Elimination contests with long-run efforts

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Abstract

We consider r-stage elimination contests in which different prizes are awarded at each stage. When an agent wins a contest at any stage, he is eliminated from competing in subsequent stages. At any stage, each agent makes a long-term effort, and his chances of winning the competition are determined by the relationship between his output and the outputs of the other competitors, where each agent's output in any stage is a function of all of his efforts prior to and including that stage. We analyze the agents' strategies in light of the prize allocation across the stages, specifically, the conditions under which agents decide to improve their outputs by exerting positive effort at various stages. We also investigate the optimal prize allocation across stages for both the agents who maximize their expected payoffs and the designer who maximizes the agents' total output.

Keywords: long-run efforts; elimination contests; prize allocations

JEL classification: D44, J31, D72, D82

1 Introduction

When an agent competes in multiple sequential contests, he does not necessarily need to make some effort in each stage, and his chance of winning may be based on the effort (output) he made in the first stage or on other efforts he made in previous stages. In other words, an agent must make at least one effort to produce some output in order to compete in multiple contests held sequentially; his chance of winning is determined by his own output as well as that of his opponents. However, each agent can improve his output at any stage by putting in more effort and thus increasing his chances of winning. In such sequential contests, an agent can only win one; once he does, he is removed from the remaining contests.

Sequential contests have numerous applications, as agents can make a single effort (output) for multiple contests. Consider, for example, a researcher who has completed a scientific paper and wishes to submit it for publication. If the paper is rejected, the researcher has the option of improving it based on the comments he received in the previous round, or not making any changes and submitting it to a new journal as it is. A similar scenario occurs when a young researcher completes his Ph.D. studies and applies to several departments at different universities. He presents his thesis, which could be followed by an interview and an oral presentation. Naturally, once he had received his offer, no other universities would consider him. If he is not accepted at one university, he may apply some of the criticism directed at his research, or he may simply move on to the next without making any new efforts.

We study such elimination contests using long-term efforts. In our elimination contests, each agent can only win in one stage (contest), and the winner of each stage receives a prize based on the agents' outputs, as determined by the Tullock success function for contests. When an agent wins a contest, he is removed from the list of agents who will compete in subsequent stages (contests). Each agent must make a single effort to produce some output and compete in multiple contests, but he can do so at any stage and thus improve his output, which means that if he adds some effort in one stage to his efforts in previous stages, all of his efforts up to and including that stage affect his output in the current stage and all subsequent stages. In our previous example, a researcher wrote a scientific paper and submitted it to a journal, but the paper could be rejected. The researcher then revises the manuscript before submitting it to another journal. Both efforts—the initial paper writing and the revision—have an impact on the quality of the paper and its chances of acceptance in the next submission, as well as any subsequent submissions that may be required.

We begin by considering our elimination contest, which consists of two stages and a general set of output functions. We provide a sufficient condition on the ratio of marginal outputs in both stages such that the agents make no additional effort in the second stage, and their output is solely determined by their efforts in the first stage. We then explicitly calculate this condition for agents with weighted additive output functions, where the marginal outputs in relation to their efforts at each stage is fixed. In particular, we obtain that agents are not active in the second stage if the marginal outputs in relation to the efforts in both stages are equal. However, when agents are active in the second stage, their total output is greater than when they are not. We also demonstrate that if agents exert positive effort in the second stage, allocating the larger prize in the second stage maximizes their total output. Because it is always best for each agent if the first prize is the lowest, we conclude that both the agents and the designer prefer that the lower prize be awarded first, followed by the higher prize. This order maximizes the agents' expected payoffs and their total output.

We then consider elimination contests with any number of stages ($r > 1$) and additive output functions, which means that the marginal output in relation to the effort in each stage is identical and equal to 1. We provide several claims that show both quality and quantity results for the agents' activity. For example, we provide a condition based on the values of prizes according to which agents do not make any effort in any stage s ($s \leq r$). On the other hand, if an agent makes a positive effort at any stage, we set a limit on his total effort up to and including that stage. Similarly, if agents make positive efforts in two consecutive stages, s and $s + 1$, we provide the exact value of their total effort up to and including stage s; on the other hand, we provide a condition under which agents do not make positive efforts in two consecutive stages.

Based on our analysis of r-stage elimination contests, we find sufficient conditions that agents make positive efforts only in the first stage and are inactive in subsequent stages. Such an elimination contest is the same as a one-stage competition with multiple prizes. Then, unlike our result for two-stage elimination contests with a weighted additive output function, we find that the total output is maximized when the first-stage prize is the highest. However, the agents still want the first prize to be the lowest. In fact, even in other elimination contests where agents put forth effort beyond the first stage, the conflict between the designer and the agents over the order of the prizes occurs in the majority, but not all, of them. It can be shown that if we have two vectors of the same prizes and the agents do not make a positive effort in the first stage, the designer and the agents may desire the same vector of prizes, namely, prizes in the same order.

To summarize, consider a researcher who submits his paper first to the most prestigious journal, and then if the paper is rejected, he may improve the paper and submit the revised version to a less prestigious journal, and so on, until the paper is accepted to a journal where its final version is significantly better than its initial version, and the paper in its current version is more likely to be accepted to a more prestigious journal, which is preferable from the researcher's standpoint. Is it better for a researcher to start with the least prestigious journal and then submit the revised version to a more prestigious journal at each subsequent stage, which may result in a higher expected payoff but is not preferable from the designer's standpoint? Our findings indicate that the answers to these questions are ambiguous and are determined by a combination of the value of the prizes (the prestige of the journals) and the marginal values of the agents' outputs (the levels of comments in the revisions and the researchers' abilities to apply them) at each stage of the elimination contests.

1.1 Related literature

Our elimination contest is part of a large body of literature on elimination contests. One of the issues with elimination tournaments is determining the optimal agent seeding. Rosen (1986) investigates an elimination tournament in which the probability of winning a match is a random function of the agents' efforts. He discovered numerically that a random seeding results in a greater total effort than a seeding in which strong agents face weak agents in the semifinals. Gradstein and Konrad (1999) investigate elimination tournaments in which homogeneous agents compete against one another in a Tullock contest and discovered that simultaneous contests are clearly superior if the contest's rules are sufficiently discriminatory (as in an all-pay auction). Groh et al. (2012) study a two-stage elimination tournament with four heterogeneous agents and discovered the optimal seeding that maximizes either the strongest agent's winning probability, total effort throughout the tournament, or the likelihood of a final between the two top agents. Cohen et al. (2023) study a model of two interdependent contests in which the winning values are determined by the types (abilities) of both winners. They found for different winning value functions how the designer seeds players according to their types in order to maximize (minimize) the total effort. In our model, as in Fu and Lu (2012), the designer has no control over agent seeding, and the winner of the previous stage is eliminated at each stage.

