BAYESIAN NASH EQUILIBRIUM EXISTENCE IN (ALMOST CONTINUOUS) CONTESTS

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Bayesian Nash Equilibrium Existence in (Almost Continuous) Contests

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Abstract

We prove the existence of a behavioral-strategy Bayesian Nash equilibrium in contests where each contestant's probability to win is continuous in efforts outside the zero-effort profile, monotone in his own effort, and greater that 1/2 if that contestant is the only one exerting positive effort. General type spaces, and in particular a continuum of information types, are allowed. As a corollary, the existence of a pure-strategy Bayesian Nash equilibrium is established in generalized Tullock contests, where the probability to win is strictly concave in one's own effort.

Journal of Economic Literature classification numbers: C72, D72, D82.

Key words: Contests, Tullock lottery, Bayesian Nash Equilibrium, equilibrium existence, absolute continuity of information, continuum of types.

1 Introduction

Krishna and Morgan (1997) brought to the fore two incomplete-information auction games, the war of attrition and the all-pay auction, which correspond to the standard notions of the second- and first-price auctions but differ from them in one major respect: all players must pay their bids and not just the winners. In these auctions, Krishna and Morgan (1997) showed the existence and uniqueness of equilibrium in Milgrom and Weber's (1982) framework of affiliated signals. Recent advances in the

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equilibrium existence literature provided important results for discontinuous Bayesian games, with a particularly strong claim in the case of all-pay auctions. Specifically, He and Yannelis (2016) and Carbonell-Nicolau and McLean (2018) have shown that when the value of winning in an all-pay auction is common, the Bayesian Nash equilibrium (henceforth, BNE) exists for *any* absolutely continuous[1] information structure.

The "all-pay" feature is also present in the model of imperfectly discriminating auctions, or contests, where the players incur the cost of their bids/efforts regardless of their ultimate winning status, but the winning player is no longer identified by the highest effort; instead, anyone typically stands a chance to be a winner, with the probability of winning being a monotonically increasing function of one's own effort. The Tullock lottery (see, e.g., Tullock (1980)) is the best known form of such a contest, where the probabilities of winning are assumed to be proportional to the players' efforts; it has been widely used in modeling R&D races, political contests, and rent-seeking and lobbying activities. In a more general variant of the Tullock lottery due to Szidarovszky and Okuguchi (1997), players' efforts are evaluated via certain concave "effort impact" functions, and the probabilities are proportional to effort impacts. Einy *et al.* (2015) considered a general class of contest success functions that possess three features that these lotteries have in common: the probability of winning is (a) continuous with respect to the efforts of all players whenever the total effort is positive; (b) non-decreasing and strictly concave in the player's own effort; and (c) equal to 1 if the player is the only one exerting positive effort. The expected payoff in the corresponding *generalized Tullock contests* is therefore necessarily discontinuous at the zero-effort profile, and continuous elsewhere.[2] This stands in contrast to all-pay auctions, which are discontinuous along entire "diagonal" curves containing equal bids.

Despite having discontinuities at a single effort profile,[3] generalized Tullock con-

[1] Absolute continuity of information, introduced by Milgrom and Weber (1986), is a mild and nearly universally assumed condition, requiring the joint distribution of the players' types to be absolutely continuous with respect to the product of its marginals. It will also be assumed throughout in this work.

[2] Under the commonly made assumption of continuous costs of effort (which is also maintained in this work), continuity of the expected payoffs for any realization of the players' types is tantamount to continuity of the success function.

