. \end{aligned}$$

Since $(v_3^\alpha - v_1^\alpha) \leq 0$ and $(v_4^\alpha - v_2^\alpha) \leq 0$ we obtain that $TE_3 - TE_1 \geq 0$.

2)

$$\begin{aligned} TE_3 - TE_2 &= \frac{(v_1^\alpha v_3^\alpha + v_4^\alpha v_2^\alpha)(v_1^\alpha v_2^\alpha + v_4^\alpha v_3^\alpha)}{(v_1^\alpha + v_4^\alpha)(v_2^\alpha + w_3^\alpha)} - \frac{(v_1^\alpha v_4^\alpha + v_3^\alpha v_2^\alpha)(v_1^\alpha v_2^\alpha + m_3^\alpha w_4^\alpha)}{(v_1^\alpha + v_3^\alpha)(v_2^\alpha + v_4^\alpha)} \\ &= \frac{(v_2^\alpha - v_1^\alpha)(v_4^\alpha - v_3^\alpha)(v_1^\alpha v_2^\alpha + v_3^\alpha v_4^\alpha)}{(v_1^\alpha + v_3^\alpha)(v_2^\alpha + v_4^\alpha)(v_1^\alpha + v_4^\alpha)(v_2^\alpha + w_3^\alpha)}. \end{aligned}$$

Since $(v_2^\alpha - v_1^\alpha) \leq 0$ and $(v_4^\alpha - v_3^\alpha) \leq 0$ we obtain that $TE_3 - TE_2 \geq 0$.

3)

$$\begin{aligned} TE_2 - TE_1 &= \frac{(v_1^\alpha v_4^\alpha + v_3^\alpha v_2^\alpha)(v_1^\alpha v_2^\alpha + m_3^\alpha w_4^\alpha)}{(v_1^\alpha + v_3^\alpha)(v_2^\alpha + v_4^\alpha)} - \frac{(v_1^\alpha v_4^\alpha + v_2^\alpha v_3^\alpha)(v_1^\alpha v_3^\alpha + v_2^\alpha v_4^\alpha)}{(v_1^\alpha + v_2^\alpha)(v_3^\alpha + v_4^\alpha)} \\ &= \frac{(v_4^\alpha - v_1^\alpha)(v_3^\alpha - v_2^\alpha)(v_1^\alpha v_4^\alpha + v_2^\alpha v_3^\alpha)}{(v_1^\alpha + v_2^\alpha)(v_1^\alpha + v_3^\alpha)(v_2^\alpha + v_4^\alpha)(v_3^\alpha + w_4^\alpha)}. \end{aligned}$$

Since $(v_4^\alpha - v_1^\alpha) \leq 0$ and $(v_3^\alpha - v_2^\alpha) \leq 0$, we obtain that $TE_2 - TE_1 \geq 0$. The above analysis implies that $TE_3 \geq TE_2 \geq TE_1$. Thus, we can conclude the following:

Proposition 6 *In our model of two interdependent contests with a multiplicative winning value function, when the relative impact of each set on the players' winning value is the same ($\alpha = \beta$), the optimal seeding for a designer who wishes to maximize the players' total effort is to place the players with the highest and the lowest types in the same set ($M : 1 - 4, W : 2 - 3$). On the other hand, the optimal seeding for a designer who wishes to minimize the total effort is to place the two players with the highest types in the same contest ($M : 1 - 2, W : 3 - 4$).*

It is worth noting that Proposition 6 considers only the order of the players' types, since the differences among the players' types do not affect the structure of the optimal seeding either for maximizing or minimizing the players' total effort. The intuitive explanation for Proposition 6 is that usually in contests the highest efforts are derived from the strongest players, namely, the players with the highest types. In the seeding $M : 1-2, W : 3-4$ the two strongest players are placed in one set (M), and therefore their winning values are their own types multiplied by some average of the two lowest types in the other set W. Thus, in this case we slightly increase the winning values of the strongest players, 1 and 2, and as such, the total effort is relatively small.

On the other hand, in the seeding $M : 1-4, W : 2-3$ the winning values of the players with the highest types, 1 and 2, significantly increase since they both win against the lower type players, and as such their winning values will be close to the product of their types. Therefore their winning values will be significantly increased and the total effort is relatively large.

