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Abstract

We study two-sided matching contests with two sets, A and B, each of which includes a finite number of heterogeneous agents with commonly known types. The agents in each set compete in Tullock contests where they simultaneously send their costly efforts, and then are assortatively matched, namely, the winner of set A is matched with the winner of set B and so on until all the agents in the set with the smaller number of agents are matched. We analyze the agents' equilibrium efforts for which an agent's match-value is either a multiplicative or an additive function of the types who are matched. We demonstrate that whether or not both sets have the same number of agents might have a critical effect on their equilibrium efforts. In particular, a little change in the size of one of the sets might have a radical effect on the agents' equilibrium efforts.

Keywords: Two-sided matching, Tullock contest

JEL classification: D44, J31, D72, D82

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1 Introduction

Consider two-sided matching contests in which two contests take place independently within two groups. At the end of these contests, the agents in both groups are assortatively matched according to the ratings of the agents in the contests. Such a two-sided matching can be found in academic life, in which one of the groups includes universities which invest in hiring the best researchers and teachers as well as in providing the best conditions for the students. Such an investment improves its rank and thus will attract better candidates. The other group includes potential international student candidates, who aspire to be admitted to higher education universities, and for this purpose they put forth their best efforts in learning, studying for entrance exams, requesting recommendations, etc. Subsequently, candidates with the best qualities will be admitted to the highest ranked/top universities. Similar two-sided matching contests can be seen among accounting or law students on the one side and firms on the other, or among models, actors, and artists on one side, and the talent agencies on the other.

In such two-sided matching contests, the form of the contest among the agents has a significant effect on their behavior and, in particular, on the results. In this paper, we involve the well-known Tullock contest (see tullock 1980) in the study of two-sided matching. Formally, we study such a two-sided matching model under complete information where there are two sets of agents, set A with m heterogeneous firms and set B with $n, n \leq m$, heterogeneous workers, each of whom has commonly known types. The firms compete against each other in a Tullock contest, and similarly the workers compete against each other in another Tullock contest. The agents simultaneously exert their efforts, and then they are assortatively matched, namely, the winner in the contest of set A is matched with the winner in the contest of set B, and so on until all the agents in the set with the smaller number of agents are matched. The agents have a match-value function that is monotonically increasing in both types of firms and workers. An agent who is matched has a payoff of his match-value minus the cost of his effort.

We begin with $2x2$ assortative matching contests in which there are two agents on each side. We first establish that there are equilibrium efforts, and we provide a necessary and sufficient condition on the match-value function such that the firm (worker) with the higher type exerts a larger effort than his opponent. For such matching contests with a multiplicative match-value function of the agents' types, we explicitly characterize the equilibrium efforts and show that the ratio of the agents' efforts in each set is equal to the ratio of their types such that the larger the type of the agent is, the larger is his equilibrium effort. On the other hand, in these matching contests with an additive value function of the agents' types, there is only a unique equilibrium in which all the agents exert an effort of zero and therefore the matching is randomized and every firm (worker) has the same probability to be matched with each of the workers (firms).

The analysis of mxn assortative matching contests is quite complex, and in order to analyze the agents' behavior when the number of agents in both sets are not the same, we focus first on the simpler case of $3x2$ assortative matching contests with three firms and two workers. We establish the existence of the equilibrium efforts, and prove that at least two firms exert positive efforts in equilibrium, but with an additive match-value function all the three firms exert positive efforts in equilibrium. On the other hand, in equilibrium, the two workers might exert an effort of zero.

We also study assortative matching contests with a larger number of agents. We first generalize some of the above results and claim that in a mxn assortative matching contest where $m > n$, at least n firms exert positive efforts in equilibrium. Likewise, with an additive match-value function where $m > n$ at least $n+1$ firms exert positive efforts in equilibrium. Then, we consider assortative matching contests with $m \geq 2$ firms and two workers. We show that when there is a multiplicative match-value function, the ratio of the workers' efforts is equal to the ratio of their types exactly as in the $2x2$ assortative matching contests. In addition, we show that with an additive match-value function, the equilibrium workers' efforts are the same.

When the number of agents in both sets are the same we generalize our result for $2x2$ assortative matching contests and show that for every $n \geq 2$, in the nxn assortative matching contest with an additive match-value function, the equilibrium efforts of all the agents are zero and therefore the matching is randomized and each firm (worker) has the same probability to be matched with each of the workers (firms).

Hence, one of the insights we can derive from our analysis of assortative matching contests is that in a two-sided matching model where agents have additive match-value functions, if the sets have the same size, the agents from both sets might be not active. Then, when a new agent joins to one of the sets, although his type (ability) is much smaller than the other agents' types, an intensive competition might begin there. On the other hand, in a two-sided matching model where the agents have multiplicative match-value functions, if a new agent joins to one of the sets, if his type is much smaller than the other agents' types, he does not affect the other agents' strategies, and actually he stays out of the competition. In other words, if an additional agent, independently of his ability, joins to one of the sets in a two-sided matching model, depending on the form of the agents' match-value function, the status quo may be either completely changed or not at all.

The rest of the paper is organized as follows: in Section 2 we present our assortative matching contest. In Sections 3 and 4 we analyze $2x2$ and $3x2$ assortative matching contests, respectively, and in Section 5 we analyze some assortative matching contests with a larger number of agents. Section 6 concludes. Some of the proofs appear in the Appendix.

Related literature

In a matching model, efforts can be exerted by either one side or both. One-sided activity is modeled in the Tullock contest (see, for example, Tullock 1980, Skaperdas 1996, Szidarovszky and Okuguchi 1997, Baye and Hoppe 2003, and Einy et al. 2015), the all-pay contest (see, for example, Baye, Kovenock, and de Vries 1996, Moldovanu and Sela 2001, 2006, Che and Gale 1998, and Siegel 2009), the rank-order tournament (see, for example, Lazear and Rosen 1981, Rosen 1986), among others. In these contests we have one set of agents and one set of prizes, and the agents exert efforts to win the prizes. In such one-sided models, the higher is the agent's effort, the higher is his probability to win a larger prize. Some examples of one-sided models include Chao and

Wilson (1987) and Wilson (1989) who considered a seller facing a continuum of customers who differ in their private valuations of service quality. They showed how customers can be matched to different service qualities by offering them price menus that induce them to reveal their types. Likewise, Fernandez and Gali (1999) compared markets to matching tournaments in a model with a continuum of uniformly distributed agents on each side where only one side is active. They found that despite wasteful signaling, tournaments may be welfare superior to markets if the active agents have budget constraints.

