CONTESTS WITH IDENTITY-DEPENDENT EXTERNALITIES

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Contests with identity-dependent externalities

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Abstract

We study lottery (Tullock) contests with identity-dependent externalities. We consider two types of players where the players of the same type have the same winning value as well as the same identity-dependent loss value. It is assumed that the identity-dependent value affects the player whether or not he participates in the contest. We examine the effects of these externalities on the participation of a player in the contest, and find that a negative loss value may only encourage him to participate in the contest and never discourages them. On the other hand, this player's loss value may only discourage the players of the other type to participate in the contest. Furthermore, independent of the players' values of winning, there are always identity-dependent loss values that will ensure the participation of all the players in the contest. Similar results hold for positive loss values.

Keywords: Lottery (Tullock) contests, identity-dependent externalities, loss values.

JEL classification: D44, O31, O33

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1 Introduction

Contestants who compete in contests usually have values of winning a single prize, sometimes of winning several prizes, and occasionally they also have values of punishments-negative prizes (see, among others, Clark and Riis 1996,1998, Modovanu and Sela 2001, Schweinzer and Segev 2012, Moldovanu et al. 2012, Fu et al. 2014, Kamijo 2016, Sela 2020, and Caso et al. 2020). In all the above cases, it is assumed that when a player does not win a contest, he is indifferent to the identity of the winner. But in many situations, a player cares about the allocation of the prize if he loses. This phenomenon is known as an identity-dependent externality. While there is extensive literature on identity-dependent externalities in mechanism design and especially in auction theory (see, among others, Funk 1996, Jehiel et al. 1996, 1999, Das Varma 2002, Aseff and Chade 2008, and Brocas 2013), the literature on identity-dependent externalities in contest theory is quite sparse. Among these are Linster (1993) who shows that with the lottery contest success function and linear costs, the system of first-order conditions for the players may be solved to derive an equilibrium in pure strategies, and Klose and Kovenock (2015) who derive necessary and sufficient conditions for the existence of equilibria with two active players in the all-pay auction under complete information and identity-dependent externalities.[1]

In this paper, we try to shed some light on this issue in contests for which purpose we study Tullock contests (see, among others, Tullock 1980, Skaperdas 1996, and Baye and Hoppe 2003, Chowdhury and Sheremeta 2011, Ewerhart 2015 and Lu et al. 2022) with identity-dependent externalities. We consider a contest with two types of players in which players of the same type have the same winning value as well as the same identity-dependent loss value. We assume that players are unable to avoid the effect of externalities simply by refusing to participate in the contest. In other words, if a player chooses not to participate in the contest, the identity-dependent loss values still affect his utility such that he may have a negative payoff.

We consider the following three cases: 1) Each player has a loss value when he does not win; 2) Each

[1] See also Esteban and Ray (1999) and Konrad (2006) who study special cases of contests with identity-dependent externalities.

player has an identity-dependent loss value when a player of his type wins, and 3) Each player has an identity-dependent loss value when a player of the other type wins. We examine how the identity-dependent loss values affect the participation of the players in the contest. While it is clear that a player's (negative) identity-dependent loss value decreases his expected payoff, it is not clear how it affects his willingness to take part in the contest, since by participating, a player may reduce his negative expected payoff compared with staying out of the contest. For example, consider two classes of pupils from the same school who participate in the same competition. While every pupil wants to win the competition, he might have a different response if he does not win. One can say that he does not care who will win as long as the winner is from his class, and, another pupil can say that if he does not win, he prefers that no one from his class will win the competition. These two pupils have different identity-dependent loss values and obviously, the effects of these loss values on the participation of the players are not the same.

Our findings indicate that although the identity-dependent (negative) loss values decrease their players' expected payoffs, regardless of their reason, they may only encourage them to participate in the contest and never discourage them to do so. However, although identity-dependent loss values of players of some type do not directly affect the payoff of players of the other type, it could only discourage them to participate in the contest. When there are positive identity-dependent loss values, the results are exactly the opposite, namely, a player's identity-dependent loss value could only discourage him to participate in the contest and never encourages him. However, it could encourage players of the other type to participate. But, the identity-dependent loss value of a player never affects the players who are the reason for the existence of this loss value. This result could be explained by the fact that if the identity-dependent loss value of a player discourages the participation of the players who are the reason for the existence of this loss value, when these players will stay out of the contest, the reason for the loss value will disappear, and this implies that these players will no longer stay out of the contest. According to our study of the effect of identity-dependent loss values on players' participation in the contest, we can conclude that this effect is ambiguous such that the number of participants might increase or decrease. On the one hand, the players' identity-dependent loss values encourage their players to participate, but, on the other hand, they discourage the players of the other type to participate in the contest. Moreover, since there is no equilibrium in which all the players stay out of the contest, independent of the players' winning values, there are always identity-dependent loss values that ensure the participation of all the players in the contest.

The rest of the paper is organized as follows: In Section 2, we study the basic model when a player has a loss value if he does not win. In Section 3, we study this model when a player has an identity-dependent loss value if a player of his own type wins, and in Section 4 for the case when a player has an identity-dependent loss value if a player of the other type wins. In Section 5 we study positive externalities. Section 6 concludes.

2 Case 1 - loss values

Consider $n+m$ players who compete in a Tullock contest. Each player $i \in A = \{1,..,n\}$ has a positive winning value of v_A if he wins and a negative loss value of $-l_A$, if he does not win. Likewise, each player $j \in B = \{1,...,m\}$ has a positive winning value of v_B if he wins and a negative loss value of $-l_B$ if he does not win. In that case, we say that all the other players are the reason for a player's loss value. Each player $i \in A$ exerts an effort of x_i and bears a cost of $c(x_i) = x_i$. Similarly, each player $j \in B$ exerts an effort of y_j and bears a cost of $c(y_j) = y_j$. Then, player $i \in A$ wins with a probability of $\frac{x_i}{X+Y}$ where $X = \sum_{k=1}^{n} x_k$, and $Y = \sum_{k=1}^{m} y_k$. Similarly, player $j \in B$ wins with a probability of $\frac{y_j}{X+Y}$. We assume that a player cannot avoid the contest such that if he decides not to participate he is affected by his loss value. A player from set A will be called a player of type A, and a player from set B will be called a player of type B. This competition will be referred to as the contest of case 1. Below, we assume a symmetric equilibrium in which all the players of the same type have the same equilibrium strategy.