We now assume, without loss of generality, that set M has a larger impact on the players' winning values than set W, namely, $\alpha > \beta$. In that case, we show that

Proposition 7 *In our model of two interdependent contests with a multiplicative winning value function, when the relative impact of set M on the players' winning value is sufficiently larger than that of set W ($\alpha >> \beta$), the optimal seeding for a designer who wishes to maximize the total effort is to place the players with the highest and lowest types in the same contest ($M : 1-4, W : 2-3$). On the other hand, the optimal seeding for a designer who wishes to minimize the total effort is to place the two players with the highest types in the same contest ($M : 3-4, W : 1-2$).*

Proof. *See Appendix.* ■

Proposition 7 demonstrates that when both sets have significantly different impacts on the players' winning values, the optimal seedings for designers who wish to maximize (or minimize) the total effort are the same. The intuitive explanation for this is that when the impact parameter α is significantly larger than the impact parameter β, given that v_1 is the highest type, the dominant variable in terms of the players' total effort is v_1^{α}. Thus, in order to maximize the players' total effort, the player with type v_1 has to be placed in set M which has the higher impact α. Furthermore, in order to increase the probability that v_1^α will be part of the players' total effort, player 1 with type v_1 should compete against the player with the lowest types v_4. Thus, the seeding $M : 1{-}4, W : 2{-}3$ maximizes the players' total effort. By a similar argument, the seeding $M : 3{-}4, W : 1{-}2$ minimizes the players' total effort.

The relevant question now for a designer who maximizes the players' total effort is whether or not the optimal seeding $M : 1 - 4, W : 2 - 3$ holds for asymmetric impacts of both sets on the players' winning values when this asymmetry is relatively small. Figure 1 shows the total effort for each of the possible seedings of the players with different types as functions of the impact of set M on the players' winning value α where the impact of the other set β is constant. According to this figure, we can see that any asymmetry between the impacts of the sets on the players' winning values does not change the type of the optimal seeding.

Figure 2, however, also shows the total effort for each of the possible seedings of four possible types of players as a function of the impact α where the other impact β is constant. We can see that the asymmetry of the impacts of both sets on the players' winning values does change the type of the optimal seeding, and there are possible values of α for which $TE_1 > TE_3$, namely, the seeding $M : 1 - 4, W : 2 - 3$ is not necessarily the optimal seeding for a designer who maximizes the players' total effort.

4.2 Seedings for additive winning value functions

Assume now that the players' winning value function belongs to the family of additive functions $f(m_i, w_j) = \alpha m_i + \beta w_j$, i, $j \in \{1, 2, 3, 4\}$. Then, by (18) we immediately obtain that

Proposition 8 *In our model of two interdependent contests with an additive winning value function, when the relative impact of both sets on the players' winning value is the same ($\alpha = \beta$), their total effort is independent of their seeding. However, when the relative impact of set M on the players' winning value is larger than that of set W ($\alpha > \beta$), a designer who wishes to maximize the* *players' total effort has to place the players with the highest types in set M ($M : 1-2, W : 3-4$), while a designer who wishes to minimize the total effort has to place the two players with the highest types in set W ($M : 3-4, W : 1-2$).*

By Proposition 8, when both sets have the same impact on the players' winning value, then the seeding has no effect on the players' total effort. The reason for this non-intuitive result is that the players' total effort is based on the difference between the players' types in each set. However, when the winning value function is additive, the differences between the players' winning values in one set do not depend at all on the types of the players in the other set such that the players' total effort in each set depends only on the own types of the players in that set. As such, unless if one set has a larger impact on the players' winning values, the seeding of the players is less relevant. Then, obviously, it is optimal for a designer who wishes to maximize the players' total effort to place the players with the highest types in the set with the higher impact parameter on the players' winning value. The opposite is true if a designer wishes to minimize the total effort, in which case he has to place the players with the lowest types in the set with the higher impact.

5 Concluding remarks

We studied the optimal seeding of players in a model of two interdependent contests as reflected through the endogenous win probabilities in each contest. These probabilities depend on the other contests' outcomes through the identity of the winner there. We showed that in such a model the optimal seeding depends on the form of the winning value function. When the mixed partial derivatives of the winning value function are positive as in the multiplicative winning value function, a strong mutual effect exists between the matched types. In that case, if the goal is to maximize the players' expected total effort, it is optimal to place the two strongest types in different sets. On the other hand, when the mixed partial derivatives of the winning value function are equal to zero as in the additive winning value function, a very weak mutual effect exists between the matched types. In that case, it is optimal to place the two strongest types in the same set, since otherwise they almost do not affect each other.