A matching model in which efforts are exerted by agents on two sides with complete information is studied by Bhaskar and Hopkins (2016) who considered a continuum of homogenous agents who are matched according to the tournament model of Lazear and Rosen (1981). Hoppe, Moldovanu, and Sela (2009) studied two-sided markets with incomplete information and a finite number of agents where the agents are matched according to the all-pay contest. Hoppe, Moldovanu, and Ozdenoren (2011) studied this model where the agents on both sides compete in the all-pay contest, but there is an infinite number of agents on each side.[1] Dizdar, Moldovanu, and Szech (2019) also studied a two-sided model with a finite number of agents where on each side the agents compete in the all-pay contest, but in contrast to Hoppe, Moldovanu and Sela (2009) who assumed that the agents' efforts are wasteful, they assumed that the efforts are not completely wasteful and that the agents' efforts generate benefits for their partners that are increasing in the level of effort. We, on the other hand, also assume that the agents are assortatively matched and that their efforts are wasteful, but unlike the above two-sided matching models, in our model the agents compete in the Tullock contest. Since, the contest success function in the Tullock contest is stochastic while in the all-pay contest it is deterministic, though we assume there is complete information, the stochastic Tullock success function generates uncertainty in the matching between the two sides.

[1] Peters (2007) showed that equilibrium efforts in a very large finite two-sided matching model can be quite different from the equilibrium efforts in the continuum model.

2 The assortative matching contest

We consider a set $A = \{1, 2, ..., m\}$ of m firms and a set $B = \{1, 2, ..., n\}$ of n workers where $n \leq m$. The firms' types are m_i, where $m_i \geq m_{i+1}$, $i = 1, ..., m-1$. The workers' types are w_j, where $w_j > w_{j+1}$, $j = 1, ..., n-1$. All these types are commonly known. The matching contest proceeds as follows: Each firm $i, i = 1, 2, ..., m$ exerts an effort x_i, and each worker $j, j = 1, 2, ..., n$ exerts an effort y_j. Efforts are submitted simultaneously. The order of the firms (workers) to be matched is determined according to the method of Clark and Riis (1998) which is as follows: The first firm to be matched is determined by the probability success function that takes into account the efforts of all the firms. Formally, firm i, $i = 1, ..., m$ wins to be the first match with probability $\frac{x_i}{\sum_{k=1}^{m} x_k}$, where x_k is firm k's effort, $k = 1, ..., m$. Then, the second firm to be matched is determined by the probability success function that is based on the efforts of all the firms excluding the effort of the first winner. Thus, firm i, $i = 1, ..., m$ wins to be the second match with probability $\sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{x_k}{\sum_{j=1}^{m} x_j} \frac{x_i}{\sum_{\substack{j=1 \\ j \neq k}}^{m} x_j}$, and so on until all the firms are ranked, and similarly, all the workers are ranked. Then the firm and the worker who win first place in their sets are matched, those who win second place in their sets are matched and so on until all the workers are matched. If firm i is matched with worker j after exerting efforts of x_i and y_j, correspondingly, the firm's utility is $f(m_i, w_j) - x_i$ and, similarly, the worker's utility is $f(m_i, w_j) - y_j$, where $f : R^2 \to R^1$ is the match-value function which is monotonically increasing in the types of the firms and the workers who are matched. We say that a matching contest has an equilibrium if every agent chooses an effort that maximizes his expected utility given the efforts of the other agents in both sets.

3 The 2x2 assortative matching contest

We next consider a set $A = \{l, h\}$ of two firms and a set $B = \{l, h\}$ of two workers. We call the types m_h and w_h the high-type firm and worker, respectively, and the other types, m_l and

w_l, the low-type firm and worker, respectively. Suppose that firm $i, i = h, l$ exerts effort x_i and worker $j, j = h, l$ exerts effort y_j, and the two firms compete against each other in a Tullock contest and the two workers compete against each other in another Tullock contest. Then, the agents are assortatively matched, namely the firm that won the contest is matched with the worker who won his contest, and, similarly, the firm that lost the contest is matched with the worker who lost his contest. In this case, the maximization problem of the high-type firm is

$$\begin{aligned}
&\max_{x_h} f(m_h, w_h)\left[\frac{x_h}{x_h+x_l}\frac{y_h}{y_h+y_l}+\frac{x_l}{x_h+x_l}\frac{y_l}{y_h+y_l}\right] \\
&+f(m_h, w_l)\left[\frac{x_h}{x_h+x_l}\frac{y_l}{y_h+y_l}+\frac{x_l}{x_h+x_l}\frac{y_h}{y_h+y_l}\right]-x_h
\end{aligned} \tag{1}$$

and that of the low-type firm is

$$\begin{aligned}
&\max_{x_l} f(m_l, w_h)\left[\frac{x_l}{x_h+x_l}\frac{y_h}{y_h+y_l}+\frac{x_h}{x_h+x_l}\frac{y_l}{y_h+y_l}\right] \\
&+f(m_l, w_l)\left[\frac{x_l}{x_h+x_l}\frac{y_l}{y_h+y_l}+\frac{x_h}{x_h+x_l}\frac{y_h}{y_h+y_l}\right]-x_l
\end{aligned} \tag{2}$$

The maximization problem of the high-type worker is

$$\begin{aligned}
&\max_{y_h} f(m_h, w_h)\left[\frac{y_h}{y_h+y_l}\frac{x_h}{x_h+x_l}+\frac{x_l}{x_h+x_l}\frac{y_l}{y_h+y_l}\right] \\
&+f(m_l, w_h)\left[\frac{y_h}{y_h+y_l}\frac{x_l}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_h}{x_h+x_l}\right]-y_h
\end{aligned} \tag{3}$$

and that of the low-type worker is

$$\begin{aligned}
&\max_{y_l} f(m_h, w_l)\left[\frac{y_h}{y_h+y_l}\frac{x_l}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_h}{x_h+x_l}\right] \\
&+f(m_l, w_l)\left[\frac{y_h}{y_h+y_l}\frac{x_h}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_l}{x_h+x_l}\right]-y_l
\end{aligned} \tag{4}$$