The maximization problem of player 1 of type A is

$$\max_{x_1} v_A \frac{x_1}{X+Y} - l_A(1 - \frac{x_1}{X+Y}) - x_1, \tag{1}$$

and that of player 1 of type B is

$$\max_{y_1} v_B \frac{y_1}{X+Y} - l_B(1 - \frac{y_1}{X+Y}) - y_1. \tag{2}$$

When the players of both types participate in the contest, the FOC of these players' maximization problems are

$$
\begin{aligned}
(v_A + l_A)\frac{X - x_1 + Y}{(X+Y)^2} &= 1 \\
(v_B + l_B)\frac{X + Y - y_1}{(X+Y)^2} &= 1.
\end{aligned}
\tag{3}
$$

By symmetry, $x = x_i$ for all $i \in A$ and $y = y_j$ for all $j \in B$. Then, the FOC are[2]

$$
\begin{aligned}
(v_A + l_A)\frac{(n-1)x + my}{(nx+my)^2} &= 1 \\
(v_B + l_B)\frac{nx + (m-1)y}{(nx+my)^2} &= 1.
\end{aligned}
\tag{4}
$$

By (4), if we divide the equations given by (4) by each other we obtain that

$$
(v_A + l_A)((n-1)x + my) = (v_B + l_B)(nx + (m-1)y).
$$

This implies that

$$
\frac{x}{y} = \frac{(v_B + l_B)(m-1) - (v_A + l_A)m}{(v_A + l_A)(n-1) - (v_B + l_B)n}. \tag{5}
$$

Substituting (5) into (4) gives us the players' equilibrium efforts:

$$
x = \frac{m+n-1}{(ml_A + nl_B + mv_A + nv_B)^2}\left(l_A + v_A\right)\left(l_B + v_B\right)\left(l_B + v_B + ml_A - ml_B + mv_A - mv_B\right), \tag{6}
$$

and

$$
y = \frac{1}{(ml_A + nl_B + mv_A + nv_B)^2}\left(l_A + v_A\right)\left(l_B + v_B\right)\left(m+n-1\right)\left(l_A + v_A - nl_A + nl_B - nv_A + nv_B\right). \tag{7}
$$

By (5), (6), and (7) we have

Proposition 1 *In the contest of case 1, the players of type A participate in the contest iff* $v_A + l_A \geq \frac{m-1}{m}(v_B + l_B)$*, and the players of type B participate in the contest iff* $v_B + l_B \geq \frac{n-1}{n}(v_A + l_A)$*. It is impossible that the both types of players do not participate in the contest. When the both types of players participate in the contest their equilibrium efforts are given by (6) and (7).*

[2] A simple derivation of (3) shows that the second-order conditions (SOC) are satisfied.

Proof. See Appendix. ■

Our goal in this paper is to analyze the effect of the loss values l_A and l_B on the players' participation in the contest. For this purpose we consider the following scenarios:

- Scenario 1: Assume that $l_A = l_B = 0$ and $v_B > \frac{m}{m-1} v_A$. Then, by Proposition 1, the players of type A stay out of the contest. Now assume that there are loss values of $l_A > l_B > 0$ such that $v_B + l_B < \frac{m}{m-1}(v_A + l_A)$. Then, by Proposition 1, all the players of type A participate. Thus, a sufficiently large l_A can encourage the players of type A to participate in the contest.

- Scenario 2: Assume that $l_A = l_B = 0$ and $v_B \geq \frac{n-1}{n} v_A$. Then, by Proposition 1, the players of type B participate in the contest. Now assume that there are loss values of $l_A > l_B > 0$ such that $v_A + l_A > \frac{n}{n-1}(v_B + l_B)$. Then, by Proposition 1, the players of type B do not participate. Thus, a sufficiently large l_A can discourage the players of type B to participate in the contest.

The above two scenarios yields

Proposition 2 *In the contest of case 1, a sufficiently high loss value of any type (A or B) may encourage the players of this type to participate in the contest, but may discourage the players of the other type to participate.*

The intuition for this result is quite clear. When the loss value of a player increases, his incentive to increase his effort in order to win increases as well. On the other hand, when a player significantly increases his effort, the incentive of the other players to compete decreases, since they should significantly increase their efforts to win, and then their expected payoffs might be negative.

Now, suppose that $n = 1$, namely, there is only one player of type A. Then, by (5), we have

$$\frac{x}{y} = \frac{(v_A + l_A)m - (v_B + l_B)(m-1)}{(v_B + l_B)}.$$

In that case, there is no symmetry between the two types since there is only one player of type A. Then, a sufficiently large l_A can encourage the player of type A to participate in the contest, but it does not affect the participation of players of type B. On the other hand, assume that $l_A = l_B = 0$ and $v_B < \frac{m}{m-1} v_A$. Then, by Proposition 1, the player of type A participates in the contest. However, for $l_B > l_A > 0$ such that $v_B + l_B > \frac{m}{m-1}(v_A + l_A)$ the player of type A does not participate. Thus, a sufficiently large l_B can discourage the player of type A to participate. We can conclude that

Proposition 3 *In the contest of case 1, when there is only one player of type A, a sufficiently high value of his loss value l_A, may encourage the participation of this player, but does not affect the participation of the players of type B. On the other hand, a sufficiently high value of the loss value l_B of the players of type B may discourage the participation of the player of type A.*

By Proposition 3, if there is only one player of type A, then the loss values of the player of type A does not affect the participation of the players of type B. This is intuitive, since, otherwise, if the players of type B will stay out of the contest, we will have a contest with only one player, which is impossible. A comparison of Propositions 2 and 3 indicates the power of a group of players over a single one. While by Proposition 2 a group of players can discourage the participation of another group of players, by Proposition 3, a single player is not able to discourage the participation of the other group of players.

It is of interest to examine the effect of the players' loss values on their expected payoffs. This effect is not straightforward since the loss values affect the players' equilibrium efforts as well. By inserting the equilibrium efforts given by (6) and (7) into (1) and (2) we obtain the players' expected payoffs. Then, the following result shows that (negative) loss values, as we assume in this section, have a negative effect on the payoff of the players who have these values.

Proposition 4 *In the contest with negative loss values (case 1), if all the players participate in the contest, the expected payoff of the players decrease in their loss values.*

Proof. See Appendix. ■

By Proposition 4, a player's loss value decreases his expected payoff, but, on the other hand, by Proposition 2, this loss value may encourage him to participate in the contest when without it he would prefer to stay out of the contest.