Another issue in elimination tournaments under incomplete information is information disclosure. Zhang and Wang (2009) look into how information revelation rules influence the existence and efficiency of equilibria in two-round elimination all-pay contests. They demonstrate that the no-revelation rule is both efficient and optimal in maximizing the total effort of the contestants. Fu and Wu (2022) consider a two-stage Tullock contest in which only a subset of the contestants advance to the final round. They investigate the optimal policy for disclosing contestants' interim status following the preliminary round, specifically identifying the conditions under which disclosure or concealment emerges as the optimal option. Information disclosure is only one factor influencing agents' performance in elimination tournaments. Cohen et al. (2018), for example, demonstrate that in two-stage elimination Tullock contests, in order to maximize the expected total effort, the designer can give the winner of the first stage a head start when competing against the other finalists in the second stage, and this always increases the players' expected total effort. Sela (2023) investigates the effect of the timing of the competitions in the first stage on the agents' expected payoffs and total effort, in which the players are divided into two all-pay contests, the winners of which interact with each other in the second stage. In our model, at each stage, all of the agents who have not won compete against one another in a single competition, so timing is irrelevant.

The issue with elimination tournaments that is more relevant to our paper is the optimal prize distribution. In one-stage contests with linear cost functions, in the most common contest forms, it is optimal to allocate the entire prize sum as a single prize, a single punishment, or a combination of a single prize and a single punishment (see, for example, Clark and Riis 1996, Barut and Kovenock 1998, Moldovanu and Sela 2001, Moldovanu et al. 2012, Liu et al. 2018, Sela 2020, and Liu and Lu 2023). Some of these works imply, for example, that if multiple prizes are awarded, the prize values must decrease, i.e., the agent in first place wins the highest prize, the agent in second place wins the second highest prize, and so on. Cohen et al. (2023), for example, show that in all-pay auctions with heterogeneous prizes, the value of the second prize may have a non-negligible effect on the identity of players with positive expected payoffs, but the values for the first (larger) prize have the greatest effect and thus should be higher than the values of the second prizes. In elimination contests, Moldovanu and Sela (2006) demonstrate, for example, that if the designer maximizes the expected highest effort and there are enough competitors, it is optimal to divide the competitors into two divisions and hold a final between the two divisional winners, with the winner receiving the entire prize sum. Fu and Lu (2012) demonstrate that the optimal contest eliminates one contestant at each stage until the final round, and the winner of the final receives the entire prize sum. Lu et al. (2022)

compare the winner-leave and loser-leave procedures for allocating a sequence of fixed prizes in multi-stage nested Tullock contests, and demonstrate that both procedures maximize effort by allocating one prize in each stage. In contrast to previous literature, we demonstrate that in order to maximize the agents' total output, the values of the prizes in our elimination contests should not necessarily decrease, and may even be increased in stages, with the first prize being the lowest. The optimal prize order is determined by the form of the agents' output functions, specifically the marginal outputs in relation to the efforts at each stage of the elimination contests.

There are one-stage models in which each agent selects more than one type of effort, and each agent's chance of winning is determined by a combination of all of these efforts (for example, Arbatskaya and Mialon 2010, and Husken 2020). However, only a few models, such as ours, account for multi-stage models in which agents can exert multiple efforts in multiple stages, and the chance of each agent winning in a single stage is determined by the agents' efforts in all preceding stages. Yildrim (2005), for example, studies two-stage contests in which agents can supplement their previous efforts after observing their opponents' most recent efforts in the intermediate stage. In his model, the winner is determined by each agent's total effort across both stages. Arbatskaya and Mialon (2012) investigate a two-stage contest model in which agents choose efforts in long-run activities, observe each other's efforts in these activities, and then select efforts in short-run activities. In their model, an agent's output is defined as a multiplicative function of long-run and short-run efforts that combine to determine the agent's likelihood of winning. In both of these multi-stage models, a single winner receives only one prize, whereas we assume multiple prizes for winners at each stage of the elimination contest.

Knyazev (2017) and Klein and Schmutzler (2017) are the most closely related papers. Knyazev (2017) considers multi-stage elimination contests in which agents' efforts at each stage produce some output for the principal. He demonstrates, similar to our findings, that the optimal prize structure can be non-monotone, meaning that at some stages, prizes decrease over time. However, his model differs significantly from our elimination contest. In his model, there is no correlation between the stages for the agents, because their winning probabilities in each stage are solely determined by their efforts in that stage, rather than, as in our model, by their efforts prior to and during the relevant stage. As a result, in his model, the unique symmetric equilibrium in every stage does not depend on the prizes at all earlier stages, whereas in our model, the symmetric equilibrium in any stage depends on all the prizes prior and including the relevant stage. Klein and Schmutzler (2017) investigate a two-stage contest in which two agents compete in a rank-order tournament at each stage. Similar to our model, they assume that the probability of winning in the second stage is determined by the agents' efforts in both stages. In their model, they take into account both general contest success and cost functions. On the other hand, in their model, there are only two players and two prizes; there is no elimination, so each player can win both prizes, whereas in our model, the winners are eliminated, so each player can only win one. In their model, our assumption that each player can only win one prize renders the analysis trivial because each player will win one prize, and their two-stage contest will be equivalent to a one-stage contest with a single prize equal to the difference between the prizes in both stages.

2 Elimination contests

Consider n symmetric agents competing sequentially in $r \geq 2$ stages. In each stage j, agents compete for a single prize v_j, $j = 1, ..., r$. an agent can only win one prize, and once he does, he is eliminated from all the contests in subsequent stages. In stage j, agent i observes the efforts of other agents prior to this stage and decides to exert an effort of $y_{ij} \geq 0$ with a one-time cost of $c(y_{ij}) = y_{ij}$. Each agent i has an output in the contest of stage j that depends on all the efforts he exerted until and including that stage; that is, agent i's output in stage j is $g((y_{ik})_{k=1}^{j}) : R^j \to R$, where $(y_{ik})_{k=1}^{j}$ is the vector that includes all the efforts of agent i until and including stage j, and the output function $g((y_{ik})_{k=1}^{j})$ is quasi-concave and monotonically increasing in each element of this vector of efforts. Thus, in particular, an agent can win in some stage $j > 1$ even if he makes no effort in that stage but makes some effort in previous stages. If all of the agents do not make a positive effort until and including a certain stage, no one wins any of the prizes until that stage, and all of the agents move on to the next stage. The lottery success function, which takes into account all of the agents' outputs in the first stage, determines the winner of the first contest. The lottery success function determines the winner of the second contest, which is based on the outputs of all the agents in the second stage except the first stage winner, and these outputs are based on the agents' efforts in both the first and second stages. The elimination process will continue until the winners of all stages are determined.[1] This competition will be referred to as the r-stage elimination contest.

Then, agent 1's maximization problem in stage $s, s = 1, 2, ..., r$, given that he participates in that stage, is

$$\max_{y_{1s}} \sum_{j=s}^{r} v_j \left(\frac{g((y_{1k})_{k=1}^{j})}{g((y_{1k})_{k=1}^{j}) + (n-j)g((y_k)_{k=1}^{j})}\right) \prod_{t=s}^{j-1} \left(1 - \frac{g((y_{1k})_{k=1}^{t})}{g((y_{1k})_{k=1}^{t}) + (n-t)g((y_k)_{k=1}^{t})}\right) - \sum_{j=s}^{r} y_{1s}, \quad (1)$$

In stage j, y_{1j} represents agent 1's effort, while y_j represents the symmetric effort of all other agents, $j = 1, ..., r$.