6 Appendix

6.1 Proof of Proposition 1

By (5), the second-order conditions (SOC) of the maximization problems (1), (2), (3) and (4) are

$$\begin{aligned}
soc_{m_h} &= \frac{-2x_l}{(x_h+x_l)^3}\left(f(m_h,w_h)\frac{y_h}{y_h+y_l}+f(m_h,w_l)\frac{y_l}{y_h+y_l}\right)\\
soc_{m_l} &= \frac{-2x_h}{(x_h+x_l)^3}\left(f(m_l,w_h)\frac{y_h}{y_h+y_l}+f(m_l,w_l)\frac{y_l}{y_h+y_l}\right)\\
soc_{w_h} &= \frac{-2y_l}{(y_h+y_l)^3}\left(g(m_h,w_h)\frac{x_h}{x_h+x_l}+g(m_l,w_h)\frac{x_l}{x_h+x_l}\right)\\
soc_{w_l} &= \frac{-2y_h}{(y_h+y_l)^3}\left(g(m_h,w_l)\frac{x_h}{x_h+x_l}+g(m_l,w_l)\frac{x_l}{x_h+x_l}\right).
\end{aligned}$$

This can be rewritten as:

$$\begin{aligned}
soc_{m_h} &= \frac{-2}{x_h+x_l}*foc_{m_h}=\frac{-2}{x_h+x_l}\left(\frac{x_l}{(x_h+x_l)^2}\left(f(m_h,w_h)\frac{y_h}{y_h+y_l}+f(m_h,w_l)\frac{y_l}{y_h+y_l}\right)\right)\\
soc_{m_l} &= \frac{-2}{x_h+x_l}*foc_{m_l}=\frac{-2}{x_h+x_l}\left(\frac{x_h}{(x_h+x_l)^2}\left(f(m_l,w_h)\frac{y_h}{y_h+y_l}+f(m_l,w_l)\frac{y_l}{y_h+y_l}\right)\right)\\
soc_{w_h} &= \frac{-2}{y_h+y_l}*foc_{w_h}=\frac{-2}{y_h+y_l}\left(\frac{y_l}{(y_h+y_l)^2}\left(g(m_h,w_h)\frac{x_h}{x_h+x_l}+g(m_l,w_h)\frac{x_l}{x_h+x_l}\right)\right)\\
soc_{w_l} &= \frac{-2}{y_h+y_l}*foc_{w_l}=\frac{-2}{y_h+y_l}\left(\frac{y_h}{(y_h+y_l)^2}\left(g(m_h,w_l)\frac{x_h}{x_h+x_l}+g(m_l,w_l)\frac{x_l}{x_h+x_l}\right)\right).
\end{aligned}$$

By (5), the existence of an interior equilibrium implies that foc_{m_h}, foc_{m_l} foc_{w_h}, foc_{w_l} are all positive, and therefore the SOC are negative. $Q.E.D.$

6.2 Proof of Proposition 3

By (9), the total effort in set M is

$$\begin{aligned}
TE_M &= x_h + x_l = \frac{m_h^{2\alpha} m_l^{\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)^2\left(w_h^{\beta}+w_l^{\beta}\right)} + \frac{m_h^{\alpha} m_l^{2\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)^2\left(w_h^{\beta}+w_l^{\beta}\right)} \\
&= \frac{\left(m_h^{2\alpha} m_l^{\alpha}+m_h^{\alpha} m_l^{2\alpha}\right)\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)^2\left(w_h^{\beta}+w_l^{\beta}\right)} = \frac{m_l^{\alpha} m_h^{\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)},
\end{aligned}$$

and in set W

$$\begin{aligned}
TE_W &= y_h + y_l = \frac{\left(m_h^{2\alpha}+m_l^{2\alpha}\right) w_h^{2\beta} w_l^{\beta}}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)^2} + \frac{\left(m_h^{2\alpha}+m_l^{2\alpha}\right) w_h^{\beta} w_l^{2\beta}}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)^2} \\
&= \frac{\left(m_h^{2\alpha}+m_l^{2\alpha}\right)\left(w_h^{2\beta} w_l^{\beta}+w_h^{\beta} w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)^2} = \frac{\left(m_h^{2\alpha}+m_l^{2\alpha}\right) w_l^{\beta} w_h^{\beta}}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)}.
\end{aligned}$$