The F.O.C. of the maximization problems (1), (2), (3), and (4) are

$$\begin{aligned}
(f(m_h, w_h)-f(m_h, w_l))\frac{x_l}{(x_h+x_l)^2}\frac{y_h-y_l}{y_h+y_l} &\leq 1 \\
(f(m_l, w_h)-f(m_l, w_l))\frac{x_h}{(x_h+x_l)^2}\frac{y_h-y_l}{y_h+y_l} &\leq 1 \\
(f(m_h, w_h)-f(m_l, w_h))\frac{y_l}{(y_h+y_l)^2}\frac{x_h-x_l}{x_h+x_l} &\leq 1 \\
(f(m_h, w_l)-f(m_l, w_l))\frac{y_h}{(y_h+y_l)^2}\frac{x_h-x_l}{x_h+x_l} &\leq 1
\end{aligned} \tag{5}$$

In an interior equilibrium, there is equality between the LHS and the RHS of (5) and then we have

Proposition 1 *The agents' equilibrium efforts in the* $2x2$ *assortative matching contest are obtained by the solution of the equations given in (5).*

Proof. See Appendix. ■

In an interior equilibrium, if we divide the LHS of the first two equations of (5) by each other, and also divide both RHS of these equations by each other, we obtain that

$$\frac{f(m_h, w_h) - f(m_h, w_l)}{f(m_l, w_h) - f(m_l, w_l)} = \frac{x_h}{x_l} \tag{6}$$

Similarly, if we divide both LHS of the last two equations of (5) by each other, and divide the RHS of these equations by each other, we obtain

$$\frac{f(m_h, w_h) - f(m_l, w_h)}{f(m_h, w_l) - f(m_l, w_l)} = \frac{y_h}{y_l} \tag{7}$$

Equations (6) and (7) yield

Proposition 2 *In the* $2x2$ *assortative matching contest, if all the agents exert positive efforts, the worker (firm) with the higher type exerts a larger effort than his opponent iff the match-value function satisfies* $\frac{df(m,w)}{dwdm} > 0$.

3.1 The 2x2 assortative matching contest with a multiplicative match-value function

We assume now that the agents' match-value function is multiplicative, namely, $f(m_i, w_j) = m_i w_j$, $i = l, h$, $j = l, h$.[2] By (5), the agents' equilibrium efforts satisfy:

$$\begin{aligned} m_h(w_h - w_l)\frac{x_l}{(x_h+x_l)^2}\frac{y_h - y_l}{y_h + y_l} &\leq 1 \\ m_l(w_h - w_l)\frac{x_h}{(x_h+x_l)^2}\frac{y_h - y_l}{y_h + y_l} &\leq 1 \\ w_h(m_h - m_l)\frac{y_l}{(y_h+y_l)^2}\frac{x_h - x_l}{x_h + x_l} &\leq 1 \\ w_l(m_h - m_l)\frac{y_h}{(y_h+y_l)^2}\frac{x_h - x_l}{x_h + x_l} &\leq 1 \end{aligned} \tag{8}$$

In an interior equilibrium, by (6) and (7), we obtain

$$\begin{aligned} \frac{y_h}{y_l} &= \frac{w_h}{w_l} \\ \frac{x_h}{x_l} &= \frac{m_h}{m_l} \end{aligned}$$

Thus, we have

Proposition 3 *In the* 2*x*2 *assortative matching contest with a multiplicative match-value function, the agents' equilibrium efforts are*

$$\begin{aligned} x_h &= \frac{m_h^2 m_l}{(m_l+m_h)^2}\frac{(w_h-w_l)^2}{(w_h+w_l)} \\ x_l &= \frac{m_h m_l^2}{(m_l+m_h)^2}\frac{(w_h-w_l)^2}{(w_h+w_l)} \\ y_h &= \frac{w_h^2 w_l}{(w_h+w_l)^2}\frac{(m_h-m_l)^2}{(m_h+m_l)} \\ y_l &= \frac{w_h w_l^2}{(w_h+w_l)^2}\frac{(m_h-m_l)^2}{(m_h+m_l)} \end{aligned} \tag{9}$$

where the worker (firm) with the larger type exerts a larger effort than his opponent.

[2] Note that our results in this section can be immediately extended to match-value functions of the form $f(m_i, w_j) = \delta(m_i)\rho(w_j)$, where δ and ρ are strictly increasing and differentiable.

In the standard Tullock contest between firms (workers) without any matching when their values of winning are m_h, m_l (w_h, w_l), the equilibrium efforts (see Tullock 1980) are

$$\begin{aligned}\widetilde{x}_h &= \frac{m_h^2 m_l}{(m_l+m_h)^2}\\ \widetilde{x}_l &= \frac{m_h m_l^2}{(m_l+m_h)^2}\end{aligned}$$

and the equilibrium efforts in the standard Tullock contest between the workers are

$$\begin{aligned}\widetilde{y}_h &= \frac{w_h^2 w_l}{(w_h+w_l)}\\ \widetilde{y}_l &= \frac{w_h w_l^2}{(w_h+w_l)}\end{aligned}$$

If we compare the agents' equilibrium efforts in the (two-sided) assortative matching contest with the (one-sided) standard Tullock contest, we obtain that each firm's effort in the $2x2$ assortative matching contest with a multiplicative match-value function is larger than in the standard Tullock contest iff

$$(w_h - w_l)^2 > (w_h + w_l)$$

Similarly, each worker's effort is larger than in the standard Tullock contest iff

$$(m_h - m_l)^2 > (m_h + m_l)$$

We can see that the agents' efforts are larger than their efforts in the Tullock contest iff the difference of their opponents' types is relatively larger with respect to their sum. In the $2x2$ assortative matching contest the agents' total effort is

$$\begin{aligned}TE &= x_h + x_l + y_h + y_l\\ &= m_h m_l \frac{(m_h - m_l)}{(m_l+m_h)^2}\frac{(w_h-w_l)^2}{(w_h+w_l)}\\ &\quad + w_h w_l \frac{(w_h-w_l)}{(w_h+w_l)^2}\frac{(m_h-m_l)^2}{(m_h+m_l)}\end{aligned}$$

Thus, when the sum of the agents' types is constant on both sides, the larger the difference of the agents' types on both sides is, the larger is the equilibrium total effort.