3 Case 2 - identity-dependent loss values

Consider $n+m$ players who compete in a Tullock contest. Each player $i \in A = \{1, .., n\}$ has a winning value of v_A if he wins, and an identity-dependent loss value of $-l_A$ if another player from set A wins. In that case, we say that all the other players in set A are the reason for the identity-dependent loss value of a player from set A. Likewise, each player $j \in B = \{1, ..., m\}$ has a winning value of v_B if he wins, and an identity-dependent loss value of $-l_B$ if another player from set B wins. Then, we say that the other players in set B are the reason for the identity-dependent loss value of a player from set B. Each player $i \in A$ exerts an effort of x_i and bears a cost of $c(x_i) = x_i$. Similarly, each player $j \in B$ exerts an effort of y_j and bears a cost of $c(y_j) = y_j$. Then, player $i \in A$ and player $j \in B$ win by the same probabilities as in the contest of case 1. We assume that a player cannot avoid the contest such that if he decides not to participate, he is affected by his identity-dependent loss value. This competition will be referred to as the contest of case 2. Below, we assume a symmetric equilibrium in which all players of the same type have the same equilibrium strategy.

The maximization problem of player 1 of type A is

$$\max_{x_1} v_A \frac{x_1}{X+Y} - l_A \sum_{j=2}^{n} \frac{x_j}{X+Y} - x_1, \tag{8}$$

and that of player 1 of type B is

$$\max_{y_1} v_B \frac{y_1}{X+Y} - l_B \sum_{j=2}^{m} \frac{y_j}{X+Y} - y_1, \tag{9}$$

When the players of both types participate in the contest, the FOC of these players' maximization problems are[3]

$$\begin{aligned} v_A \frac{X - x_1 + Y}{(X+Y)^2} + l_A \frac{X - x_1}{(X+Y)^2} &= 1 \\ v_B \frac{X + Y - y_1}{(X+Y)^2} + l_B \frac{Y - y_1}{(\sum_{k=1}^{n} x_k + \sum_{k=1}^{m} y_k)^2} &= 1. \end{aligned} \tag{10}$$

[3] A simple derivation of (10) shows that the second-order conditions (SOC) are satisfied.

By symmetry, $x = x_i$ for all $i \in A$, and $y = y_j$ for all $j \in B$. Then the FOC are

$$\begin{aligned} v_A \frac{(n-1)x + my}{(nx+my)^2} + l_A \frac{(n-1)x}{(nx+my)^2} &= 1 \\ v_B \frac{nx + (m-1)y}{(nx+my)^2} + l_B \frac{(m-1)y}{(nx+my)^2} &= 1. \end{aligned} \tag{11}$$

If we divide the equations given by (11) by each other we obtain that

$$(v_A + l_A)(n-1)x + v_A my = (v_B + l_B)(m-1)y + v_B nx.$$

This implies that

$$\frac{y}{x} = \frac{(n-1)(v_A + l_A) - nv_B}{(m-1)(v_B + l_B) - mv_A}. \tag{12}$$

Substituting (12) into (11) yields

$$\begin{aligned} x = {} & \frac{l_B + v_B - ml_B + mv_A - mv_B}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2} \cdot \\ & \left(\begin{array}{c} ml_A l_B - l_A v_B - l_B v_A - v_A v_B - l_A l_B + nl_A l_B + ml_A v_B + ml_B v_A + nl_A v_B \\ + nl_B v_A + mv_A v_B + nv_A v_B - mnl_A l_B - mnl_A v_B - mnl_B v_A \end{array} \right), \end{aligned} \tag{13}$$

and

$$\begin{aligned} y = {} & \frac{l_A + v_A - nl_A - nv_A + nv_B}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2} \cdot \\ & \left(\begin{array}{c} ml_A l_B - l_A v_B - l_B v_A - v_A v_B - l_A l_B + nl_A l_B + ml_A v_B + ml_B v_A + nl_A v_B \\ + nl_B v_A + mv_A v_B + nv_A v_B - mnl_A l_B - mnl_A v_B - mnl_B v_A \end{array} \right). \end{aligned} \tag{14}$$

By (12), (13), and (14) we have

Proposition 5 *In the contest of case 2, the players of type A participate in the contest iff* $v_A \geq \frac{m-1}{m}(v_B + l_B)$*, and the players of type B participate iff* $v_B \geq \frac{n-1}{n}(v_A + l_A)$*. It is not possible that the both types of players do not participate in the contest. When the both types of players participate, their equilibrium efforts are given by (13) and (14).*

Proof. See Appendix. ■

In order to examine the effects of the identity-dependent loss values l_A and l_B on the players' participation in the contest, consider the following scenarios:

- Scenario 3: Assume that $l_A = 0$ and $v_B > \frac{m}{m-1}v_A$. Then, by Proposition 5, the players of type A do not participate in the contest. In that case, the players of type A participate in the contest if $v_B + l_B < \frac{m}{m-1}v_A$. Thus, l_A does not affect the participation of the players of type A in the contest.

- Scenario 4: Assume that $l_A = 0$ and $v_B \geq \frac{n-1}{n}v_A$. Then, by Proposition 5, the players of type B participate in the contest. Now, assume that there is an identity-dependent loss value $l_A > 0$ such that $v_A + l_A > \frac{n}{n-1}v_B$. Then, by Proposition 5, the players of type B do not participate in the contest. Thus, a sufficiently large l_A can discourage the participation of the players of type B in the contest.

By the above two scenarios we can conclude that

Proposition 6 *In the contest of case 2, a sufficiently high identity-dependent loss value of a player of any type (A or B) does not affect the participation of the players of the same type, but may discourage the players of the other type to participate in the contest.*

The intuitive explanation for this result is as follows: a player has a loss value if one of the players of the same type wins. However, by our assumption of symmetric-type equilibrium, if this player stays out of the contest, all the other players of the same type stay out of the contest and therefore each of them is not going to win. As such, increasing the loss value of a player will not change his expected payoff if he already stays out of the contest. Thus, an identity-dependent loss value of a player does not affect the participation of the players of the same type. On the other hand, if all the players participate, by increasing the identity-dependent loss value of a player his incentive to increase his effort increases which may discourage the players of the other type to participate in the contest.