Thus, the total effort in both sets is

$$\begin{aligned}
TE &= TE_M + TE_W = x_h + x_l + y_h + y_l \\
&= \frac{m_h^{\alpha} m_l^{\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)} + \frac{\left(m_h^{2\alpha}+m_l^{2\alpha}\right) w_h^{\beta} w_l^{\beta}}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)} \\
&= \frac{m_l^{\alpha} m_h^{\alpha}\left(w_h^{2\beta}+w_l^{2\beta}\right)+\left(m_h^{2\alpha}+m_l^{2\alpha}\right) w_h^{\beta} w_l^{\beta}}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)} \\
&= \frac{m_l^{\alpha} m_h^{\alpha} w_h^{2\beta}+m_l^{\alpha} m_h^{\alpha} w_l^{2\beta}+w_h^{\beta} w_l^{\beta} m_h^{2\alpha}+w_h^{\beta} w_l^{\beta} m_l^{2\alpha}}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)} \\
&= \frac{(m_h^{\alpha} w_h^{\beta}+m_l^{\alpha} w_l^{\beta})(m_h^{\alpha} w_l^{\beta}+m_l^{\alpha} w_h^{\beta})}{\left(m_h^{\alpha}+m_l^{\alpha}\right)\left(w_h^{\beta}+w_l^{\beta}\right)}.
\end{aligned}$$

Q.E.D.

6.3 Proof of Proposition 4

Below, we calculate the terms $\tilde{w}$ and $\tilde{m}$ that appear in the equilibrium efforts given by (15). By definition,

$$\begin{aligned}\tilde{m} &= \frac{x_h}{x_h+x_l}m_h+\frac{x_l}{x_h+x_l}m_l\\ \tilde{w} &= \frac{y_h}{y_h+y_l}w_h+\frac{y_l}{y_h+y_l}w_l.\end{aligned}$$

Inserting (13) and (14) yields

$$\begin{aligned}\tilde{w} &= \frac{y_h}{y_h+y_l}w_h+\frac{y_l}{y_h+y_l}w_l=\frac{\alpha\tilde{m}+\beta w_h}{\beta w_h+\beta w_l+2\alpha\tilde{m}}w_h+\left(1-\frac{\alpha\tilde{m}+\beta w_h}{\beta w_h+\beta w_l+2\alpha\tilde{m}}\right)w_l \qquad (21)\\ &= \frac{\beta\left(w_h^2+w_l^2\right)+\alpha\tilde{m}\left(w_h+w_l\right)}{\beta w_h+\beta w_l+2\alpha\tilde{m}}=\frac{\beta\left(w_h^2+w_l^2\right)+\alpha\left(\frac{x_h}{x_h+x_l}m_h+\frac{x_l}{x_h+x_l}m_l\right)\left(w_h+w_l\right)}{\beta w_h+\beta w_l+2\alpha\left(\frac{x_h}{x_h+x_l}m_h+\frac{x_l}{x_h+x_l}m_l\right)}\\ &= \frac{\alpha\left(w_h+w_l\right)\left(x_hm_h+x_lm_l\right)+\beta\left(w_h^2+w_l^2\right)\left(x_h+x_l\right)}{2\alpha\left(x_hm_h+x_lm_l\right)+\beta\left(w_h+w_l\right)\left(x_h+x_l\right)}.\end{aligned}$$

By (15), we have

$$\begin{aligned}x_h+x_l &= \frac{\left(\alpha m_h+\beta\tilde{w}\right)^2\left(\alpha m_l+\beta\tilde{w}\right)}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}+\frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)^2}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2} \qquad (22)\\ &= \frac{\left(\alpha m_h+\beta\tilde{w}\right)^2\left(\alpha m_l+\beta\tilde{w}\right)+\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)^2}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}\\ &= \frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)}{\alpha\left(m_h+m_l\right)+2\beta\tilde{w}},\end{aligned}$$

and

$$\begin{aligned}x_hm_h+x_lm_l &= \frac{\left(\alpha m_h+\beta\tilde{w}\right)^2\left(\alpha m_l+\beta\tilde{w}\right)}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}m_h+\frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)^2}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}m_l \qquad (23)\\ &= \frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)\left(\alpha\left(m_h^2+m_l^2\right)+\beta\tilde{w}\left(m_h+m_l\right)\right)}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}.\end{aligned}$$