3.2 The 2x2 assortative matching contest with an additive match-value function

We next assume that the agents' match-value function is additive, namely, $f(m_i, w_j) = m_i + w_j$, $i = l, h$, $j = l, h$.[3] By (5), the agents' equilibrium efforts satisfy:

$$\begin{aligned}(w_h - w_l)\frac{x_l}{(x_h + x_l)^2}\frac{y_h - y_l}{y_h + y_l} &\leq 1 \\ (w_h - w_l)\frac{x_h}{(x_h + x_l)^2}\frac{y_h - y_l}{y_h + y_l} &\leq 1 \\ (m_h - m_l)\frac{y_l}{(y_h + y_l)^2}\frac{x_h - x_l}{x_h + x_l} &\leq 1 \\ (m_h - m_l)\frac{y_h}{(y_h + y_l)^2}\frac{x_h - x_l}{x_h + x_l} &\leq 1\end{aligned} \tag{10}$$

In an interior equilibrium, by (6) and (7),

$$x_h = x_l, y_h = y_l$$

which contradicts equality in the equations of (10). Thus, in this case we have only a corner equilibrium.

Proposition 4 *In the assortative 2x2 matching contest with an additive match-value function the equilibrium efforts are*

$$x_h = x_l = y_h = y_l = 0$$

and therefore the matching is randomized and every firm (worker) has the same probability to be matched with each of the workers (firms)

In the next section we show that if the numbers of firms and workers are not the same, in contrast to Proposition 4, they compete in the contests and exert positive efforts.

[3] Note that our results in this section can be immediately extended to match-value functions having the form $f(m_i, w_j) = \delta(m_i) + \rho(w_j)$, where δ and ρ are strictly increasing and differentiable.

4 The 3x2 assortative matching contest

We now consider the simplest case of two sets with a different number of agents where in set $A = \{h, m, l\}$ there are three firms and in set $B = \{h, l\}$ there are two workers. The firms' types are m_h, m_m and m_l, where $m_h \geq m_m \geq m_l$, and the workers' types are w_h and w_l where $w_h \geq w_l$. Suppose that firm $i, i = h, m, l$ exerts effort x_i and worker $j, j = h, l$ exerts effort y_j. Then, the maximization problem of firm h is

$$\max_{x_h} f(m_h, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_h}{x_h+x_m+x_l}+\frac{y_l x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right] \tag{11}$$
$$+f(m_h, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_h}{x_h+x_m+x_l}+\frac{y_h x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right]-x_h$$

the maximization problem of firm m is

$$\max_{x_m} f(m_m, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_m}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{y_l x_l}{x_h+x_m+x_l}\frac{x_m}{x_m+x_h}\right)\right] \tag{12}$$
$$+f(m_m, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_m}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{y_h x_l}{x_h+x_m+x_l}\frac{x_m}{x_m+x_h}\right)\right]-x_m$$

and the maximization problem of firm l is

$$\max_{x_l} f(m_l, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_l}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_l}{x_l+x_h}\right)\right] \tag{13}$$
$$+f(m_l, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_l}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_l}{x_l+x_h}\right)\right]-x_l$$

Similarly, the maximization problem worker h is

$$\max_{y_h} f(m_h, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_h}{x_h+x_m+x_l}+\frac{y_l x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right] \tag{14}$$
$$+f(m_m, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_m}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{y_l x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right]$$
$$+f(m_l, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_l}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right]-y_h$$

and the maximization problem of worker l is

$$\max_{y_l} f(m_h, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_h}{x_h+x_m+x_l}+\frac{y_h x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right] \tag{15}$$
$$+f(m_m, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_m}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{y_h x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right]$$
$$+f(m_l, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_l}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right]-y_l$$

The F.O.C. of firm h's maximization problems is

$$f(m_h, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h(x_m+x_l)}{(x_h+x_m+x_l)^2}-\frac{y_l x_l}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_m}-\frac{y_l x_m}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_l}\right)\right] \tag{16}$$
$$+f(m_h, w_h)\left[\frac{y_l}{y_h+y_l}\left(\frac{x_m}{(x_h+x_m)^2}\frac{x_l}{x_h+x_m+x_l}+\frac{x_l}{(x_h+x_l)^2}\frac{x_m}{x_h+x_m+x_l}\right)\right]$$
$$+f(m_h, w_l)\left[\frac{1}{y_h+y_l}\frac{y_l(x_m+x_l)}{(x_h+x_m+x_l)^2}-\frac{y_h x_l}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_m}-\frac{y_h x_m}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_l}\right)\right]$$
$$+f(m_h, w_l)\left[\frac{y_h}{y_h+y_l}\left(\frac{x_m}{(x_h+x_m)^2}\frac{x_l}{x_h+x_m+x_l}+\frac{x_l}{(x_h+x_l)^2}\frac{x_m}{x_h+x_m+x_l}\right)\right]\leq 1$$

The F.O.C. of firm m's maximization problem is

$$f(m_m, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h(x_l+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_l x_h}{(x_h+x_m+x_l)^2}\frac{x_m}{x_l+x_m}-\frac{y_l x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{x_m+x_h}\right)\right] \tag{17}$$
$$+f(m_m, w_h)\left[\frac{y_l}{y_h+y_l}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{(x_l+x_m)^2}+\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{(x_m+x_h)^2}\right)\right]$$
$$+f(m_m, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l(x_l+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_h x_h}{(x_h+x_m+x_l)^2}\frac{x_m}{x_l+x_m}-\frac{y_h x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{x_m+x_h}\right)\right]$$
$$+f(m_m, w_l)\left[\frac{y_h}{y_h+y_l}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{(x_l+x_m)^2}+\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{(x_m+x_h)^2}\right)\right]$$
$$\leq 1$$

and the F.O.C. of firm l's maximization problem is

$$f(m_l, w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h(x_m+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_l x_h}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_m}-\frac{y_l x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_h}\right)\right] \tag{18}$$
$$+f(m_l, w_h)\left[\frac{y_l}{y_h+y_l}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{(x_l+x_m)^2}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{(x_l+x_h)^2}\right)\right]$$
$$+f(m_l, w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l(x_m+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_h x_h}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_m}-\frac{y_h x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_h}\right)\right]$$
$$+f(m_l, w_l)\left[\frac{y_h}{y_h+y_l}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{(x_l+x_m)^2}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{(x_l+x_h)^2}\right)\right]$$
$$\leq 1$$