[3] We stress the discontinuity of payoffs in contests because that is, in general, the primary reason for equilibrium non-existence. Indeed, in Bayesian games where the payoffs are continuous

tests with incomplete information have seen rather limited BNE existence results. The earlier works on incomplete information Tullock contests in which BNE existence was shown considered uncertainty only on some parameters and assumed specific forms of information endowments, such as one-sided incomplete information on continuously distributed valuation (Hurley and Shogren (1998a)), discrete valuations (Hurley and Shogren (1998b), Malueg and Yates (2004) and Schoonbeek and Winkel (2006)), a continuously distributed common valuation (Harstad (1995), Wärneryd (2012) and Rentschler (2009)), and continuously and independently distributed marginal costs (Fey (2008), Ryvkin (2010), Wasser (2013) and Ewerhart (2014)).[4] Two later works that addressed BNE existence, Einy *et al.* (2015) and Ewerhart and Quartieri (2020), admitted all modes of incomplete information on both the valuations and the costs, but restricted the cardinality of type sets. Specifically, the assumption on generalized Tullock contests in Einy *et al.* (2015) amounted to the type sets of any player being at most countable, while Ewerhart and Quartieri (2020) worked with finitely many states of nature.[5]

This work will show that a pure-strategy BNE exists in generalized Tullock contests for general, possibly uncountable, type-spaces. In fact, BNE existence, albeit in behavioral and not necessarily pure strategies, will be established for much more general contests, where there may be no winner with positive probability, the probabilities of winning/costs need not be concave/convex in one's own effort, and even a player who is the only one exerting positive effort may lose (though with probability less than 1/2). It is important to note that, although the discontinuity of these contests is very mild, being confined to the zero-effort profile, the recent results that guarantee BNE existence in discontinuous Bayesian games, of He and Yannelis (2016) and Carbonell-Nicolau and McLean (2018), can only be applied to contests – and we will show how – when the value of winning is common and there is always a winner. That is partially because most of the results require upper semi-continuity

for every realization of players' types, the classical results of Milgrom and Weber (1986) and Balder (1988) guarantee BNE existence under quite mild conditions (such as compactness of action sets and absolutely continuous information).

[4]This latter list and the classifications therein are taken, with some minor modifications, from Ewerhart and Quartieri (2020).

[5]Compared with Einy *et al.* (2015), Ewerhart and Quartieri (2020) worked with the smaller domain of Szidarovszky and Okuguchi's (1997) variant of success functions, but allowed player- and type-dependent budget caps on each player's set of admissible efforts.

of the payoffs' sum for every realization of players' types; but such a sum inherits the discontinuity at zero possessed by the contest success function if the value for winning is not common or if there may be no winner (in the common-value case where there is always a winner, the sum of the expected shares of the value is equal to just the value, hence independent of the players' efforts and trivially continuous).

It turns out that a BNE can be arrived at by an extension of the method that has been employed in Einy *et al.* (2013), Ewerhart (2014), and Ewerhart and Quartieri (2020) in their treatment of simpler cases. The method consists of approximating the contest by a sequence of "constrained" contests, where all efforts are required to be above a small positive constant. The positive constraints make all payoffs continuous in efforts for any realization of types, and therefore the constrained contests fall under the scope of the familiar BNE existence result of Balder (1988) that imposes no restriction on the type-spaces. Because the sets of behavioral strategies in the contest are compact in a certain "weak" topology defined in Balder (1988), constrained BNEs have a weak accumulation point as the constraints become vanishingly small. It is this accumulation point that is a candidate for a BNE in the (unconstrained) contests. The weak topology on behavioral strategies, however, is quite unwieldy, and makes ascertaining that the accumulation point is indeed a BNE a rather heavy technical task, which will take up the bulk of our proof.

The paper is organized as follows. In section 2 we offer a general model of a contest that subsumes generalized Tullock contests but still allows discontinuity only at the zero-effort profile. Section 3 contains our results on BNE existence, beginning with common-value contests in order to illustrate the power and limitations of directly applying ready-made results, and then considering general contests. All proofs are gathered in Section 4. Section 5 discusses various extensions of our contest framework that are compatible with BNE existence. The Appendix expands on some indirectly relevant technical points.