Now suppose that $n = 1$, namely, there is only one player of type A. Then, by (12), we have

$$\frac{y}{x} = \frac{v_B}{mv_A - (v_B + l_B)(m-1)}.$$

Now, since there is only one player of type A, there is no symmetry between the two types of players, and actually, in that case, l_A does not play any role. However, l_B may affect the participation of the player of type A. Suppose that $l_B = 0$ and $v_A > \frac{m-1}{m}v_B$, then, by Proposition 5, the player of type A participates in the contest. However, for $l_B > 0$ such that $v_B + l_B > \frac{m}{m-1} v_A$, by Proposition 5, the player of type A does not participate in the contest. Thus, a sufficiently large value of l_B may discourage the participation of the player of type A. By the above analysis, we can conclude that

Proposition 7 *In the contest of case 2, when there is one player of type A, a sufficiently high identity-dependent loss value l_B may discourage the participation of the player of type A, but does not affect the participation of the players of type B.*

Notice that when there is only one player of type A, similar to case 1, this player is not able to affect the participation of the players of type B. The following example illustrates a simple contest of case 2 where each of the two sets A and B includes two players.

Example 1 *Consider a contest of case 2 where $v_A = v_B = 4$, $l_B = 1, m = 2, n = 2$. Let $l_A < 2v_B - v_A = 4$. Then, by Proposition 5, all the players participate in the contest, and by (13)and (14) their equilibrium efforts are*

$$\begin{aligned} x &= -\frac{3}{4\left(l_A - 7\right)^2}\left(5l_A - 44\right) \\ y &= \frac{1}{4}\frac{l_A - 4}{\left(l_A - 7\right)^2}\left(5l_A - 44\right) \end{aligned}$$

These equilibrium efforts as functions of the identity-dependent loss value l_A are presented in the following figure:

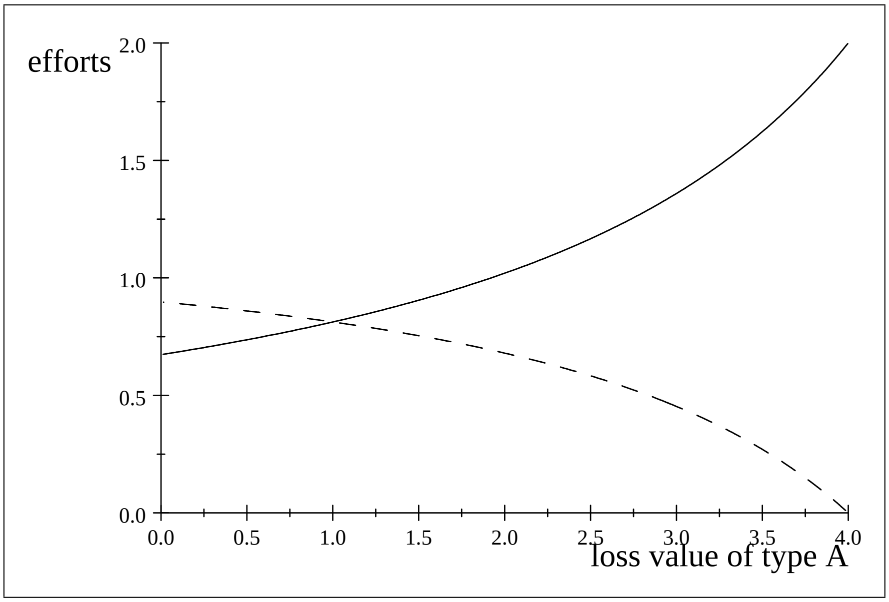


Figure 1: Equilibrium efforts in case 2

We can see that the symmetric equilibrium effort of the players of type A (solid line) increases in the identity-dependent loss value l_A, while the symmetric equilibrium effort of the players of type B (dashed line) decreases in l_A. By (8), the expected payoff of a player of type A is

$$u_A = \frac{3}{4\left(l_A - 7\right)^2}\left(2l_A^2 - 17l_A + 12\right)$$

This utility as a function of the identity-dependent loss value l_A is presented in the following figure:

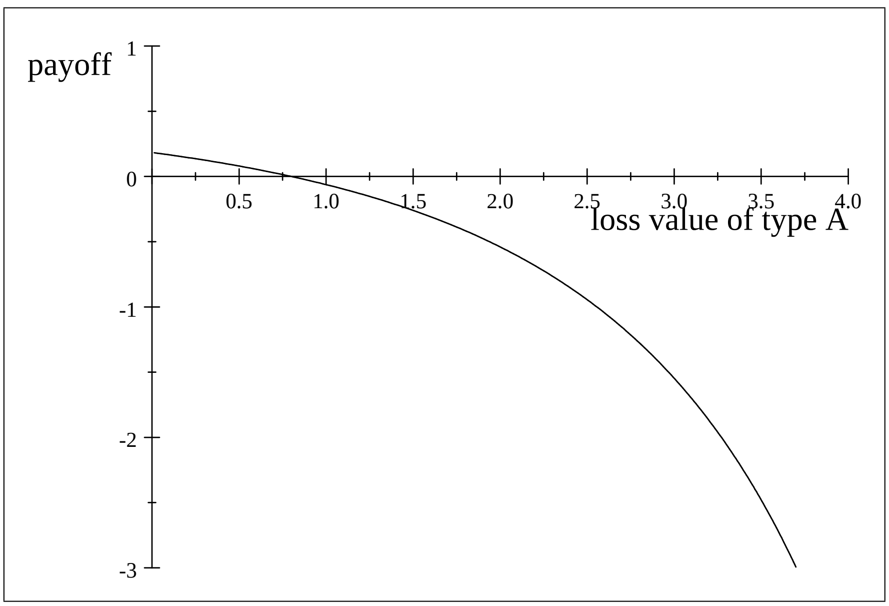


Figure 2: The payoff of type A in case 2

We can see that while the equilibrium effort of a player of type A increases in his identity-dependent loss value l_A, his expected payoff decreases.

In the above example, a player's identity-dependent loss value decreases his expected payoff. On the other hand, by Proposition 6, this identity-dependent loss value may encourage this player to participate in the contest, while without this identity-dependent loss value, this player may prefer to stay out of the contest.