Similarly, the F.O.C of worker h's maximization problems is

$$\begin{aligned} & f(m_h, w_h)\left[\frac{y_l}{(y_h+y_l)^2}\left(\frac{x_h}{x_h+x_m+x_l}-\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right] \\ & +f(m_m, w_h)\left[\frac{y_l}{(y_h+y_l)^2}\left(\frac{x_m}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}-\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right] \\ & +f(m_l, w_h)\left[\frac{y_l}{(y_h+y_l)^2}\left(\frac{x_l}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right] \leq 1 \end{aligned} \tag{19}$$

and the F.O.C. of worker l's maximization problems is

$$\begin{aligned} & f(m_h, w_l)\left[\frac{y_h}{(y_h+y_l)^2}\left(\frac{x_h}{x_h+x_m+x_l}-\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right] \\ & +f(m_m, w_l)\left[\frac{y_h}{(y_h+y_l)^2}\left(\frac{x_m}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}-\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right] \\ & +f(m_l, w_l)\left[\frac{y_h}{(y_h+y_l)^2}\left(\frac{x_l}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right] \leq 1 \end{aligned} \tag{20}$$

Then, we have an interior equilibrium.

Proposition 5 *The equilibrium efforts of the* $3x2$ *assortative matching contest are obtained by the solution of the equations given in (16), (17), (18), (19), and (20).*

Proof. See Appendix. ■

We showed that in the $2x2$ assortative matching contest with an additive match-value function the agents from both sets do not exert efforts in equilibrium. However, this does not occur in the $3x2$ matching contests.

Proposition 6 *In the* $3x2$ *assortative matching contest, at least two firms exert positive efforts in equilibrium.*

Proof. Suppose that in the $3x2$ assortative matching contest all the three firms do not exert any effort. In such a case, it is obvious that also the two workers do not have an incentive to exert positive efforts. Therefore, every firm is matched with each of the two workers with the probability of $\frac{1}{3}$ and then a firm has a positive expected payoff. In addition, a firm is not matched at all with the probability of $\frac{1}{3}$ and then it has an expected payoff of zero. Thus, if one firm exerts a positive effort that approaches zero, given that its opponents do not exert any effort, its expected payoff significantly increases since it is matched with the probability of $\frac{1}{2}$ with each of the workers who both exert an effort of zero such that each has the same chance to win as well as to lose. Therefore there is no equilibrium in which all the three firms do not exert any effort. ■

By Proposition 6, there is no equilibrium in the $3x2$ assortative matching contest in which all the three firms exert an effort of zero. However, the following example shows that there is an equilibrium in which both workers exert an effort of zero.

Example 1 *Assume a $3x2$ matching contest with three symmetric firms where $m = m_h = m_m = m_l$ and two asymmetric workers where $w_h \geq w_l$. By symmetry of the firms, assume that every firm exerts the same effort x and worker $j, j = h, l$ exerts effort y_j. By (16), (17), and (18), the firms have the same F.O.C. which is given by*

$$\begin{aligned} & f(m, w_h)\left[\frac{2}{9x}\frac{y_h}{y_h + y_l} + \frac{1}{18x}\frac{y_l}{y_h + y_l}\right] \\ & + f(m, w_l)\left[\frac{2}{9x}\frac{y_l}{y_h + y_l} + \frac{1}{18}\frac{y_h}{y_h + y_l}\right] \\ = & \; 1 \end{aligned}$$

By symmetry of the firms, the workers' F.O.C. (19) and (20) are

$$\begin{aligned} 3f(m, w_h)\left[\frac{y_l}{(y_h + y_l)^2}(\frac{1}{3} - (\frac{1}{6} + \frac{1}{6})\right] - 1 &< 0 \\ 3f(m, w_l)\left[\frac{y_h}{(y_h + y_l)^2}(\frac{1}{3} - (\frac{1}{6} + \frac{1}{6})\right] - 1 &< 0 \end{aligned}$$

Thus, the equilibrium efforts of the workers are $y_l = y_h = 0$, and $\frac{y_l}{y_h + y_l} = \frac{y_h}{y_h + y_l} = \frac{1}{2}$. Then, the identical effort of all three firms is $x = \frac{5}{18}\frac{f(m,w_h)+f(m,w_l)}{2}$.

By Proposition 6, in any $3x2$ assortative matching contest at least two firms exert positive efforts in equilibrium. The following example shows that in a $3x2$ assortative matching contest with a multiplicative match-value function, it is possible that exactly two firms exert positive efforts and the third one exerts an effort of zero, or, alternatively, stays out of the contest.

Example 2 *Suppose that in a $3x2$ assortative matching contest, firms h and m have the same type, and firm l exerts an effort of $x_l = 0$. Then, by the the equilibrium efforts in the $2x2$ assortative matching contest given by (9), we obtain that the equilibrium efforts of the workers satisfy $\frac{y_h}{y_l} = \frac{w_h}{w_l}$ and that the equilibrium efforts of the firms that participate are*

$$x_m = x_h = x = \frac{m_h}{4} \frac{(w_h - w_l)^2}{(w_h + w_l)}$$

By (18), the F.O.C. of firm l's maximization problem is

$$\begin{aligned} FOC_3 &= \left(\frac{x_m + x_h}{(x_h + x_m)^2} \frac{1}{y_h + y_l}(f(m_l, w_h)y_h + f(m_l, w_l)y_l)\right. \\ &\quad +\left(\frac{x_h}{x_h + x_m} \frac{1}{x_m} + \frac{x_m}{x_h + x_m} \frac{1}{x_h}\right) \frac{1}{y_h + y_l}(f(m_l, w_h)y_l + f(m_l, w_l)y_h) \\ &\quad -1 \end{aligned}$$

Inserting the equilibrium efforts of the other agents yields

$$\begin{aligned} FOC_3 &= \frac{1}{2x} \frac{1}{1 + \frac{w_l}{w_h}} m_l(w_h + \frac{w_l^2}{w_h}) + \frac{1}{x} \frac{1}{1 + \frac{w_l}{w_h}} m_l(2w_l) \\ &= \frac{1}{x} \frac{w_h}{w_h + w_l} m_l(2w_l + w_h + \frac{w_l^2}{w_h}) = \frac{4m_l}{m_h} \frac{w_h}{(w_h - w_l)^2}(2w_l + w_h + \frac{w_l^2}{w_h}) - 1 \\ &= \frac{4m_l}{m_h} \frac{(w_h + w_l)^2}{(w_h - w_l)^2} - 1 \end{aligned}$$

Thus, if $\frac{m_l}{m_h}$ is sufficiently small, FOC_3 is negative, which implies that firm l stays out of the contest.