3 Two-stage elimination contests

We first focus on n agents who compete in two-stage elimination contests. By (1), the maximization problem of agent 1 in the first stage is

$$\max_{y_{11}} v_1\left(\frac{g(y_{11})}{g(y_{11}) + (n-1)g(y_1)}\right) + v_2\left(1 - \frac{g(y_{11})}{g(y_{11}) + (n-1)g(y_1)}\right)\frac{g(y_{11}, y_{12})}{g(y_{11}, y_{12}) + (n-2)g(y_1, y_2)} - y_{11} - y_{12}, \quad (2)$$

where y_j represents the symmetric effort of all agents in stage $j = 1, 2$ (excluding agent 1). Agent 2's second-stage maximization problem, given that he did not win the first stage, is

$$\max_{y_{12}} v_2 \frac{g(y_{11}, y_{12})}{g(y_{11}, y_{12}) + (n-2)g(y_1, y_2)} - y_{12}. \quad (3)$$

Denote $g_{y_1}(y_1, y_2) = \frac{d}{dy_1} g(y_1, y_2)$ and $g_{y_2}(y_1, y_2) = \frac{d}{dy_2} g(y_1, y_2)$. Then we can show that

Proposition 1 *In a two-stage elimination contest with an output function $g(y_1, y_2)$, if $v_1(n-1) \geq v_2$ and $ng_{y_2}(y_1, y_2) \leq (n-1)g_{y_1}(y_1, y_2)$ for all $y_1, y_2 \in R_+$, in the symmetric equilibrium, the agents do not make a positive effort in the second stage, namely, $y_2 = 0$.*

Proposition 1 states that if $(n-1)$ times of the marginal output of the effort in first stage cannot exceed n times of the marginal output of the effort in second stage, no agent will make a positive effort in the second stage. We will show later that even if the marginal output of the effort in the first stage y_1 is smaller

[1] A contest with such a metohd of prize allocation is also called "multi-stage nested Tullock contest " (see Clark and Riss 1996, Fu et al. 2014, and Lu et al. 2022).

than the marginal output of the effort in the second stage y_2 for all $y_1, y_2 \in R$[redacted], this may not be enough to ensure that the agents improve their initial outputs by making additional efforts in the second stage. To determine the threshold for investing effort in the second stage, we use a weighted additive output function $g(y_1, y_2) = \alpha y_1 + \beta y_2$ where $\alpha, \beta \in R_+$. Then, in the second stage, we determine whether it is necessary and sufficient for agents to improve their initial outputs by exerting additional effort.

Proposition 2 *In a two-stage elimination contest with a weighted additive output function $g(y_1, y_2) = \alpha y_1 + \beta y_2$, if*

1) $v_1(n-1) \geq v_2$,

and

2) $\frac{\beta}{\alpha} > \frac{1}{n^2}\left(n(n-1) + \frac{1}{v_2}(v_1(n-1) - v_2)\frac{(n-1)^2}{n-2}\right) > 1$,

the following symmetric equilibrium efforts in both stages

$$\begin{aligned} y_1 &= \frac{\beta}{n^2\beta - n(n-1)\alpha}((n-1)v_1 - v_2) \\ y_2 &= \left(-\frac{\alpha}{n^2\beta - n\alpha(n-1)}((n-1)v_1 - v_2) + \frac{v_2}{(n-1)^2}(n-2)\right). \end{aligned} \tag{4}$$

In this scenario, each agent's effort in the first stage (y_1) decreases in the marginal output functions β, while their effort in the second stage (y_2) increases in β.

If, on the other hand, $\frac{\beta}{\alpha} \leq \frac{1}{n^2}\left(n(n-1) + \frac{1}{v_2}(v_1(n-1) - v_2)\frac{(n-1)^2}{n-2}\right)$, the symmetric equilibrium efforts in both stages are

$$\begin{aligned} y_1 &= \frac{1}{n^3 - n^2}\left(v_1(n-1)^2 + v_2(1 - 3n + n^2)\right) \\ y_2 &= 0. \end{aligned} \tag{5}$$

We now study the optimal prize order in two-stage elimination contests with weighted additive output functions. We begin with the following example:

Example 1 *Consider a two-stage elimination contest with five agents and a weighted additive output function,*

$$g(y_1, y_2) = y_1 + 3y_2.$$

The two prizes have the values $v_1 = 2, v_2 = 1$. According to Proposition 2, the symmetric equilibrium efforts in both stages are

$$y_1 = \frac{3}{23},\ y_2 = \frac{27}{220},$$

then, the agents' total output is

$$E = 5y_1 + 4(3y_2) = 2.125.$$

If we replace the values of the prizes such that $v_1 = 1, v_2 = 2$, according to Proposition 2, the symmetric equilibrium efforts are

$$\widetilde{y}_1 = \frac{6}{55},\ \widetilde{y}_2 = \frac{149}{440},$$

then, the agents' total output is

$$\widetilde{E} = 5y_1 + 4(3y_2) = 4.609$$

Thus, the optimal order of the prizes that maximizes the agents' total output is the lower prize $v_1 = 1$ in the first stage, and the higher prize $v_2 = 2$ in the second stage.

In cases where the second stage's marginal output, β, exceeds the first stage's marginal output, α, designers may prioritize the second stage competition to maximize total output. In this case, the higher prize may be allocated in the second stage. The following result establishes a sufficient condition for allocating the higher prize in the second stage to maximize total output.

Proposition 3 *In a two-stage elimination contest with a weighted additive output function, $g(y_1, y_2) = \alpha y_1 + \beta y_2$, if $\frac{\beta}{\alpha} \geq \frac{(n-1)^2}{n(n-2)}$, the marginal output of the second prize v_2 is larger than that of the first prize v_1. In particular, if agents make positive efforts in the second stage, allocating the larger prize in the second stage maximizes total output. Then there is $v_2 > v_1$, but $v_1(n-1) > v_2$.*

Note that $\frac{\beta}{\alpha} = 1 \geq \frac{(n-1)^2}{n(n-2)}$, namely, even when the marginal output function of the efforts in the first and second stages are the same, when the agents do not make a positive effort in the second stage, according to Proposition 2, if the objective is to maximize the agents' total output, it is optimal to allocate the larger prize in the second stage.

3.1 Multi-stage elimination contests

In this section we consider r-stage elimination contests where $2 \leq r < n$. Specifically, we concentrate on the additive output function:

$$g((y_j)_{j=1}^r) = \sum_{j=1}^{r} y_j.$$

Then, by (1),agent 1's maximization problem in stage $s, s = 1, 2, ..., r$, is

$$\max_{y_{1s}} \sum_{j=s}^{r} v_j \Big(\frac{\sum_{k=1}^{j} y_{1k}}{\sum_{k=1}^{j} y_{1k} + (n-i)(\sum_{k=1}^{j} y_k)} \Big) \prod_{t=1}^{j-1} \Big(1 - \frac{\sum_{k=1}^{t} y_{1k}}{\sum_{k=1}^{t} y_{1k} + (n-t)(\sum_{k=1}^{t} y_k)} \Big) - \sum_{j=s}^{r} y_{1j}, \tag{6}$$

where $y_{1j}, j = 1, ..., r$ is agent 1's effort in stage j, and y_j is the symmetric effort of agent i, $i = 2, ..., n$ in stage j, $j = 1, ..., r$.