Inserting (22) and (23) into (21) yields

$$
\begin{aligned}
\tilde{w} &= \frac{\alpha\left(w_h+w_l\right)\left(x_h m_h+x_l m_l\right)+\beta\left(w_h^2+w_l^2\right)\left(x_h+x_l\right)}{2\alpha\left(x_h m_h+x_l m_l\right)+\beta\left(w_h+w_l\right)\left(x_h+x_l\right)} \\
&= \frac{\alpha\left(w_h+w_l\right)\frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)\left(\alpha\left(m_h^2+m_l^2\right)+\beta\tilde{w}\left(m_h+m_l\right)\right)}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}+\beta\left(w_h^2+w_l^2\right)\frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)}{\alpha\left(m_h+m_l\right)+2\beta\tilde{w}}}{2\alpha\frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)\left(\alpha\left(m_h^2+m_l^2\right)+\beta\tilde{w}\left(m_h+m_l\right)\right)}{\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)^2}+\beta\left(w_h+w_l\right)\frac{\left(\alpha m_h+\beta\tilde{w}\right)\left(\alpha m_l+\beta\tilde{w}\right)}{\alpha\left(m_h+m_l\right)+2\beta\tilde{w}}} \\
&= \frac{\beta\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)\left(w_h^2+w_l^2\right)+\alpha\left(\alpha\left(m_h^2+m_l^2\right)+\beta\tilde{w}\left(m_h+m_l\right)\right)\left(w_h+w_l\right)}{\beta\left(\alpha\left(m_h+m_l\right)+2\beta\tilde{w}\right)\left(w_h+w_l\right)+2\alpha\left(\alpha\left(m_h^2+m_l^2\right)+\beta\tilde{w}\left(m_h+m_l\right)\right)} \\
&= \frac{\alpha\beta\left(m_h+m_l\right)\left(w_h^2+w_l^2\right)+2\beta^2\tilde{w}\left(w_h^2+w_l^2\right)+\alpha^2\left(m_h^2+m_l^2\right)\left(w_h+w_l\right)+\alpha\beta\tilde{w}\left(m_h+m_l\right)\left(w_h+w_l\right)}{\alpha\beta\left(m_h+m_l\right)\left(w_h+w_l\right)+2\beta^2\tilde{w}\left(w_h+w_l\right)+2\alpha^2\left(m_h^2+m_l^2\right)+2\alpha\beta\tilde{w}\left(m_h+m_l\right).}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left(\left(\alpha\beta\left(m_h+m_l\right)\left(w_h+w_l\right)+2\alpha^2\left(m_h^2+m_l^2\right)\right)+\left(2\alpha\beta\left(m_h+m_l\right)+2\beta^2\left(w_h+w_l\right)\right)\tilde{w}\right)\tilde{w} \\
= &\left(\alpha\beta\left(m_h+m_l\right)\left(w_h^2+w_l^2\right)+\alpha^2\left(m_h^2+m_l^2\right)\left(w_h+w_l\right)\right) \\
&+\left(2\beta^2\left(w_h^2+w_l^2\right)+\alpha\beta\left(m_h+m_l\right)\left(w_h+w_l\right)\right)\tilde{w},
\end{aligned}
$$

or, alternatively,

$$
\begin{aligned}
&\left(\alpha\beta\left(m_h+m_l\right)\left(w_h+w_l\right)+2\alpha^2\left(m_h^2+m_l^2\right)-2\beta^2\left(w_h^2+w_l^2\right)-\alpha\beta\left(m_h+m_l\right)\left(w_h+w_l\right)\right)\tilde{w} \\
&+\left(2\alpha\beta\left(m_h+m_l\right)+2\beta^2\left(w_h+w_l\right)\right)\tilde{w}^2-\left(\alpha\beta\left(m_h+m_l\right)\left(w_h^2+w_l^2\right)+\alpha^2\left(m_h^2+m_l^2\right)\left(w_h+w_l\right)\right) \\
= &\ 0.
\end{aligned}
$$