4 Case 3 - identity-dependent loss values

Consider $n+m$ players who compete in a Tullock contest. Each player $i \in A = \{1,..,n\}$ has a winning value of v_A if he wins, and an identity-dependent loss value of $-l_A$ if a player from set B wins. In that case, we say that all the players from set B are the reason for the identity-dependent loss value of a player from set A**.** Likewise, each player $j \in B = \{1,...,m\}$ has a winning value of v_B if he wins, and an identity-dependent loss value of $-l_B$ if a player from set A wins. Then, we say that all the players from set A are the reason for the identity-dependent loss value of a player from set B**.** Each player $i \in A$ exerts an effort of x_i and bears a cost of $c(x_i) = x_i$. Similarly, each player $j \in B$ exerts an effort of $y_j$and bears a cost of $c(y_j) = y_j$. Then, player $i \in A$ and player $j \in B$ win by the same probabilities as in the contest of cases 1 and 2. We assume that a player cannot avoid the contest such that if he decides not to participate he is affected by his identity-dependent loss value. This competition will be referred to as the contest of case 3. Below, we assume a symmetric equilibrium in which all the players of the same type have the same equilibrium strategy.

The maximization problem of player 1 of type A is

$$\max_{x_1} v_A \frac{x_1}{X+Y} - l_A \sum_{j=1}^{m} \frac{y_j}{X+Y} - x_1, \tag{15}$$

and that of player 1 of type B is

$$\max_{y_1} v_B \frac{y_1}{X+Y} - l_B \sum_{i=1}^{n} \frac{x_i}{X+Y} - x_1. \tag{16}$$

When the players of both types participate in the contest, the FOC of these players' maximization problems are[4]

$$\begin{aligned} v_A \frac{X - x_1 + Y}{(X+Y)^2} + l_A \frac{Y}{(X+Y)^2} &= 1 \\ v_B \frac{X + Y - y_1}{(X+Y)^2} + l_B \frac{X}{(X+Y)^2} &= 1. \end{aligned} \tag{17}$$

By symmetry, $x = x_i$ for all $i \in A$, and $y = y_j$ for all $j \in B$. Then, the FOC are

$$\begin{aligned} v_A \frac{(n-1)x + my}{(nx+my)^2} + l_A \frac{my}{(nx+my)^2} &= 1 \\ v_B \frac{nx + (m-1)y}{(nx+my)^2} + l_B \frac{nx}{(nx+my)^2} &= 1. \end{aligned} \tag{18}$$

If we divide the equations given by (18) by each other we obtain that

$$v_A((n-1)x + my) + l_A my = v_B(nx + (m-1)y) + l_B nx.$$

This implies that

$$\frac{x}{y} = \frac{(m-1)v_B - m(v_A + l_A)}{(n-1)v_A - n(v_B + l_B)}. \tag{19}$$

Substituting (19) into (18) yields

$$x = \frac{v_A(v_B + ml_A + mv_A - mv_B)\left(mv_A - nl_B - v_A + nv_A + mnl_A + mnl_B\right)}{\left(mv_A + nv_B + mnl_A + mnl_B\right)^2}, \tag{20}$$

and

$$y = \frac{v_B\left(v_A + nl_B - nv_A + nv_B\right)\left(mv_A - nl_B - v_A + nv_A + mnl_A + mnl_B\right)}{\left(mv_A + nv_B + mnl_A + mnl_B\right)^2}. \tag{21}$$

By (19), (20), and (21) we have

Proposition 8 *In the contest of case 3, the players of type A participate in the contest iff $v_A + l_A \geq \frac{m-1}{m} v_B$, and the players of type B participate iff $v_B + l_B \geq \frac{n-1}{n} v_A$. It is not possible that the both types of players do not participate in the contest. When the both types of players participate in the contest their equilibrium efforts are given by (20) and (21).*

[4] A simple derivation of (17) shows that the second-order conditions (SOC)) are satisfied.

Proof. See Appendix. ■

In order to examine the effects of the identity-dependent loss values l_A and l_B on the players' participation in the contest, consider the following scenarios:

- Scenario 5 : Assume that $l_A = l_B = 0$ and $v_B > \frac{m}{m-1}v_A$. Then, by Proposition 8, all the players of type A stay out of the contest. Now, assume that there are identity-loss values $l_A, l_B > 0$ such that $v_B + l_B < \frac{m}{m-1}(v_A + l_A)$. Then, by Proposition 8, the players of type A participate in the contest. Thus, a sufficiently large l_A may encourage the players of type A to participate in the contest.

- Scenario 6: We can see that l_A does not affect the participation of the players of type B.

By the above two scenarios, we can conclude that

Proposition 9 *In the contest of case 3, a sufficiently high identity-dependent loss value of any type (A or B) may encourage the players of the same type to participate in the contest, but does not affect the participation of the players of the other type.*

The intuitive explanation for this result is as follows: If a player of type A has a loss value of l_B that increases, then he increases his effort to win the contest. However, if this implies that the players of type B will leave the competition, then each of the players of type B does not win and then the effect of l_B on the player of type A has immediately vanished. As such, the player of type A has an incentive to decrease his effort which implies that the players of type B will return to be active in the contest. Therefore, the identity-dependent loss value of any type (A or B) does not affect the participation of the players of the other type. On the other hand, If a player of type A stays out of the contest and his loss value of l_B increases, then his incentive to increase his effort to win the contest increases as well, Thus, a high identity-dependent loss value of any type (A or B) may encourage the players of the same type to participate in the contest.

Now suppose that $n = 1$, namely, there is only one player of type A. Then, by (19), we have

$$\frac{x}{y} = \frac{(m-1)v_B - m(v_A + l_A)}{v_A - (v_B + l_B)}.$$

We can see that the identity-dependent loss value l_B does not affect the participation of both types. However, the identity-dependent loss value l_A may affect the participation of the player of type A. Suppose that $l_A > 0$ and $v_B > \frac{m}{m-1} v_A$, then, by Proposition 8, the player of type A does not participate in the contest. However, for $l_A > 0$ such that $v_B < \frac{m}{m-1} v_A + l_A$, by Proposition 8, the player of type A participates. Thus, a sufficiently large l_A may encourage the player of type A to participate in the contest.