The subsequent three-stage elimination contest shows that agents are motivated to exert a positive effort solely during the initial stage when the prize values are non-increasing throughout the stages.

Example 2 *Consider a three-stage elimination contest with prizes v_1, v_2, v_3. If the output function is additive, agent 1's maximization problem in stage 3 is*

$$\max_{y_{13}} v_3 \frac{y_{11} + y_{12} + y_{13}}{(y_{11} + y_{12} + y_{13}) + (n-3)(y_1 + y_2 + y_3)} - y_{13}.$$

By symmetry, the FOC with respect to y_{13} is

$$\Gamma_{13} = v_3 \frac{(n-3)}{(n-2)^2} \frac{1}{y_1 + y_2 + y_3} - 1 \leq 0.$$

Likewise, agent 1's maximization problem in stage 2 is

$$\begin{aligned}
&\max_{y_{12}} v_2 \frac{y_{11} + y_{12}}{y_{11} + y_{12} + (n-2)(y_1 + y_2)} \\
&+ v_3 \Big(1 - \frac{y_{11} + y_{12}}{y_{11} + y_{12} + (n-2)(y_1 + y_2)}\Big) \frac{y_{11} + y_{12} + y_{13}}{y_{11} + y_{12} + y_{13} + (n-3)(y_1 + y_2 + y_3)} \\
&-(y_{12} + y_{13}).
\end{aligned}$$

By symmetry, the FOC with respect to y_{12} is

$$\begin{aligned}
\Gamma_{12} &= v_3 \frac{(n-3)}{(n-2)^2 (y_1 + y_2 + y_3)} \frac{n-2}{n-1} + \Big(v_2 \frac{(n-2)}{(n-1)^2} - v_3 \frac{1}{(n-1)^2}\Big) \frac{1}{y_1 + y_2}\Big) \\
&= \Big(v_2 \frac{(n-2)}{(n-1)^2} - v_3 \frac{1}{(n-1)^2}\Big) \frac{1}{y_1 + y_2} + v_3 \frac{n-2}{n-1} \frac{(n-3)}{(n-2)^2} \frac{1}{y_1 + y_2 + y_3} - 1 \\
&\leq 0,
\end{aligned}$$

and, agent 1's maximization problem in stage 1 is

$$\max_{y_{11}} v_1(\frac{y_{11}}{y_{11}+(n-1)y_1}) + v_2(1-\frac{y_{11}}{y_{11}+(n-1)y_1})\frac{y_{11}+y_{12}}{y_{11}+y_{12}+(n-2)(y_1+y_2)}$$
$$+v_3(1-\frac{y_{11}}{y_{11}+(n-1)y_1})(1-\frac{y_{11}+y_{12}}{y_{11}+y_{12}+(n-2)(y_1+y_2)})\frac{y_{11}+y_{12}+y_{13}}{y_{11}+y_{12}+y_{13}+(n-3)(y_1+y_2+y_3)}$$
$$-(y_{11}+y_{12}+y_{13}).$$

By symmetry, the FOC with respect to y_{11} is

$$\begin{aligned}\Gamma_{11} &= \frac{n-1}{n}((v_2\frac{(n-2)}{(n-1)^2}-v_3\frac{1}{(n-1)^2})\frac{1}{y_1+y_2}+v_3\frac{n-2}{n-1}\frac{(n-3)}{(n-2)^2}\frac{1}{y_1+y_2+y_3}) \\ &\quad +\frac{1}{y_1}(v_1\frac{n-1}{n^2}-v_2\frac{n-1}{n^2}\frac{1}{n-1}-v_3\frac{n-1}{n^2}\frac{n-2}{n-1}\frac{1}{n-2})-1 \\ &\leq 0.\end{aligned}$$

Let $v=v_1=v_2=v_3$. Then, the solution is

$$\begin{aligned}y_1 &= \frac{1}{n^2}\frac{v}{n^2-3n+2}\left(3n^3-15n^2+20n-6\right) \\ y_2 &= y_3=0,\end{aligned}$$

which yields for $n \geq 4$,

$$\begin{aligned}\Gamma_{11} &= 0 \\ \Gamma_{12} &= -\frac{n^3-6n^2+8n-2}{3n^3-9n^2+8n-2}<0 \\ \Gamma_{13} &= v\frac{(n-3)}{(n-2)^2}\frac{1}{\frac{1}{n^2}\frac{v}{n^2-3n+2}(3n^3-15n^2+20n-6)}-1=-\frac{2n^3-11n^2+14n-4}{3n^3-12n^2+14n-4}<0.\end{aligned}$$

Consequently, in the symmetric equilibrium, the agents only exert a positive effort during the initial stage when the prizes are identical across the stages. Similarly, it can be verified that the agents only exert positive effort when the prizes increase across the stages.

In Example 2, the outcome that agents make positive efforts only in the initial stage appears to be straightforward. A unit of effort in the initial stage influences an agent's marginal revenue in all subsequent stages, whereas a unit of effort in any subsequent stage only affects an agent's revenue in that stage and the subsequent corresponding stages. However, the impact of the first stage's effort on the agent's revenue is significant in all stages except the first, with some probability but no certainty. For example, the effect of effort in the first stage on an agent's revenue in stage $s > 1$ is only applicable if the agent did not win in the previous stage. On the other hand, the effect of an agent's effort in stage s on his revenue in that stage is effective with a probability of one, so it is unclear whether the effect of the effort in the first stage on an agent's revenue is greater than the effect of an agent's effort in any subsequent stage.

The following result reveals several key details about the subgame perfect equilibrium of our elimination contests with an additive output function. In particular, it provides sufficient conditions for an agent to make or not make a positive effort at any stage ($r > s > 1$).

Proposition 4 *In the symmetric equilibrium of an r-stage elimination contest with an additive output function and prizes of $\{w_i\}_{i=1}^r$, we have:*

1) An agent does not make a positive effort in stage r if $v_{r-1} \geq v_r$.