Rearranging the last equation yields the following quardratic equation of the parameter $\tilde{w}$,

$$
\begin{aligned}
&\left(2\alpha\beta\left(m_h+m_l\right)+2\beta^2\left(w_h+w_l\right)\right)\tilde{w}^2+\left(2\alpha^2\left(m_h^2+m_l^2\right)-2\beta^2\left(w_h^2+w_l^2\right)\right)\tilde{w} \\
&-\left(\alpha\beta\left(m_h+m_l\right)\left(w_h^2+w_l^2\right)+\alpha^2\left(m_h^2+m_l^2\right)\left(w_h+w_l\right)\right) \\
= &\ 0.
\end{aligned}
$$

The solution of this equation is

$$
\begin{aligned}
\tilde{w} = &\ \frac{\beta^2\left(w_h^2+w_l^2\right)-\alpha^2\left(m_h^2+m_l^2\right)}{2\beta\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)} \\
&\pm\frac{\sqrt{\left(\left(\alpha m_h+\beta w_h\right)^2+\left(\alpha m_l+\beta w_l\right)^2\right)\left(\left(\alpha m_h+\beta w_l\right)^2+\left(\alpha m_l+\beta w_h\right)^2\right)}}{2\beta\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)}.
\end{aligned}
$$

Since $\tilde{w}$ is positive, we have only one feasible solution which is

$$\begin{aligned}
\tilde{w} &= \frac{\sqrt{\left((\alpha m_h+\beta w_h)^2+(\alpha m_l+\beta w_l)^2\right)\left((\alpha m_h+\beta w_l)^2+(\alpha m_l+\beta w_h)^2\right)}}{2\beta\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)} \\
&\quad +\frac{\beta^2\left(w_h^2+w_l^2\right)-\alpha^2\left(m_h^2+m_l^2\right)}{2\beta\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)}.
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
\tilde{m} &= \frac{\sqrt{\left((\alpha m_h+\beta w_h)^2+(\alpha m_l+\beta w_l)^2\right)\left((\alpha m_h+\beta w_l)^2+(\alpha m_l+\beta w_h)^2\right)}}{2\alpha\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)} \\
&\quad +\frac{\alpha^2\left(m_h^2+m_l^2\right)-\beta^2\left(w_h^2+w_l^2\right)}{2\alpha\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)}.
\end{aligned}$$

Q.E.D.

6.4 Proof of Proposition 5

By (15), we obtain the total effort in set M,

$$\begin{aligned}
x_h+x_l &= \frac{(\alpha m_h+\beta\tilde{w})^2(\alpha m_l+\beta\tilde{w})}{(\alpha(m_h+m_l)+2\beta\tilde{w})^2}+\frac{(\alpha m_h+\beta\tilde{w})(\alpha m_l+\beta\tilde{w})^2}{(\alpha(m_h+m_l)+2\beta\tilde{w})^2} \\
&= \frac{(\alpha m_h+\beta\tilde{w})^2(\alpha m_l+\beta\tilde{w})+(\alpha m_h+\beta\tilde{w})(\alpha m_l+\beta\tilde{w})^2}{(\alpha(m_h+m_l)+2\beta\tilde{w})^2} \\
&= \frac{(\alpha m_h+\beta\tilde{w})(\alpha m_l+\beta\tilde{w})((\alpha m_h+\beta\tilde{w})+(\alpha m_l+\beta\tilde{w}))}{(\alpha(m_h+m_l)+2\beta\tilde{w})^2} \\
&= \frac{(\alpha m_h+\beta\tilde{w})(\alpha m_l+\beta\tilde{w})(\alpha m_h+\alpha m_l+2\beta\tilde{w})}{(\alpha(m_h+m_l)+2\beta\tilde{w})^2} \\
&= \frac{(\alpha m_h+\beta\tilde{w})(\alpha m_l+\beta\tilde{w})}{(\alpha(m_h+m_l)+2\beta\tilde{w})}.
\end{aligned}$$

Inserting (16) into the above equation gives us

$$x_h+x_l=\frac{\beta^2\left(w_h^2+w_l^2\right)+\alpha\beta(m_h+m_l)w_h+\alpha\beta(m_h+m_l)w_l+2\alpha^2m_hm_l}{2\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)}.$$

Similarly, the total effort in set W is

$$\begin{aligned}
y_h+y_l &= \frac{(\alpha\tilde{m}+\beta w_h)(\alpha\tilde{m}+\beta w_l)}{(2\alpha\tilde{m}+\beta(w_h+w_l))} \\
&= \frac{\alpha^2\left(m_h^2+m_l^2\right)+\alpha\beta(w_h+w_l)m_h+\alpha\beta(w_h+w_l)m_l+2\beta^2w_hw_l}{2\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)}
\end{aligned}$$