The above example shows that in a $3x2$ assortative matching contest with a multiplicative match-value function there is an equilibrium in which only two firms participate. However, in any $3x2$ assortative matching contest with an additive match-value function all the firms take part in the contest.

Proposition 7 *In a $3x2$ assortative matching contest with an additive match-value function all the three firms exert positive efforts in equilibrium.*

Proof. Suppose that one of the three firms exerts an effort of zero. Then we actually have a $2x2$ assortative matching contest, and by Proposition 4, all the firms do not exert any effort. In that case, each of the firms has a probability of $\frac{1}{3}$ to be matched with each of the two workers and a probability of $\frac{1}{3}$ not to be matched at all. Thus, if one of the firms exerts any positive effort that approaches zero it significantly increases its expected payoff since then it is matched for sure with a probability of $\frac{1}{2}$ with each of the workers. Consequently, in any equilibrium, all the three firms participate and exert positive efforts. ■

5 The mxn assortative matching contest

Consider now the general case when there is a set $A = \{1, 2, ..., m\}$ of $m \geq 2$ firms and a set $B = \{1, 2, ..., n\}$ of $n \geq 2$ workers where $n \leq m$. The firms' types are m_i, where $m_i \geq m_{i+1}$, $i = 1, ..., m-1$. The workers' types are w_j, where $w_j > w_{j+1}$, $j = 1, ..., n-1$. An immediate generalization of Propositions (6) and (7) is

Proposition 8 *In a mxn assortative matching contest where $m > n$, at least n firms exert positive efforts in equilibrium. Likewise, in a mxn assortative matching contest with an additive match-value function where $m > n$ at least $n+1$ firms exert positive efforts in equilibrium.*

Consider now that $n = 2$ such that the firms' types are m_i, where $m_i \geq m_{i+1}$, $i = 1, ..., m-1$, and the workers' types are w_h and w_l, where $w_h \geq w_l$. Then, for interior equilibrium, we have the following two results :

Proposition 9 *In a $mx2$ assortative matching contest with a multiplicative match-value function the efforts of the workers satisfy*

$$w_h y_l - w_l y_h = 0$$

Proof. See Appendix. ■

Proposition 10 *In a $mx2$ assortative matching contest with an additive match-value function the equilibrium workers' efforts satisfy*

$$y_h = y_l$$

Proof. See Appendix. ■

Last, assume that $m = n$. In such a symmetric assortative matching contest with an additive match-value function, firm $i, 1 \leq i \leq n$, obtains an expected payoff that is equal to its own type m_i independent of the equilibrium efforts. In other words, independent of the ranking of firm i it has for sure a payoff of m_i and in addition a payoff that is equal to the type of the matched worker. Thus, the real prize for firm i is the type of the matched worker, and therefore each firm actually has n possible prizes which are functions of the workers' types with these prizes being contingent on the result in the contest. Thus, all the firms actually have the same n prizes such that they are actually symmetric agents who have symmetric equilibrium efforts. This argument holds for the workers as well such that they also exert symmetric equilibrium efforts. Thus, if the heterogeneous workers exert the same effort, each firm has n identical prizes since it has the same probability to be matched with each of the workers, independent of the result of the contest. As such, if n firms have n identical prizes, each of them does not have an incentive to exert an effort, and therefore each of the firms exert an effort of zero. Likewise, each of the workers exerts an effort of zero, and we have

Proposition 11 *In the nxn assortative matching contest with an additive match-value function, for every $n \geq 2$, the equilibrium efforts of all the agents are zero and therefore the matching is randomized and each firm (worker) has the same probability to be matched with each of the workers (firms).*

6 Concluding remarks

We studied assortative matching contests in which there are two sets of agents. In each set the agents compete against each other in a Tullock contest, and then according to the results of both Tullock contests, the agents from both sets are assortatively matched, such that the first agents from both sets are matched, the second agents are matched, and so on until all the agents from the smaller set are matched. Every two agents who are matched win a reward according to a match-value function that depends on both agents' types. Such symmetric assortative matching contests in which the number of agents in both sets is the same, have an equilibrium in which all the agents in both sets do not exert efforts and as such the agents are randomly matched. These matching contests may have other equilibrium strategies which depend on the form of the match-value function. However, when the number of agents in both sets is not the same, independent of the form of the match-value function, there is no equilibrium in which all the agents do not exert efforts, although it is possible that the agents of only one set do not exert efforts in equilibrium. Therefore if the agents' efforts have some positive effect, the sizes of the sets should be different.

When we compare the total effort in these two-sided assortative matching contests with the one-sided standard Tullock contest, we can see that if the variance of the agents' types in both sets is relative large, the total effort might be larger than in the one-sided Tullock contest and vice versa when the variance of the agents' types in both sets is relatively small. The reason is that when the variance of the agents' types in one set is relatively large, the agents of the other set have a high incentive to compete against each other, while in the one-sided Tullock contest, similarly to any other one-sided contest, if the variance of the agents' type is large, the competition between the agents is weak.