2) An agent's total effort over all the stages of the contest satisfies

$$\sum_{i=1}^{r} y_i \geq \frac{(n-r)v_r}{(n-r+1)^2}.$$

3) If an agent makes a positive effort in stage $r-1 \geq s \geq 1$ then his total effort until and includes stage s satisfies

$$\sum_{i=1}^{s} y_i \leq \frac{(n-s)v_s - \sum_{i=s+1}^{r} v_i}{(n-s+1)}.$$

4) An agent does not make a positive effort in stage $r-1 \geq s \geq 1$ if

$$(n-s)v_s \leq \sum_{i=s+1}^{r} v_i.$$

5) If an agent makes a positive effort in stage $s+1$, he makes also some positive effort before stage $s+1$ if

$$(n-s)v_s > \sum_{i=s+1}^{r} v_i.$$

6) If an agent makes a positive effort in two successive stages s and $s+1$, $1 \leq s \leq r-2$, then his total effort until stage s satisfies

$$\sum_{i=1}^{s} y_i = \frac{(n-s)v_s - \sum_{i=s+1}^{r} v_i}{(n-s+1)}.$$

7) An agent does not make a positive effort in two successive stages s and $s+1$, $1 \leq s \leq r-1$, if

$$(n-s)((n-s+1)v_{s+1} - (n-s)v_s) - \sum_{i=s+1}^{r} v_i \leq 0.$$

We discuss the meanings of the claims in Proposition 4 in the order (1) - (7). (1) This result is derived from Proposition 2 regarding two-stage elimination contests, which states that agents do not make a positive effort in the final stage if the previous stage's prize is greater than the last stage's prize. (2) This result provides a lower bound for each agent's total effort in the contest, which is solely determined by the final stage prize value. (3) If an agent makes a positive effort at some stage, this result provides an upper bound on his total effort up to and including that stage, which is less than the prize value at that stage. (4) This is a condition according to which agents do not make any effort in any stage if the prize in that stage is significantly smaller than the sum of the prizes in the subsequent stages. (5) This is a condition in which if agents make a positive effort in one stage, they must have made some efforts in previous stages. (6) If agents make positive efforts in two consecutive stages, s and $s+1$, this result specifies the exact value of their total effort up to and including stage s. (7) This condition provides a condition under which agents do not make positive efforts in two successive stages, s and $s+1$, especially, this happens when the winning value in stage s is significantly higher than in stage s, namely, $\frac{v_s}{v_{s+1}} > \frac{n-s+1}{n-s}$.

The findings in Proposition 4 provide some basic facts on the agents' behavior in elimination contests, and different combinations of them can lead to new results. For example, by combining (1), which states that if the values of the prizes decrease, agents do not make a positive effort in the last stage, and (6), which states that if the value of the prizes decrease, agents do not make positive efforts in two consecutive stages, we get that

Proposition 5 *In a three-stage elimination contest with prizes $v_1 \geq v_2 \geq v_3$, agents only make a positive effort in the first stage.*

Claims (4) and (5) in Proposition 4, similar to Proposition 5, provide conditions under which an agent makes a positive effort only during the first stage.

Proposition 6 *In the r-stage elimination contest with additive output functions, agents make a positive effort in the first stage only if*

1) $(n-1)v_1 > \sum_{i=2}^{r} v_i$.

2) for all $2 \leq s < r-1$, $(n-s)v_s \leq \sum_{i=s+1}^{r} v_i$.

If agents make efforts only in the first stage, by (6), the maximization problem of agent 1 is

$$\max_{y_{11}} \sum_{j=1}^{r} v_j\left(\frac{y_{11}}{y_{11}+(n-j)y_1}\right)\prod_{t=1}^{j-1}\left(1-\frac{y_{11}}{y_{11}+(n-j+t)y_1}\right)-y_{11},$$

where y_{11} represents agent 1's effort in the first stage, while y_1 represents the symmetric equilibrium effort of the other agents in the same stage. In that case, with only positive efforts in the first stage, our elimination contest is equivalent to a one-stage competition with multiple prizes. According to Sela (2020), the symmetric equilibrium effort in the first stage is

$$y_1 = \sum_{i=1}^{r} v_i \frac{1-(H_n - H_{n-i})}{n}. \tag{7}$$

where

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

Then, for all $j = 1, ..., n-1$ we have

$$\frac{dy}{dv_i} - \frac{dy}{dv_{i+1}} = \frac{H_{n-i} - H_{n-i-1}}{n} = \frac{1}{n(n-i)}.$$

Thus, just as in a one-shot contest with multiple prizes, where it is optimal to allocate the entire prize sum as a single prize if the goal is to maximize the agents' total effort (see Clark and Riis 1996), in our elimination contest with additive output functions, winning in the j-th stage has a greater effect on the agents' symmetric equilibrium effort than winning in the $j+1$-th stage. That is, the marginal effect of a prize v_k, $2 \leq k < r$ on the agents' equilibrium effort is greater than the marginal effect of each of the prizes $v_s, k < s \leq r$.

By the symmetry of the agents, each agent's expected payoff is

$$\begin{aligned} u &= \sum_{i=1}^{r} v_i \frac{1}{n-i+1} \prod_{j=1}^{i-1} \frac{n-j}{n-j+1} - y_1 \\ &= \sum_{i=1}^{r} v_i \frac{1}{n} - \sum_{i=1}^{r} v_i \frac{1-(H_n - H_{n-i})}{n} = \sum_{i=1}^{r} v_i \frac{(H_n - H_{n-i})}{n}, \end{aligned} \tag{8}$$

where y_1 is the symmetric equilibrium effort. Then, we obtain that

$$\frac{du_i}{dv_i} - \frac{du_i}{dv_{i+1}} = \frac{H_{n-i} - H_{n-i-1}}{n} = -\frac{1}{n(n-i)} < 0.$$

Thus, we can conclude that

Proposition 7 *In an r-stage elimination contest with an additive output function, if the agents only make a positive effort in the first stage, their total effort is maximized when the first-stage prize is the highest. The agents, on the other hand, maximize their expected payoffs when the lowest prize is awarded in the first stage and the prize values increase in the stages.*

In fact, the conflict between the designer and the agents over the order of the prizes applies to the majority of elimination contests, but not all of them. If we have two vectors of the same prizes, and the agents do not make a positive effort in the first stage, the designer and the agents may want the same vector of prizes, as shown in the following example:

Example 3 *Consider a three-stage elimination contest with four agents in the following two scenarios:*

Case A: the prizes are $v_1 = 20, v_2 = 1, v_3 = 1$. *Then, by (7), the agents' symmetric equilibrium effort in the first stage of the elimination contest is:*

$$\begin{aligned} y_1 &= \sum_{i=1}^{r} v_i \frac{1-(H_n - H_{n-i})}{n} \\ &= 20(\frac{3}{16}) + (\frac{5}{48}) - (\frac{1}{48}) = \frac{23}{6}, \end{aligned}$$

and in the other two stages, the agents do not make any effort, that is,

$$y_2 = y_3 = 0.$$

Thus, the total effort in this case is

$$TE = 15\frac{2}{6},$$

and, each agent's expected payoff is

$$u = \frac{20}{4} + \frac{1}{3}\frac{3}{4} + \frac{1}{2}\frac{3}{4}\frac{2}{3} - \frac{23}{6} = \frac{5}{3}.$$

Case B: Consider a different prize order: $v_1 = 1, v_2 = 1, v_3 = 20$. *It is easily verified that in that case, the agents choose not to compete in the first two stages, resulting in their equilibrium efforts in the three stages*

$$\widetilde{y}_1 = 0, \widetilde{y}_2 = 0, \widetilde{y}_3 = \frac{15}{4}$$

Each agent's total effort is

$$\widetilde{TE} = \frac{60}{4} = 15,$$

and, each agent's expected payoff is

$$\widetilde{u} = \frac{20}{4} - \frac{60}{16} = \frac{5}{4}$$

If an agent switches and decides to compete for one of the previous prizes, he wins at no cost, but his payoff is smaller and equal to 1*. Comparing the agents' total effort and expected payoff, we can conclude that the agents and the designer prefer the same vector of prizes, with the highest prize awarded in the first stage.*

The last example shows that agents do not always want to make a positive effort in the first stage, and they may prefer to start competing later. Furthermore, if the number of prizes is large in comparison to the number of agents, and the agents do not make positive efforts in the final stages, the prizes in the final stages reduce the agents' efforts in the first stages as well as their overall effort. In such a case, the designer may choose to reduce the number of stages so that the agents compete only for the large prizes awarded in the final stages. Then, the designers and agents may prefer the same order of the prizes in the elimination contests.