2 Contests with incomplete information

2.1 The model

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize. For each $i \in N$, a measurable type-space $(T_i, \mathcal{T}_i)$ constitutes i's information endowment, with a common prior probability p on the product space $(T, \mathcal{T}) := (\times_{i \in N} T_i, \otimes_{i \in N} \mathcal{T}_i)$ of all type profiles. In common with much of the literature, it will be assumed that p is absolutely continuous w.r.t. the product of its marginals, $\otimes_{i \in N} p_i$.

Upon privately observing their respective types, players simultaneously choose their effort levels. It is assumed that effort choices may not exceed some universal bound $M > 0$ that will be fixed throughout,[6] and hence any effort profile $x = (x_1, ..., x_n)$ is an element of the cube $[0, M]^n$. The state-dependent value for the prize of each player $i \in N$ is given by $V_i : T \to \mathbb{R}_{++}$, i.e., if $t \in T$ is the realized type profile then player i's value is $V_i(t) > 0$. The state-dependent cost of effort of each player $i \in N$ is described by $c_i : T \times [0, M] \to \mathbb{R}_+$. The following assumptions will be made on the functions V_i and c_i :

(i) V_i is $\mathcal{T}$-measurable and c_i is $\mathcal{T} \otimes \mathcal{B}([0, M])$-measurable[7];

(ii) V_i and $\sup_{x_i \in [0,M]} c_i(\cdot, x_i)$ are p-integrable;

(iii) for any $t \in T$, the function $c_i(t, \cdot)$ is non-decreasing and continuous.

The prize is awarded in a probabilistic fashion, according to a *success function* $\rho : [0, M]^n \to [0, 1]^n$. For each $i \in N$ and $x \in [0, M]^n$, $\rho_i(x)$ is the probability that player i will be the recipient of the prize when x is the realized action profile; we will assume that $\rho_i(x)$ is a sub-probability vector, i.e., $\sum_{i \in N} \rho_i(x) \leq 1$. Thus, we allow the possibility that the prize is withheld from the players with some probability, for some (or even all) effort profiles.

Denote by $\mathbf{0} \in \mathbb{R}^n$ the zero vector, and, for any effort profile $x \in [0, M]^n$, $i \in N$ and $y_i \in [0, M]$, let $(y_i, x_{-i}) \in [0, M]^n$ be the profile obtained from x by replacing x_i with y_i. We impose the following conditions on ρ:

(iv) ρ is continuous on $[0, M]^n \setminus \{\mathbf{0}\}$;

(v) $\rho_i(y_i, x_{-i})$ is non-decreasing in i's effort y_i, for any $i \in N$ and any fixed $x_{-i} \in [0, M]^{n-1}$.

Definition 1. An incomplete-information *contest* is given by the collection $G = (N, (T_i, \mathcal{T}_i)_{i \in N}, p, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$ of the above-described attributes, such that (**i**)–(**v**) are satisfied. If the allowable efforts of each player are additionally constrained

[6] See Section 5.3 on the need for this assumption, and the option for its removal.

[7] Here and henceforth, given a Borel set $S \subset \mathbb{R}_+^m$ for some $m \geq 1$, $\mathcal{B}(S)$ will denote the σ-algebra of Borel subsets of S. The measurability of real-valued functions is w.r.t. the Borel σ-algebra on their stated range.

to lie in an interval $[m, M]$, where $0 \leq m < M$, the resulting contest will be denoted by $G(m)$ and called a *constrained contest*, with $G(0) = G$.

For any realized type profile $t \in T$ and any effort profile $x \in [0, M]^n$, the payoff of each player $i \in N$ in a contest G is given by his expected share of the prize's value net of his cost of effort, namely,

$$u_i(t, x) = \rho_i(x) \cdot V_i(t) - c_i(t, x_i). \tag{1}$$

Below we list some obvious implications of conditions (**i**) – (**v**).