7 Appendix

7.1 Proof of Proposition 1

The S.O.C. of the maximization problems (1), (2), (3), and (4) are

$$(f(m_h,w_h)-f(m_h,w_l))\frac{-2x_l}{(x_h+x_l)^3}\frac{y_h-y_l}{y_h+y_l}$$
$$(f(m_l,w_h)-f(m_l,w_l))\frac{-2x_h}{(x_h+x_l)^3}\frac{y_h-y_l}{y_h+y_l}$$
$$(f(m_h,w_h)-f(m_l,w_h))\frac{-2y_l}{(y_h+y_l)^3}\frac{x_h-x_l}{x_h+x_l}$$
$$(f(m_h,w_h)-f(m_l,w_h))\frac{-2y_h}{(y_h+y_l)^3}\frac{x_h-x_l}{x_h+x_l}$$

which can be rewritten as

$$\frac{-2}{(x_h+x_l)}\left[(f(m_h,w_h)-f(m_h,w_l))\frac{x_l}{(x_h+x_l)^2}\frac{y_h-y_l}{y_h+y_l}\right]$$
$$\frac{-2}{(x_h+x_l)}\left[(f(m_l,w_h)-f(m_l,w_l))\frac{x_h}{(x_h+x_l)^2}\frac{y_h-y_l}{y_h+y_l}\right]$$
$$\frac{-2}{(y_h+y_l)}\left[(f(m_h,w_h)-f(m_l,w_h))\frac{y_l}{(y_h+y_l)^2}\frac{x_h-x_l}{x_h+x_l}\right]$$
$$\frac{-2}{(y_h+y_l)}\left[(f(m_h,w_l)-f(m_l,w_l))\frac{y_h}{(y_h+y_l)^2}\frac{x_h-x_l}{x_h+x_l}\right]$$

Since in an interior equilibrium each of the terms inside the parentheses is positive according to the F.O.C. given in (5), we obtain that each of the equations of the S.O.C. is negative and therefore the solution obtained by the equations of the F.O.C is an equilibrium.

7.2 Proof of Proposition 5

In an interior equilibrium, by (16), the F.O.C. of firm h's maximization problem is

$$\begin{aligned} FOC_h &= f(m_h,w_h)(foc_1+foc_2+foc_3+foc_4+foc_5) \\ &\quad +f(m_h,w_l)(foc_6+foc_7+foc_8+foc_9+foc_{10}) \\ &= 1 \end{aligned}$$

where

$$\begin{aligned}
foc_1 &= \frac{x_m + x_l}{(x_h + x_m + x_l)^2} \frac{y_h}{y_h + y_l} \\
foc_2 &= -\frac{x_l}{(x_h + x_m + x_l)^2} \frac{x_h}{x_h + x_m} \frac{y_l}{y_h + y_l} \\
foc_3 &= -\frac{x_m}{(x_h + x_m + x_l)^2} \frac{x_h}{x_h + x_l} \frac{y_l}{y_h + y_l} \\
foc_4 &= \frac{x_m}{(x_h + x_m)^2} \frac{x_l}{x_h + x_m + x_l} \frac{y_l}{y_h + y_l} \\
foc_5 &= \frac{x_l}{(x_h + x_l)^2} \frac{x_m}{x_h + x_m + x_l} \frac{y_l}{y_h + y_l} \\
foc_6 &= \frac{x_m + x_l}{(x_h + x_m + x_l)^2} \frac{y_l}{y_h + y_l} \\
foc_7 &= -\frac{x_l}{(x_h + x_m + x_l)^2} \frac{x_h}{x_h + x_m} \frac{y_h}{y_h + y_l} \\
foc_8 &= -\frac{x_m}{(x_h + x_m + x_l)^2} \frac{x_h}{x_h + x_l} \frac{y_h}{y_h + y_l} \\
foc_9 &= \frac{x_m}{(x_h + x_m)^2} \frac{x_l}{x_h + x_m + x_l} \frac{y_h}{y_h + y_l} \\
foc_{10} &= \frac{x_l}{(x_h + x_l)^2} \frac{x_m}{x_h + x_m + x_l} \frac{y_h}{y_h + y_l}
\end{aligned}$$

The S.O.C. of firm h's maximization problem is

$$\begin{aligned}
SOC_h &= f(m_h, w_h)(soc_1 + soc_2 + soc_3 + soc_4 + soc_5) \\
&\quad + f(m_h, w_l)(soc_6 + soc_7 + soc_8 + soc_9 + soc_{10})
\end{aligned}$$

where

$$
\begin{aligned}
soc_1 &= \frac{-2(x_h+x_m+x_l)(x_l+x_m)}{(x_h+x_m+x_l)^4}\frac{y_h}{y_h+y_l} \\
soc_2 &= (\frac{2(x_h+x_m+x_l)x_l}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_m} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_l}{y_h+y_l} \\
soc_3 &= (\frac{2(x_h+x_m+x_l)x_m}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_l}{y_h+y_l} \\
soc_4 &= (\frac{-2(x_h+x_m)x_m}{(x_h+x_m)^4}\frac{x_l}{x_h+x_m+x_l} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_l}{y_h+y_l} \\
soc_5 &= (\frac{-2(x_h+x_l)x_l}{(x_h+x_l)^4}\frac{x_m}{x_h+x_m+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_l}{y_h+y_l} \\
soc_6 &= \frac{-2(x_h+x_m+x_l)(x_l+x_m)}{(x_h+x_m+x_l)^4}\frac{y_l}{y_h+y_l} \\
soc_7 &= (\frac{2(x_h+x_m+x_l)x_l}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_m} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_h}{y_h+y_l} \\
soc_8 &= (\frac{2(x_h+x_m+x_l)x_m}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_h}{y_h+y_l} \\
soc_9 &= (\frac{-2(x_h+x_m)x_m}{(x_h+x_m)^4}\frac{x_l}{x_h+x_m+x_l} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_h}{y_h+y_l} \\
soc_{10} &= (\frac{-2(x_h+x_l)x_l}{(x_h+x_l)^4}\frac{x_m}{x_h+x_m+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_h}{y_h+y_l}
\end{aligned}
$$