4 Parallel contests

So far, we have assumed that the agents compete in sequential r contests, with the winner of each contest being eliminated from the next contests after winning, so that each agent can only win one prize. Consider n agents competing in r parallel contests with prizes v_j, where $j = 1, ..., r$. Each agent i makes a single effort of y_i for all contests, and unlike in the elimination contest, he can win multiple prizes. Each stage includes a Tullock contest, and this competition will be referred to as the parallel r contests.

The maximization problem of agent 1 is

$$\max_{y_1} \sum_{j=1}^{r} v_j \frac{y_1}{\sum_{i=1}^{n} y_i} - y_1. \tag{9}$$

Agents use long-run efforts in both parallel and elimination contests. Parallel contests, on the other hand, allow an agent to win multiple prizes, as opposed to elimination contests, which only allow for one. Despite this, the following result shows that agents prefer elimination contests over parallel contests.

Proposition 8 *If the output function in an elimination contest with r stages is additive, each agent's expected payoff is higher than in parallel r contests with the same prizes over the stages.*

The intuitive explanation is that when an agent wins only one prize, his expected revenue is lower than when he wins multiple prizes; however, his equilibrium effort is lower when he wins only one prize. As a result, each agent's expected payoff in an elimination contest with r stages is greater than in parallel r contests.

5 Conclusion

We study elimination contests in which the winner of the previous stage is eliminated and all other agents advance to the next stage. However, we assume that an agent's probability of winning in some stage is determined by the agents' efforts in all previous stages, including the current stage. As a result, an agent does not have to make a positive effort at any stage to win, as long as he made some efforts in previous stages. According to these assumptions, the contests across the stages are correlated, which means that any decision made at any stage influences the outcomes of all subsequent contests.

We assumed that each elimination contest had a single winner who was eliminated from the subsequent stages. If we instead assume that there are multiple winners in each stage and that all of the winners in one stage are eliminated from the competitions in subsequent stages, all of our results will remain valid. Notice that the way we assign $k > 1$ prizes of the same value to k winners in some stages is equivalent to allocating these k prizes through our elimination contest with k stages, where there is only one winner in each stage.

We examine the agents' symmetric equilibrium strategies, specifically which stages they choose to make a positive effort and which they do not, where their strategies are determined by the form of their output function and the values of the prizes in each stage. If the values of prizes decrease in stages and the marginal output of effort does not increase in stages, agents have a greater incentive to make positive efforts in the early stages. However, we show that in two-stage elimination contests, if the marginal output of the effort in the second stage is sufficiently larger than the marginal output of the first stage, then each agent makes a positive effort in both stages, and the agents who maximize their expected payoff prefer that the lower prize be allocated in the first stage, as does the designer if he wants to maximize the agents' total output.

For elimination contests with more than two stages and agents with additive output functions, i.e., the marginal output is identical for any effort regardless of the stage, we provide conditions under which agents either make or do not make a positive effort in any stage, as well as some limits to the agents' total output up to any stage depending on whether the agents make some effort in that stage.

The designer, who wants to maximize the agents' total output, and the agents, who want to maximize their expected payoff, typically have different preferences for the order of the prizes over the stages, but we show that these preferences can coincide, resulting in an optimal prize order for both the designer and the agents.

6 Appendix

6.1 Proof of Proposition 1

When the output function is $g(y_1, y_2)$, By symmetry of the agents, the FOC of the maximization problem (2) of an agent with respect to y_1 is

$$\Gamma_{y_1} = v_1 \frac{n-1}{n^2 g(y_1)} - v_2 \frac{1}{n^2 g(y_1)} + v_2 \frac{(n-2) g_{y_1}(y_1, y_2)}{n(n-1) g(y_1, y_2)} - 1 \leq 0,$$

and the FOC with respect to y_2 is

$$\Gamma_{y_2} = v_2 \frac{(n-2) g_{y_2}(y_1, y_2)}{(n-1)^2 g(y_1, y_2)} - 1 \leq 0.$$

Suppose that $\Gamma_{y_2} = 0$. If $n\ g_{y_2}(y_1, y_2) \leq (n-1) g_{y_1}(y_1, y_2)$, we obtain that

$$v_2 \frac{(n-2) g_{y_1}(y_1, y_2)}{n(n-1) g(y_1, y_2)} - 1 > \Gamma_{y_2} = 0,$$

which implies that

$$\Gamma_{y_1} > \frac{1}{n^2 y_1}(v_1(n-1) - v_2) > 0.$$

The last inequality yields that $y_2 \geq 0$ and $y_1 \to \infty$ which is not a possible solution. Thus, we can conclude that if $v_1(n-1) \geq v_2$ and $n g_{y_2}(x_1, y_2) \leq (n-1) g_{y_1}(x_1, y_2)$ then Γ_{y_2} has to be lower than zero, implying that $y_2 = 0$.

6.2 Proof of Proposition 2

When the output function is weighted additive of the form $g(y_1, y_2) = \alpha y_1 + \beta y_2$, the FOC of the maximization problem (2) with respect to y_1 is

$$v_1 \frac{n-1}{n^2 y_1} - v_2 \frac{1}{n^2 y_1} + v_2 \frac{\alpha(n-2)}{n(n-1)(\alpha y_1 + \beta y_2)} - 1 = 0. \tag{10}$$

and the FOC with respect to y_2 is

$$v_2 \frac{\beta(n-2)}{(n-1)^2(\alpha y_1 + \beta y_2)} - 1 = 0. \tag{11}$$

If we insert (11) into (10), we obtain that

$$v_1 \frac{n-1}{n^2 y_1} - v_2 \frac{1}{n^2 y_1} + \frac{((n-1)\alpha - n\beta)}{n\beta} = 0.$$

This implies that the equilibrium effort in the first stage is

$$y_1 = \frac{\beta}{n^2\beta - n(n-1)\alpha} \left((n-1)v_1 - v_2\right),$$

and the equilibrium effort in the second stage is

$$y_2 = \left(-\frac{\alpha}{n^2\beta - n\alpha\,(n-1)} \left(v_1\,(n-1) - v_2\right) + \frac{v_2}{(n-1)^2}\,(n-2)\right).$$

We obtain that $y_2 > 0$ iff

$$\frac{\beta}{\alpha} > \frac{1}{n^2} \left(n\,(n-1) + \frac{1}{v_2}\left(v_1\,(n-1) - v_2\right) \frac{(n-1)^2}{n-2}\right).$$

Otherwise, we obtain that equilibrium effort in the second stage is $y_2 = 0$ and the equilibrium effort in the first stage is

$$y_1 = \frac{1}{n^3 - n^2} \left(v_1(n-1)^2 + v_2(1 - 3n + n^2)\right).$$

So far, we obtained that when the output function is weighted additive of the form $g(y_1, y_2) = \alpha y_1 + \beta y_2$, if agents do not exert any additional effort in the second stage, by (5), the equilibrium effort in the first stage is

$$y_1 = \frac{1}{n^3 - n^2} \left(v_1(n-1)^2 + v_2(1 - 3n + n^2)\right),$$

while if agents exert additional efforts in the second stage, by (4), the equilibrium efforts in stages 1 and 2 are

$$
\begin{aligned}
y_{1a} &= \frac{\beta}{n^2\beta - n(n-1)\alpha}\left((n-1)v_1 - v_2\right) \\
y_{2a} &= \left(-\frac{\alpha}{n^2\beta - n\alpha\left(n-1\right)}\left(v_1\left(n-1\right) - v_2\right) + \frac{v_2}{\left(n-1\right)^2}\left(n-2\right)\right).
\end{aligned}
$$

It can be verified that y_{1a} decreases in β, while y_{2a} increases in β.