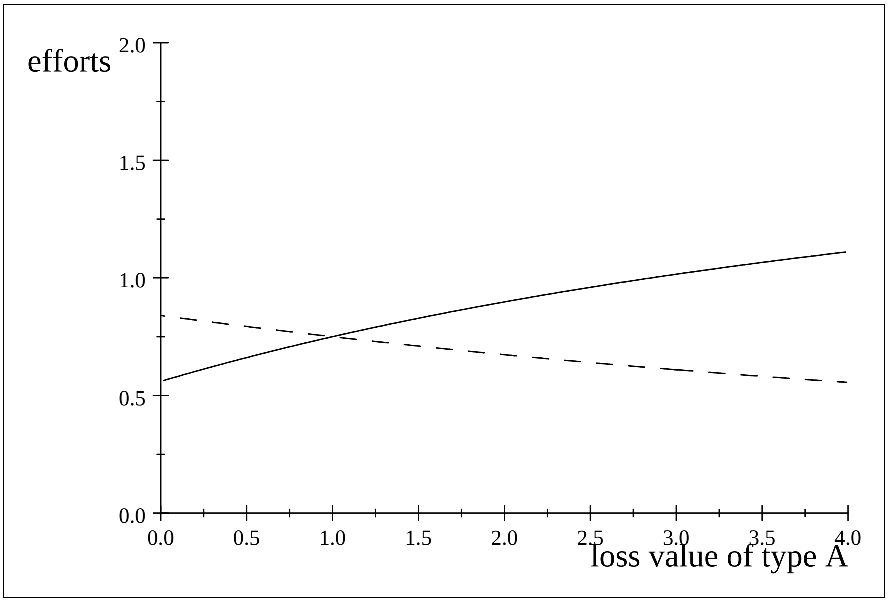
Proposition 10 *In the contest of case 3, when there is only one player of type A, the identity-dependent loss value l_B has no effect on the participation of both types. On the other hand, a sufficiently high value of l_A may encourage the participation of the player of type A, but does not affect the participation of the players of type B.*

Similar to cases 1 and 2, a single player is not able to affect the participation of the players of the other type. The following example illustrates a simple contest of case 3 in which each of the two sets A and B includes two players.

Example 2 *Consider a contest of case 2 where $v_A = v_B = 4$, $l_B = 1, m = 2, n = 2$. Let $l_A < 2v_B - v_A = 4$. Then, by Proposition 8, all the players participate in the contest, and by (20)and (21) their equilibrium efforts are*

$$\begin{aligned} x &= \frac{1}{4\left(l_A+5\right)^2}\left(2l_A+4\right)\left(4l_A+14\right) \\ y &= \frac{1}{16\left(l_A+5\right)^2}\left(96l_A+336\right) \end{aligned}$$

These equilibrium efforts as functions of the identity-dependent loss value l_A are presented in the following figure:



Fifure 3: Equilibrium efforts in case 3

We can see that the symmetric equilibrium effort of type A (solid-line) increases in the identity-dependent loss value l_A*, while the symmetric equilibrium effort of type B (dashed-line) decreases. By (15), the expected payoff of a player of type A is*

$$u_A = -\frac{3}{(l_A+5)^2}\left(l_A^2 + 4l_A - 2\right)$$

This utility as a function of the identity-dependent loss value l_A *is given in the following figure:*

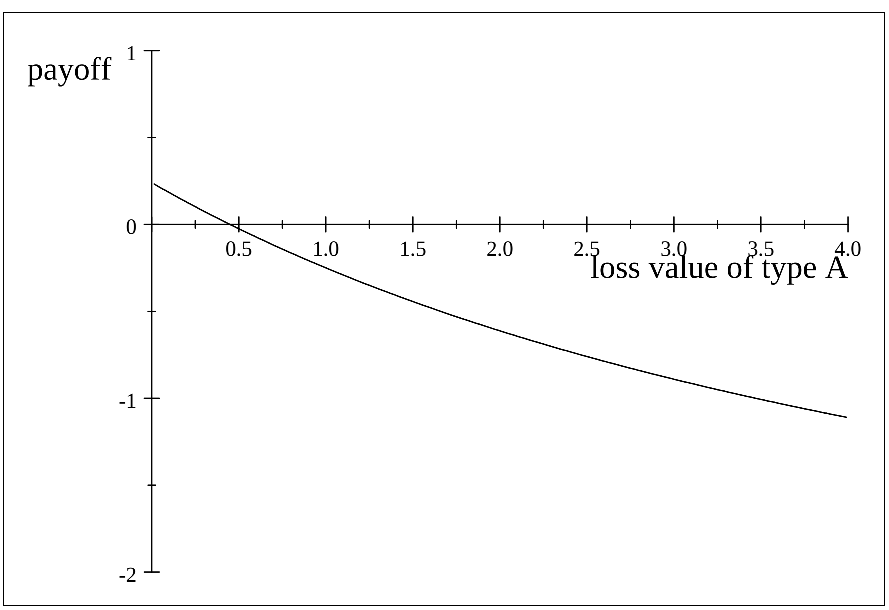


Figure 4: The payoff of type A in case 3

Similar to case 2, while the equilibrium effort of type A increases in his identity-dependent loss value l_A, his expected payoff decreases.

In the above example, a player's identity-dependent loss value decreases his expected payoff. However, by Proposition 9, his identity-dependent loss value may encourage him to participate in the contest, while without this identity-dependent loss value, he may prefer to stay out of the contest.

5 Positive externalities

So far we have assumed negative identity-dependent loss values, but if we would instead assume positive ones, as shown in the following, we will obtain the opposite results to those we already obtained by Propositions 1, 5, and 8.

1) In the contest of case 1, consider the following scenarios:

- Scenario 7 : Assume that $l_A = l_B = 0$ and $v_B < \frac{m}{m-1}v_A$. Then, by Proposition 1, the players of type A participate in the contest. Now, assume that there are loss values of $l_A > l_B > 0$ such that $v_B - l_B < \frac{m}{m-1}(v_A - l_A)$. Then, the players of type A will stay out of the contest. Thus, a sufficiently large l_A can discourage the players of type A to participate in the contest.

- Scenario 8: Assume that $l_A = l_B = 0$ and $\frac{n}{n-1}v_B < v_A$. Then, by Proposition 1, the players of type B do not participate in the contest. Now, assume that there are identity-dependent loss values $l_A > l_B > 0$ such that $v_B - l_B > \frac{n-1}{n}(v_A - l_A)$. Then, sufficiently high l_A can encourage the players of type B to participate in the contest. Thus, we can conclude that

Proposition 11 *In the contest of case 1, a sufficiently high (positive) identity-dependent loss value of any type (A or B) may discourage the players of the same type to participate, and may also encourage the players of the other type to participate in the contest.*

2) In the contest of case 2, consider the following scenarios:

- Scenario 9: By Proposition 5, we can see that l_A does not affect the participation in the contest of the players of type A.