We have the following relations among the elements of the FOC_h and those of the SOC_h:

$$
\begin{aligned}
soc_1 &= \frac{-2}{x_h+x_m+x_l} foc_1 \\
soc_2 &= (\frac{-2}{x_h+x_m+x_l} + \frac{x_m}{x_h(x_h+x_m)}) foc_2 > \frac{-2}{x_h+x_m+x_l} foc_2 \\
soc_3 &= (\frac{-2}{x_h+x_m+x_l} + \frac{x_l}{x_h(x_h+x_l)}) foc_3 > \frac{-2}{x_h+x_m+x_l} foc_3 \\
soc_4 &= (\frac{-2}{x_h+x_m} - \frac{1}{x_h+x_m+x_l}) foc_4 < \frac{-2}{x_h+x_m+x_l} foc_4 \\
soc_5 &= (\frac{-2}{x_h+x_l} - \frac{1}{x_h+x_m+x_l}) foc_5 < \frac{-2}{x_h+x_m+x_l} foc_4 \\
soc_6 &= \frac{-2}{x_h+x_m+x_l} foc_6 \\
soc_7 &= (\frac{-2}{x_h+x_m+x_l} + \frac{x_m}{x_h(x_h+x_m)}) foc_7 > \frac{-2}{x_h+x_m+x_l} foc_7 \\
soc_8 &= (\frac{-2}{x_h+x_m+x_l} + \frac{x_l}{x_h(x_h+x_l)}) foc_8 > \frac{-2}{x_h+x_m+x_l} foc_8 \\
soc_9 &= (\frac{-2}{x_h+x_m} - \frac{1}{x_h+x_m+x_l}) foc_9 < \frac{-2}{x_h+x_m+x_l} foc_9 \\
soc_{10} &= (\frac{-2}{x_h+x_l} - \frac{1}{x_h+x_m+x_l}) foc_{10} < \frac{-2}{x_h+x_m+x_l} foc_{10}
\end{aligned}
$$

Since $foc_j, j = 2, 3, 7, 8$ are negative and $foc_j, j = 1, 4, 5, 6, 9, 10$ are positive, we obtain that

$$SOC_h < \frac{-2}{x_h+x_m+x_l} FOC_h < 0$$

Similarly, it can be shown that the S.O.C. of the maximization problems of firms m and l are negative as well.

Now, in an interior equilibrium, by (19), the F.O.C. of worker h's maximization problem is

$$
\begin{aligned}
foc_h \quad = \quad & f(m_h, w_h)\left[\frac{y_l}{(y_h+y_l)^2}\frac{x_h}{x_h+x_m+x_l}\right] \\
& -f(m_h, w_h)\left[(\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l})\right] \\
& +f(m_m, w_h)\left[\frac{y_l}{(y_h+y_l)^2}\frac{x_m}{x_h+x_m+x_l}\right] \\
& -f(m_m, w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m})\right] \\
& +f(m_l, w_h)\left[\frac{y_l}{(y_h+y_l)^2}\frac{x_l}{x_h+x_m+x_l}\right] \\
& -f(m_l, w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l})\right] = 1
\end{aligned}
$$

The S.O.C. of worker h's maximization problem is

$$
\begin{aligned}
soc_h = - \quad & f(m_h, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\frac{x_h}{x_h+x_m+x_l}\right] \\
& +f(m_h, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}(\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l})\right] \\
& -f(m_m, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\frac{x_m}{x_h+x_m+x_l}\right] \\
& +f(m_m, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m})\right] \\
& -f(m_l, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\frac{x_l}{x_h+x_m+x_l}\right] \\
& +f(m_l, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l})\right]
\end{aligned}
$$

Then, we obtain that

$$soc_h = -foc_h\frac{2(y_h+y_l)}{(y_h+y_l)^2} < 0$$

Similarly, it can be shown that the S.O.C. of the maximization problem of worker l is negative as well.

7.3 Proof of Proposition 9

The maximization problem of worker h is

$$\max_{y_h} \sum_{i=1}^{m} m_i w_h \begin{bmatrix} \frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{bmatrix}$$

where $Pr(\text{firm } i \text{ wins first place})$ is the probability that firm i wins first place, and $Pr(\text{firm } i \text{ wins second place})$ is the probability that firm i wins second place.

Similarly, the maximization problem of worker l is

$$\max_{y_l} \sum_{i=1}^{m} m_i w_l \begin{bmatrix} \frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{bmatrix}$$

If we subtract the F.O.C. of these workers' maximization problems from each other we obtain that

$$\begin{aligned} \Delta FOC \quad = \quad & \sum_{i=1}^{m} \frac{m_i(w_h y_l - w_l y_h)}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins first place}) \\ & - \sum_{i=1}^{m} \frac{m_i(w_h y_l - w_l y_h)}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins second place}) \end{aligned}$$

Thus, when $w_h y_l - w_l y_h = 0$, we obtain that $\Delta FOC = 0$, which implies that in equilibrium $w_h y_l = w_l y_h$.

7.4 Proof of Proposition 10

The maximization problem of worker h is

$$\max_{y_h} \sum_{i=1}^{m} (m_i + w_h) \begin{bmatrix} \frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{bmatrix}$$

where $Pr(\text{firm } i \text{ wins first place})$ is the probability that firm i wins first place, and $Pr(\text{firm } i \text{ wins second place})$ is the probability that firm i wins second place. Similarly, the maximization problem of worker l is

$$\max_{y_l} \sum_{i=1}^{m} (m_i + w_l) \begin{bmatrix} \frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{bmatrix}$$

If we subtract the F.O.C. of these workers' maximization problems from each other we obtain that

$$\begin{aligned}\Delta FOC &= \sum_{i=1}^{m}(m_i + w_h)\frac{y_l}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins first place}) \\ &\quad -\sum_{i=1}^{m}(m_i + w_h)\frac{y_l}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins second place}) \\ &\quad -\sum_{i=1}^{m}(m_i + w_l)\frac{y_h}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins first place}) \\ &\quad +\sum_{i=1}^{m}(m_i + w_l)\frac{y_h}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins second place})\end{aligned}$$

Since for $j = h, i$ we have

$$\begin{aligned}\sum_{i=1}^{m} w_j \Pr(\text{firm } i \text{ wins the first place}) &= w_j \\ \sum_{i=1}^{m} w_j \Pr(\text{firm } i \text{ wins the second place}) &= w_j\end{aligned}$$

we obtain that

$$\begin{aligned}\Delta FOC &= m\frac{y_l - y_h}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins the first place}) \\ &\quad -m\frac{y_l - y_h}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins the second place}) \\ &\quad \sum_{i=1}^{m} m_i\frac{y_l - y_h}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins the first place}) \\ &\quad -\sum_{i=1}^{m} m_i\frac{y_l - y_h}{(y_h + y_l)^2}\Pr(\text{firm } i \text{ wins the second place})\end{aligned}$$

Thus, when $y_l = y_h$, we obtain that $\Delta FOC = 0$, which implies that in equilibrium $y_l = y_h$.

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