6.3 Proof of Proposition 3

When the output function is weighted additive of the form $g(y_1, y_2) = \alpha y_1 + \beta y_2$, and the agents make positive efforts in the second stage, by (4), the total output is

$$
\begin{aligned}
E &= n\alpha y_1 + (n-1)\beta y_2 \\
&= n\alpha\frac{\beta}{n^2\beta - n(n-1)\alpha}\left((n-1)v_1 - v_2\right) + (n-1)\beta\left(-\frac{\alpha}{n^2\beta - n\alpha\left(n-1\right)}\left(v_1\left(n-1\right) - v_2\right) + \frac{v_2}{\left(n-1\right)^2}\left(n-2\right)\right) \\
&= \frac{1}{n}\frac{\beta}{n\left(n-1\right)\left(\alpha - n\alpha + n\beta\right)}(\alpha v_1(1-n)^2 + \alpha v_2(1-3n+3n^2-n^3) + \beta v_2(-2n^2+n^3)).
\end{aligned}
$$

Thus, the marginal impact of the second prize on the total output is larger than that of the first prize if

$$\alpha(1-3n+3n^2-n^3) + \beta(-2n^2+n^3) \geq \alpha(n-1)^2,$$

where this last inequality is satisfied if

$$\frac{\beta}{\alpha} \geq \frac{(n-1)^2}{n(n-2)}.$$

Because that

$$
\begin{aligned}
& \frac{1}{n^2}\left(n\left(n-1\right) + \frac{1}{v_2}\left(v_1\left(n-1\right) - v_2\right)\frac{\left(n-1\right)^2}{n-2}\right) - \frac{(n-1)^2}{n(n-2)} \\
= & \frac{1}{n^2 v_2}\frac{n-1}{n-2}v_1(n-1)^2 - v_2(2n-1)) > 0,
\end{aligned}
$$

the threshold for making a positive effort in the second stage $\frac{1}{n^2}\left(n\left(n-1\right) + \frac{1}{v_2}\left(v_1\left(n-1\right) - v_2\right)\frac{(n-1)^2}{n-2}\right)$ is larger than the threshold for the second prize having a greater marginal effect on total output than the first, $\frac{(n-1)^2}{n(n-2)}$. Thus, to maximize the agents' total output, the higher prize should be allocated in the second stage where, $v_2 > v_1$ but still $v_1(n-1)$ should be larger than v_2.

6.4 Proof of Proposition 4

The maximization problem of agent 1 in stage $s, s = 1, 2, ..., r$ given that he plays in that stage is

$$\max_{y_{1s}} \sum_{j=s}^{r} v_j(\frac{\sum_{k=1}^{j} y_{1k}}{\sum_{k=1}^{j} y_{1k} + (n-j)(\sum_{k=1}^{j} y_k)}) \prod_{t=s}^{j-1}(1 - \frac{\sum_{k=1}^{t} y_{1k}}{\sum_{k=1}^{t} y_{1k} + (n-t)(\sum_{k=1}^{t} y_k)}) - \sum_{j=s}^{r} y_{1j}.$$

Denote for all $1 \le s \le r$,

$$\widetilde{y}_{1s} = \sum_{k=1}^{s} y_{1k},\ \widetilde{y}_s = \sum_{k=1}^{s} y_k,$$

In other words, $\widetilde{y}_{1s}$ represents agent 1's total effort up to and including stage s, whereas $\widetilde{y}_s$ represents the symmetric total effort of all other agents excluding agent 1. Then, agent 1's maximization problem in stage s can be written as follows:

$$\begin{aligned}
&\max_{y_{1s}} v_s(\frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s}) \\
&+ v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1} + (n-s-1)\widetilde{y}_{s+1}})(1 - \frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s}) \\
&+ \sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j} + (n-j)\widetilde{y}_j}) \prod_{t=s}^{j-1}(1 - \frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t} + (n-t)\widetilde{y}_t}) \\
&- \sum_{j=s}^{r} y_{1j}.
\end{aligned}$$

The FOC of agent 1's maximization problem in stage s with respect to y_{1s} is

$$\begin{aligned}
\Gamma_{1s} \quad = \quad & \frac{d}{d_{y_{1s}}}(v_s(\frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s}) + v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1} + (n-s-1)\widetilde{y}_{s+1}})\frac{d}{d_{y_{1s}}}(1 - \frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s})) \qquad (12) \\
& + \frac{d}{d_{y_{1s}}}((1 - \frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s}))(\sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j} + (n-j)\widetilde{y}_j}) \prod_{t=s+1}^{j-1}(1 - \frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t} + (n-t)\widetilde{y}_t})) \\
& + (1 - \frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s})\frac{d}{d_{y_{1s}}}(v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1} + (n-s-1)\widetilde{y}_{s+1}})) \\
& + (1 - \frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s} + (n-s)\widetilde{y}_s})\frac{d}{d_{y_{1s}}}(\sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j} + (n-j)\widetilde{y}_j}) \prod_{t=s+1}^{j-1}(1 - \frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t} + (n-t)\widetilde{y}_t})) \\
& -1 \le 0
\end{aligned}$$

Similarly, the maximization problem of agent 1 in stage $s+1$ given that he plays in that stage is

$$\begin{aligned}
&\max_{y_{1s+1}} v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1} + (n-s-1)\widetilde{y}_{s+1}}) \\
&+ \sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j} + (n-j)\widetilde{y}_j}) \prod_{t=s+1}^{j-1}(1 - \frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t} + (n-t)\widetilde{y}_t})) \\
&- \sum_{j=s+1}^{r} y_{1j},
\end{aligned}$$

and the FOC of agent 1's maximization problem in stage $s+1$ with respect to y_{1s+1} is

$$\begin{aligned}\Gamma_{1s+1} &= \frac{d}{d_{y_{1s+1}}}(v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1}+(n-s-1)\widetilde{y}_{s+1}})) \\ &+\frac{d}{d_{y_{1s+1}}}\sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j}+(n-j)\widetilde{y}_j})\prod_{t=s+1}^{j-1}(1-\frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t}+(n-t)\widetilde{y}_t})) \\ &-1 \leq 0 \end{aligned} \tag{13}$$

Assuming $s=r-1$ and $s+1=r$, agent 1's FOCs in stages $r-1$ and r are identical to those in the two-stage elimination contest. By Proposition 2, we can conclude that (1) holds, which states that agents do not make a positive effort in stage r if $v_{r-1} \geq v_r$. Claim (2) follows directly from the FOC in stage r.