Thus, the total effort in both sets is

$$\begin{aligned} TE &= x_h + x_l + y_h + y_l \\ &= \frac{\beta^2\left(w_h^2+w_l^2\right)+\alpha\beta(m_h+m_l)w_h+\alpha\beta(m_h+m_l)w_l+2\alpha^2 m_h m_l}{2\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)} \\ &\quad+\frac{\alpha^2\left(m_h^2+m_l^2\right)+\alpha\beta(w_h+w_l)m_h+\alpha\beta(w_h+w_l)m_l+2\beta^2 w_h w_l}{2\left(\alpha\left(m_h+m_l\right)+\beta\left(w_h+w_l\right)\right)} \\ &= \frac{\alpha m_h+\alpha m_l+\beta w_h+\beta w_l}{2}. \end{aligned}$$

Q.E.D.

6.5 Proof of Proposition 7

Assume that β is any constant and that α approaches infinity. Then, by (20), given that $v_1 > v_2 > v_3 > v_4$, we obtain that

1)

$$\lim_{\alpha\to\infty}\frac{TE_3}{TE_1}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_1^\alpha v_3^\beta+v_4^\alpha v_2^\beta\right)\left(v_1^\alpha v_2^\beta+v_4^\alpha v_3^\beta\right)}{\left(v_1^\alpha+v_4^\alpha\right)\left(v_2^\beta+v_3^\beta\right)}}{\frac{\left(v_1^\alpha v_4^\beta+v_2^\alpha v_3^\beta\right)\left(v_1^\alpha v_3^\beta+v_2^\alpha v_4^\beta\right)}{\left(v_1^\alpha+v_2^\alpha\right)\left(v_3^\beta+v_4^\beta\right)}}=\frac{\left(v_3^\beta+v_4^\beta\right)v_2^\beta}{\left(v_2^\beta+v_3^\beta\right)v_4^\beta}>1.$$

2)

$$\lim_{\alpha\to\infty}\frac{TE_3}{TE_2}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_1^\alpha v_3^\beta+v_4^\alpha v_2^\beta\right)\left(v_1^\alpha v_2^\beta+v_4^\alpha v_3^\beta\right)}{\left(v_1^\alpha+v_4^\alpha\right)\left(v_2^\beta+v_3^\beta\right)}}{\frac{\left(v_1^\alpha v_4^\beta+v_3^\alpha v_2^\beta\right)\left(v_1^\alpha v_2^\beta+v_3^\alpha v_4^\beta\right)}{\left(v_1^\alpha+v_3^\alpha\right)\left(v_2^\beta+v_4^\beta\right)}}=\frac{\left(v_2^\beta+v_4^\beta\right)v_3^\beta}{\left(v_2^\beta+v_3^\beta\right)v_4^\beta}>1.$$

3)

$$\lim_{\alpha\to\infty}\frac{TE_3}{TE_4}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_1^\alpha v_3^\beta+v_4^\alpha v_2^\beta\right)\left(v_1^\alpha v_2^\beta+v_4^\alpha v_3^\beta\right)}{\left(v_1^\alpha+v_4^\alpha\right)\left(v_2^\beta+v_3^\beta\right)}}{\frac{\left(v_3^\alpha v_1^\beta+v_4^\alpha v_2^\beta\right)\left(v_4^\alpha v_1^\beta+v_3^\alpha v_2^\beta\right)}{\left(v_3^\alpha+v_4^\alpha\right)\left(v_1^\beta+v_2^\beta\right)}}=\lim_{\alpha\to\infty}\frac{v_1^\alpha}{v_3^\alpha}\frac{\left(v_1^\beta+v_2^\beta\right)v_3^\beta v_2^\beta}{\left(v_2^\beta+v_3^\beta\right)v_1^\beta v_2^\beta}=\infty.$$