- Scenario 10: Assume that $l_A = 0$ and $\frac{n}{n-1}v_B < v_A$. Then, by Proposition 5, the players of type B do not participate in the contest. Now, assume that there is an identity-dependent loss value $l_A > 0$ such that $v_A - l_A < \frac{n}{n-1}v_B$. Then, by Proposition 5, the players of type B participate in the contest. Thus, a sufficiently large l_A can encourage the players of type B to participate in the contest. We can conclude that

Proposition 12 *In the contest of case 2, a sufficiently high (positive) identity-dependent loss value of any type (A or B) does not affect the participation of the players of the same type, but may encourage the players of the other type to participate in the contest.*

3) In the contest of case 3, consider the following scenarios:

- Scenario 11 : Assume that $l_A = 0$ and $v_B < \frac{m}{m-1}v_A$. Then, by Proposition 8, the players of type A participate in the contest. Now, assume that there are identity-dependent loss values of $l_A, l_B > 0$ such that $v_B - l_B > \frac{m}{m-1}(v_A - l_A)$. Then, by Proposition 8, the players of type A stay out of the the contest. Thus, a sufficiently large l_A may discourage the players of type A to participate in the contest.

- Scenario 12: By Proposition 8, we can see that l_A does not affect the participation of the players of type B.

We can conclude that

Proposition 13 *In the contest of case 3, a sufficiently high (positive) identity-dependent loss value of any type (A or B) may discourage the players of the same type to participate, but does not affect the participation in the contest of players of the other type.*

6 Conclusion

When the externalities are negative, by Propositions 2, 6, and 9, we can conclude as follows:

- Sufficiently high identity-dependent loss values of the players of the same type may encourage them to participate in the contest and may discourage players of the other type to participate in the contest. Since there is no equilibrium in which all the players stay out of the contest, there are always identity-dependent loss values that ensure that all the players participate in the contest, namely, these identity-dependent loss values encourage the players with the same type to participate in the contest, but do not discourage the players of the other type of players to participate in the contest.

- If a player has an identity-dependent loss value when a player of his own type wins, then his identity-dependent loss value does not affect the participation of the players of his own type, since if all the players of his own type do not participate, each of them is not going to win such that his identity-dependent loss value does not affect his incentive to win the contest.

- If a player has an identity-dependent loss value when a player of the other type wins, then his identity-dependent loss value does not affect the participation of the players of the other type, since if those players will stay out of the contest, his identity-dependent loss value does not affect his incentive to win the contest, and therefore it also does not affect the incentive of the players of the other type to participate in the contest.

- If there is only one player of one type, he does not affect the participation of the players of the other type. Otherwise, he would stay without competitors, and then there is no equilibrium with only one player.

We can see that regardless of the reason for a player's negative identity-dependent loss value, although this loss value decreases his expected payoff, it could encourage him to participate in the contest, but never discourages him to participate. However, although this loss value does not directly affect the payoff of the players of the other type, it could discourage them to participate in the contest. Similarly, we can see that regardless of the reason for a player's positive identity-dependent loss value, although this loss value increases his expected payoff, it could discourage him to participate in the contest, but never encourages him to participate. However, although this loss value does not directly affect the payoff of the players of the other type, it could encourage them to participate in the contest.

In our model, we assumed two types of players with identity-dependent loss values, and we could explain the behavior of the players concerning their participation in the contest. Obviously, if we will increase the number of players' types their behavior will be more interesting and more complex, but the analysis of the players' equilibrium efforts will be much more complex as well.

7 Appendix

7.1 Proof of Proposition 1

Assume that our assumption does not hold, namely, $v_A + l_A < \frac{m-1}{m}(v_B + l_B)$. Then, by (6), we have

$$\begin{aligned} x &= \frac{m+n-1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(l_B + v_B + ml_A - ml_B + mv_A - mv_B) \\ &= \frac{m+n-1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(m(v_A + l_A) - (m-1)(v_B + l_B)) \\ &< \frac{m+n-1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(m(v_A + l_A) - m(v_A + l_A)) = 0 \end{aligned}$$

Thus, the players of type A participate in the contest iff $v_A + l_A \geq \frac{m-1}{m}(v_B + l_B)$. Similarly, assume that our assumption does not hold, namely, $v_B + l_B < \frac{n-1}{n}(v_A + l_A)$. Then, by (7), we have

$$\begin{aligned} y &= \frac{1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(m+n-1)(l_A + v_A - nl_A + nl_B - nv_A + nv_B) \\ &= \frac{1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(m+n-1)(n(v_B + l_B) - (n-1(v_A + l_A)) \\ &< \frac{1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(m+n-1)(n(v_B + l_B) - (n(v_B + l_B) = 0. \end{aligned}$$

Thus, the players of type B participate in the contest iff $v_B + l_B \geq \frac{n-1}{n}(v_A + l_A)$. Notice that the conditions $v_B + l_B \geq \frac{m}{m-1}(v_A + l_A)$ and $v_A + l_A \geq \frac{n}{n-1}(v_B + l_B)$ are not satisfied together, namely, there is no situation in which no player participates in the contest.

7.2 Proof of Proposition 4

By (1), the expected payoff of every player of type A is

$$u_A = v_A \frac{x_1}{X+Y} - l_A(1 - \frac{x_1}{X+Y}) - x_1$$

where by (6) and (7)

$$\begin{aligned}
x_1 &= \frac{m+n-1}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(l_B + v_B + ml_A - ml_B + mv_A - mv_B) \\
\sum_{k=1}^{n} x_k &= \frac{n(m+n-1)}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(l_B + v_B + ml_A - ml_B + mv_A - mv_B) \\
\sum_{k=1}^{m} y_k &= \frac{m}{(ml_A + nl_B + mv_A + nv_B)^2}(l_A + v_A)(l_B + v_B)(m+n-1)(l_A + v_A - nl_A + nl_B - nv_A + nv_B)
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
\frac{du_A}{dl_A} &= -(l_B + v_B)^2 \frac{m+n-1}{(ml_A + nl_B + mv_A + nv_B)^3} \cdot \\
&\quad \left(nl_B - ml_A - mv_A + nv_B + m^2 l_A + n^2 l_B + m^2 v_A + n^2 v_B + 3mnl_A - mnl_B + 3mnv_A - mnv_B\right)
\end{aligned}$$