We now proceed to prove the remaining claims (3)-(7). Note that

$$\begin{aligned} &\frac{d}{d_{y_{1s}}}v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1}+(n-s-1)\widetilde{y}_{s+1}}) \\ =\ &\frac{d}{d_{y_{1s+1}}}v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1}+(n-s-1)\widetilde{y}_{s+1}}), \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{d_{y_{1s}}}\sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j}+(n-j)\widetilde{y}_j})\prod_{t=s+1}^{j-1}(1-\frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t}+(n-t)\widetilde{y}_t})) \\ =\ &\frac{d}{d_{y_{1s+1}}}\sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j}+(n-j)\widetilde{y}_j})\prod_{t=s+1}^{j-1}(1-\frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t}+(n-t)\widetilde{y}_t})). \end{aligned}$$

From (12) and (13) we obtain

$$\begin{aligned}\Gamma_{1s} &= \frac{d}{d_{y_{1s}}}(v_s(\frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s}+(n-s)\widetilde{y}_s})+v_{s+1}(\frac{\widetilde{y}_{1s+1}}{\widetilde{y}_{1s+1}+(n-s-1)\widetilde{y}_{s+1}})\frac{d}{d_{y_{1s}}}(1-\frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s}+(n-s)\widetilde{y}_s})) \\ &+\frac{d}{d_{y_{1s}}}((1-\frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s}+(n-s)\widetilde{y}_s}))\sum_{j=s+2}^{r} v_j(\frac{\widetilde{y}_{1j}}{\widetilde{y}_{1j}+(n-j)\widetilde{y}_j})\prod_{t=s+1}^{j-1}(1-\frac{\widetilde{y}_{1t}}{\widetilde{y}_{1t}+(n-t)\widetilde{y}_t})) \\ &+(1-\frac{\widetilde{y}_{1s}}{\widetilde{y}_{1s}+(n-s)\widetilde{y}_s})(\Gamma_{1s+1}+1)-1. \end{aligned}$$

By symmetry, $\widetilde{y}_{1t}=y_{1t}$ for all $t=1,...,r$. Therefore, if agent 1 chooses a positive effort in stage s, then we have

$$\begin{aligned} 0 &= \Gamma_{1s}=\frac{(n-s)}{(n-s+1)^2\widetilde{y}_s}(v_s-\frac{v_{s+1}}{n-s}-\frac{v_{s+2-....}}{n-s}-\frac{v_{s+3}}{(n-s)}-....-\frac{v_r}{n-s})+\frac{n-s}{n-s+1}(\Gamma_{1s+1}+1)-1 \\ &\leq \frac{1}{(n-s+1)^2\widetilde{y}_s}((n-s)v_s-v_{s+1}-v_{s+2}-....-v_r)-\frac{1}{n-s+1}. \end{aligned}$$

The first inequality holds because $\Gamma_{1s+1} \leq 0$ which implies that $\Gamma_{1s+1} + 1 \leq 1$. As a result, we obtain that claim (3) holds, namely, each agent's total effort until and including stage s satisfies

$$\widetilde{y_s} \leq \frac{(n-s)v_s - \sum_{j=s+1}^{r} v_j}{(n-s+1)}. \tag{14}$$

According to the last inequality, if $(n-s)v_s \leq \sum_{j=s+1}^{r} v_j$, then $\widetilde{y_s} \leq 0$. This leads to claim (4), which states that agent 1 does not make a positive effort in stage s.

If agents make a positive effort in stage $s+1$ we obtain that

$$\begin{aligned} 0 \quad &\geq \quad \Gamma_{1s} = \frac{(n-s)}{(n-s+1)^2\widetilde{y_s}}(v_s - \frac{v_{s+1}}{n-s} - \frac{v_{s+2-....}}{n-s} - \frac{v_{s+3}}{(n-s)} - .... - \frac{v_r}{n-s}) + \frac{n-s}{n-s+1}(\Gamma_{1s+1}+1) - 1 \\ &= \quad \frac{1}{(n-s+1)^2\widetilde{y_s}}((n-s)v_s - v_{s+1} - v_{s+2} - .... - v_r) - \frac{1}{n-s+1}. \end{aligned}$$

This implies that

$$\widetilde{y_s} \geq \frac{(n-s)v_s - \sum_{j=s+1}^{r} v_j}{(n-s+1)},$$

which gives us claim (5). If agents make positive efforts in both stages, s and $s+1$, we get equality instead of inequality in (14), resulting in claim (6).

If agents makes positive efforts in stages s and $s+1$ then we obtain that

$$\widetilde{y_s} = \frac{(n-s)v_s - v_{s+1} - v_{s+2} - ... - v_r}{n-s+1},$$

and

$$\widetilde{y}_{s+1} \leq \frac{(n-s-1)v_{s+1} - v_{s+2} - ... - v_r}{n-s}.$$

Thus,

$$\begin{aligned} \widetilde{y}_{s+1} - \widetilde{y_s} \quad &\leq \quad \frac{(n-s-1)v_{s+1} - v_{s+2} - ... - v_r}{n-s} - \frac{(n-s)v_s - v_{s+1} - v_{s+2} - ... - v_r}{n-s+1} \\ &= \quad \frac{(n-s)((n-s+1)v_{s+1} - (n-s)v_s) - \sum_{i=s+1}^{r} v_i}{(n-s)(n-s+1)}. \end{aligned}$$

Because $\widetilde{y}_{s+1} - \widetilde{y_s} \geq 0$, claim (7) is valid.

6.5 Proof of Proposition 8

The FOC of the maximization problem (9) is

$$\sum_{i=1}^{r} v_i \frac{\sum_{j=2}^{n} y_j}{(\sum_{j=1}^{n} y_j)^2} - 1 = 0.$$

By symmetry of the agents, their equilibrium effort in the parallel r contests is

$$y = (\sum_{i=1}^{r} v_i)\frac{n-1}{n^2},$$

and each agent's expected payoff is

$$\widetilde{u} = (\sum_{i=1}^{r} v_i)\frac{1}{n} - (\sum_{i=1}^{r} v_i)\frac{n-1}{n^2} = \frac{1}{n^2}(\sum_{i=1}^{r} v_i).$$

On the other hand, by (8), each agent's expected payoff in the elimination contest with r stages and additive output functions is

$$u = \sum_{i=1}^{r} v_i \frac{(H_n - H_{n-i})}{n}.$$

Then, we have

$$\begin{aligned} u - \widetilde{u} &= \sum_{i=1}^{r} v_i \frac{(H_n - H_{n-i})}{n} - \frac{1}{n^2}(\sum_{i=1}^{r} v_i) \\ &> \sum_{i=1}^{r} v_i (\frac{1}{n^2} - \frac{1}{n^2}) = 0. \end{aligned}$$

As a result, each agent's expected payoff in an elimination contest with r stages is greater than in parallel r contests.

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