Fact 1. For any $i \in N$, the payoff function $u_i : T \times [0, M]^n \to \mathbb{R}$ has the following properties:

(**a**) u_i is $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable;

(**b**) $u_i(t, \cdot)$ is continuous on $[0, M]^n \setminus \{\mathbf{0}\}$ for any $t \in T$;

(**c**) there exists a p-integrable function $\varphi_i : T \to \mathbb{R}$ such that $|u_i(t, x)| \leq \varphi_i(t)$ for any $t \in T$ and $x \in [0, M]^n$ (one may take, e.g., $\varphi_i(t) := V_i(t) + \sup_{x_i \in [0,M]} c_i(t_i, x_i)$);

(**d**) $\liminf_{y_i \to x_i+} u_i(t, (y_i, x_{-i})) \geq u_i(t, x)$ for any $t \in T$ and any $x \in [0, M]^n$.

2.2 Bayesian strategies and equilibrium

Given $0 \leq m < M$, a *pure (Bayesian) strategy* of player $i \in N$ in a constrained contest $G(m)$ is a $\mathcal{T}_i$-measurable function $s_i : T_i \to [m, M]$; that is, upon leaning his type t_i, player i chooses effort $s_i(t_i)$. A more general concept of a behavioral strategy allows randomness in the type-dependent choice of effort. Formally, a *behavioral strategy* of $i \in N$ in $G(m)$ is a mapping $\sigma_i : T_i \times \mathcal{B}([m, M]) \to [0, 1]$, such that $\sigma_i(t_i, \cdot)$ is a probability measure on $[m, M]$ for every $t_i \in T_i$ and $\sigma_i(\cdot, A)$ is $\mathcal{T}_i$-measurable for every $A \in \mathcal{B}([m, M])$.

We denote by $S_i(m)$ the set of pure strategies of player i in $G(m)$, and by $\Sigma_i(m)$ the set of his behavioral strategies. The product sets $S(m) = \times_{i=1}^n S_i(m)$ and $\Sigma(m) = \times_{i=1}^n \Sigma_i(m)$ contain the corresponding strategy profiles. Any pure strategy $s_i \in S_i(m)$ is clearly identifiable with a behavioral strategy $\sigma_i^{s_i} \in \Sigma_i(m)$ given by $\sigma_i^{s_i}(t_i, A) = \delta_{s_i(t_i)}(A)$ for any $A \in \mathcal{B}([m, M])$, where $\delta_{s_i(t_i)}$ stands for the Dirac measure concentrated on $s_i(t_i)$. Finally, note that $S_i(m)$ and $\Sigma_i(m)$ for $m > 0$ are subsets of $S_i(0)$ and of $\Sigma_i(0)$, respectively.

For any strategy profile $\sigma = (\sigma_i)_{i \in N} \in \Sigma(0)$, the expected payoff of player $i \in N$ is given by

$$U_i(\sigma) = \int_T \int_{[0,M]^n} u_i(t, x)\, \sigma_1(t_1, dx_1) ... \sigma_n(t_n, dx_n) p(dt). \tag{2}$$

Also, denote by $(\sigma_i', \sigma_{-i}) \in \Sigma(0)$ the profile that is obtained from σ by replacing σ_i with some $\sigma_i' \in \Sigma_i(0)$. Clearly, if $s = (s_i)_{i \in N} \in S(0)$ then

$$U_i(s) = \int_T u_i(t, (s_1(t_1), ..., s_n(t_n)))\, p(dt),$$

and

$$U_i(s_i, \sigma_{-i}) = \int_T \int_{[0,M]^{n-1}} u_i(t, (s_i(t_i), x_{-i})) \left[\prod_{j \neq i} \sigma_j(t_j, dx_j) \right] p(dt). \tag{3}$$

Definition 2. For any $0 \leq m < M$, a behavioral strategy profile $\sigma^* = (\sigma_i^*)_{i \in N} \in \Sigma(m)$ constitutes a *Bayesian Nash equilibrium* (or *BNE*, for short) of a contest $G(m)$ if

$$U_i(\sigma^*) \geq U_i(\sigma_i, \sigma_{-i}^*) \tag{4}$$

for every player $i \in N$ and every $\sigma_i \in \Sigma_i(m)$. If, in addition, $\sigma^* \in S(m)$, then it is a *pure-strategy* BNE.[8]