4)

$$\lim_{\alpha\to\infty}\frac{TE_3}{TE_5}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_1^\alpha v_3^\beta+v_4^\alpha v_2^\beta\right)\left(v_1^\alpha v_2^\beta+v_4^\alpha v_3^\beta\right)}{\left(v_1^\alpha+v_4^\alpha\right)\left(v_2^\beta+w_3^\beta\right)}}{\frac{\left(v_2^\alpha v_1^\beta+v_4^\alpha v_3^\beta\right)\left(v_4^\alpha v_1^\beta+v_2^\alpha v_3^\beta\right)}{\left(v_2^\alpha+v_4^\alpha\right)\left(v_1^\beta+v_3^\beta\right)}}=\lim_{\alpha\to\infty}\frac{v_1^\alpha}{v_2^\alpha}\frac{\left(v_1^\beta+v_3^\beta\right)v_3^\beta v_2^\beta}{\left(v_2^\beta+w_3^\beta\right)v_1^\beta v_3^\beta}=\infty.$$

5)

$$\lim_{\alpha\to\infty}\frac{TE_3}{TE_6}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_1^\alpha v_3^\beta+v_4^\alpha v_2^\beta\right)\left(v_1^\alpha v_2^\beta+v_4^\alpha v_3^\beta\right)}{\left(v_1^\alpha+v_4^\alpha\right)\left(v_2^\beta+w_3^\beta\right)}}{\frac{\left(v_2^\alpha v_1^\beta+v_3^\alpha v_4^\beta\right)\left(v_3^\alpha v_1^\beta+v_2^\alpha v_4^\beta\right)}{\left(v_2^\alpha+w_3^\alpha\right)\left(v_1^\beta+v_4^\beta\right)}}=\lim_{\alpha\to\infty}\frac{v_1^\alpha}{v_2^\alpha}\frac{\left(v_1^\beta+v_4^\beta\right)v_3^\beta v_2^\beta}{\left(v_2^\beta+w_3^\beta\right)v_1^\beta v_4^\beta}=\infty.$$

Thus, when α converges to infinity, the optimal seeding for a designer who wishes to maximize the players' total effort is $M:1-4$, $W:2-3$.

Likewise, by (20), given that $v_1>v_2>v_3>v_4$, we obtain that

6)

$$\lim_{\alpha\to\infty}\frac{TE_4}{TE_5}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_3^\alpha v_1^\beta+v_4^\alpha v_2^\beta\right)\left(v_4^\alpha v_1^\beta+v_3^\alpha v_2^\beta\right)}{\left(v_3^\alpha+v_4^\alpha\right)\left(v_1^\beta+v_2^\beta\right)}}{\frac{\left(v_2^\alpha v_1^\beta+v_4^\alpha v_3^\beta\right)\left(v_4^\alpha v_1^\beta+v_2^\alpha v_3^\beta\right)}{\left(v_2^\alpha+v_4^\alpha\right)\left(v_1^\beta+v_3^\beta\right)}}=\lim_{\alpha\to\infty}\frac{v_3^\alpha}{v_2^\alpha}\frac{\left(v_1^\beta+v_3^\beta\right)v_1^\beta v_2^\beta}{\left(v_1^\beta+v_2^\beta\right)v_1^\beta v_3^\beta}=0.$$

7)

$$\lim_{\alpha\to\infty}\frac{TE_4}{TE_6}=\lim_{\alpha\to\infty}\frac{\frac{\left(v_3^\alpha v_1^\beta+v_4^\alpha v_2^\beta\right)\left(v_4^\alpha v_1^\beta+v_3^\alpha v_2^\beta\right)}{\left(v_3^\alpha+v_4^\alpha\right)\left(v_1^\beta+v_2^\beta\right)}}{\frac{\left(v_2^\alpha v_1^\beta+v_3^\alpha v_4^\beta\right)\left(v_3^\alpha v_1^\beta+v_2^\alpha v_4^\beta\right)}{\left(v_2^\alpha+w_3^\alpha\right)\left(v_1^\beta+v_4^\beta\right)}}=\lim_{\alpha\to\infty}\frac{v_3^\alpha}{v_2^\alpha}\frac{\left(v_1^\beta+v_4^\beta\right)v_1^\beta v_2^\beta}{\left(v_1^\beta+v_2^\beta\right)v_1^\beta v_4^\beta}=0.$$

Since we already found that $\lim_{\alpha\to\infty}\frac{TE_4}{TE_3}=0$, and similarly it can be verified that

$$\lim_{\alpha\to\infty}\frac{TE_4}{TE_2}=\lim_{\alpha\to\infty}\frac{TE_4}{TE_1}=0,$$

we obtain that when α converges to infinity, the optimal seeding for a designer who wishes to minimize the players' total effort is $M:3-4$, $W:1-2$.

Q.E.D.

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