In order to show that $\frac{du_A}{dl_A} < 0$ it is sufficient to show that

$$q = nl_B - ml_A - mv_A + nv_B + m^2 l_A + n^2 l_B + m^2 v_A + n^2 v_B + 3mnl_A - mnl_B + 3mnv_A - mnv_B > 0$$

we can write

$$q = S + T$$

where

$$\begin{aligned}
S &= (n + n^2)(v_B + l_B) + (m^2 - m)(v_A + l_A) \\
T &= 3mnl_A - mnl_B + 3mnv_A - mnv_B
\end{aligned}$$

Clearly, $S > 0$. Now, by Proposition 1, $\frac{m}{m-1}(v_A + l_A) \geq (v_B + l_B)$. Thus, since $3 > \frac{m}{m-1}$ for all $m = 2, 3, ...$, we obtain that

$$T = 3mn(v_A + l_A) - mn(v_B + l_B) > 0$$

Therefore, $q = S + T > 0$ which implies that $\frac{du_A}{dl_A} < 0$. In other words, the expected payoff of every player in set A decreases in his loss value l_A. Similarly, it can be shown that the expected payoff of every player in set B decreases in his loss value l_B.

7.3 Proof of Proposition 5

Denote

$$\begin{aligned} Q &= ml_Al_B - l_Av_B - l_Bv_A - v_Av_B - l_Al_B + nl_Al_B + ml_Av_B + ml_Bv_A + \\ &\quad nl_Av_B + nl_Bv_A + mv_Av_B + nv_Av_B - mnl_Al_B - mnl_Av_B - mnl_Bv_A \end{aligned} \tag{22}$$

By our assumptions that $v_A < \frac{m-1}{m}(v_B + l_B)$ and $v_B \geq \frac{n-1}{n}(v_A + l_A)$, we obtain that

$$\begin{aligned} l_A &\leq \frac{n}{n-1}v_B - v_A \\ l_B &\leq \frac{m}{m-1}v_A - v_B \end{aligned}$$

Rearranging (22) and substituting the above inequalities into (22) yields

$$\begin{aligned} Q &= v_Av_B(m+n-1) - (v_Al_B + v_Bl_A + l_Al_B)(m-1)(n-1) \\ &\geq v_Av_B(m+n-1) - (v_A(\frac{m}{m-1}v_A - v_B) + v_B(\frac{n}{n-1}v_B - v_A) \\ &\quad +(\frac{n}{n-1}v_B - v_A)(\frac{m}{m-1}v_A - v_B))(m-1)(n-1) \\ &= 0 \end{aligned}$$

Now, assume that our assumption does not hold, namely, $v_A < \frac{m-1}{m}(v_B + l_B)$. Then, by (13), we have

$$\begin{aligned} x &= \frac{l_B + v_B - ml_B + mv_A - mv_B}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2}Q \\ &= \frac{mv_A - (m-1)(v_B + l_B)}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2}Q \\ &< \frac{mv_A - mv_A}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2}Q = 0 \end{aligned}$$

Thus, the players of type A participate in the contest iff $v_A \geq \frac{m-1}{m}(v_B + l_B)$. Similarly, assume that $v_B < \frac{n-1}{n}(v_A + l_A)$. Then, by (14), we have

$$
\begin{aligned}
y &= \frac{l_A + v_A - nl_A - nv_A + nv_B}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2} Q \\
&= \frac{nv_B - (n-1)(v_A + l_A)}{(ml_A + nl_B + mv_A + nv_B - mnl_A - mnl_B)^2} Q \\
&< \frac{nv_B - nv_B}{(ml_A + nl_B + mv_A + nv_B)^2} Q = 0
\end{aligned}
$$

Thus, the players of type B participate in the contest iff $v_B \geq \frac{n-1}{n}(v_A + l_A)$. Notice that there is an equilibrium in which both kinds of players exert positive efforts iff both assumptions $v_A \geq \frac{m-1}{m}(v_B + l_B)$ and $v_B \geq \frac{n-1}{n}(v_A + l_A)$ hold.

7.4 Proof of Proposition 8

Denote

$$
\begin{aligned}
Q &= mv_A - nl_B - v_A + nv_A + mnl_A + mnl_B \\
&= (m + n - 1)v_A + mnl_A + (m-1)nl_B
\end{aligned}
$$

It can be verified that $Q > 0$. Assume that our assumption does not hold, namely, $v_A + l_A < \frac{m-1}{m} v_B$. Then, by (20), we have

$$
\begin{aligned}
x &= \frac{v_A(v_B + ml_A + mv_A - mv_B)(mv_A - nl_B - v_A + nv_A + mnl_A + mnl_B)}{(mv_A + nv_B + mnl_A + mnl_B)^2} \\
&= \frac{v_A(m(v_A + l_A) - (m-1)v_B)}{(mv_A + nv_B + mnl_A + mnl_B)^2} Q \\
&< \frac{v_A(m(v_A + l_A) - m(v_A + l_A))}{(mv_A + nv_B + mnl_A + mnl_B)^2} Q = 0
\end{aligned}
$$

Thus, the players of type A participate in the contest iff $v_A + l_A \geq \frac{m-1}{m} v_B$. Similarly, assume that our assumption does not hold, namely, $v_B + l_B < \frac{n-1}{n} v_A$. Then, by (21), we have

$$
\begin{aligned}
y &= \frac{v_B\,(v_A + nl_B - nv_A + nv_B)\,(mv_A - nl_B - v_A + nv_A + mnl_A + mnl_B)}{(mv_A + nv_B + mnl_A + mnl_B)^2} \\
&= \frac{v_B\,(n(v_B + l_B) - (n-1)v_A)}{(mv_A + nv_B + mnl_A + mnl_B)^2} Q \\
&< \frac{n(v_B + l_B) - n(v_B + l_B)}{(ml_A + nl_B + mv_A + nv_B)^2} Q = 0
\end{aligned}
$$

Thus, the players of type B participate in the contest iff $v_B + l_B \geq \frac{n-1}{n} v_A$, and all the players participate iff $v_A + l_A \geq \frac{m-1}{m} v_B$ and $v_B + l_B \geq \frac{n-1}{n} v_A$.

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