2.3 Special cases: sure-prize, pre-Tullock, generalized Tullock contests

Einy *et al.* (2015) introduced a class of contests that satisfy – in addition to conditions (**i**)–(**v**) – the following requirements, for any $i \in N$:

(vi) $\rho_i(y_i, x_{-i})$ is *strictly concave* in i's effort y_i, for any fixed $x_{-i} \in [0, M]^{n-1}$;

(**vii**) for any $t \in T$, the function $c_i(t, \cdot)$ is *convex*;

(**viii**) there exists $\widehat{\rho}_i > \frac{1}{2}$ such that $\rho_i(x_i, \mathbf{0}_{-i}) \geq \widehat{\rho}_i$ for any $0 < x_i \leq M$ (that is, if i is the only player exerting positive effort then his probability to receive the prize exceeds $\frac{1}{2}$ by a constant margin);[9]

[8] That is, it is only meant that the strategies comprising the BNE are pure; unilateral deviatons by players, required to be unprofitable by (4), may be to behavioral strategies.

[9] Einy *et al.* (2015) in fact required that $\widehat{\rho}_i = 1$ for each $i \in N$, i.e., that the only player exerting positive effort receives the prize with certainty. However, the full force of that assumption is not needed in our results, and hence (**viii**) is stated in a weaker form.

and

(**ix**) $\rho(x)$ is a probability vector for every $[0, M]^n$ (that is, the prize is being awarded with certainty).

Conditions (**vi**)-(**ix**) naturally generalize a widely used and simple model of *Tullock lottery*, where $\rho = \rho^T$ is given, for each $x \in [0, M]^n \setminus \{\mathbf{0}\}$ and $i \in N$, by

$$\rho_i^T(x) = \frac{x_i}{\sum_{j=1}^n x_j}, \tag{5}$$

and costs are linear in effort. Of course, success function ρ^T also obeys (**iv**) and (**v**), and is, in fact, a showcase of the tension between simplicity (namely, probabilities of winning being proportional to efforts) and discontinuity at zero. More generally, conditions (**iv**) – (**vi**), (**viii**) and (**ix**) are satisfied by any ρ that is given, for each $x \in [0, M]^n \setminus \{\mathbf{0}\}$ and $i \in N$, by

$$\rho_i(x) = \frac{g_i(x_i)}{\sum_{j=1}^n g_j(x_j)}, \tag{6}$$

where, for every $j \in N$, the measurable effort-impact function $g_j : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, continuous, concave, and $g_j(0) = 0$. Thus, generalized Tullock contests include the incomplete-information version of Szidarovszky and Okuguchi (1997) model (where the functions $g_1, ..., g_n$ are in addition twice continuously differentiable). In particular, the commonly assumed success function that is given by (6) for $g_i(x_i) = x_i^r$ also adheres to the above specification when the "impact parameter" r is in $(0, 1]$.

Definition 3. A contest G (or, if constrained, contest $G(m)$) is called:

(1) a *sure-prize* contest is it satisfies (**ix**);

(2) a *pre-Tullock* contest if it satisfies (**viii**);

(3) a *generalized Tullock* contest if it satisfies (**vi**)–(**viii**).

Definition 3(3) utilizes the term coined by Einy *et al.* (2015) but gives it a slightly broader sense, as those authors required their generalized Tullock contests to also be sure-prize.

We now observe that, for generalized Tullock contests, attention can be *a priori* confined to pure-strategy BNE, for a fairly obvious reason. Let us call strategy $\sigma_i \in \Sigma_i(0)$ *almost pure* if $\sigma_i(\cdot, t_i)$ is a Dirac measure for p_i-almost every t_i; clearly, such a strategy can be replaced by a (genuine) pure strategy without affecting any of the expected payoffs. Then:

Fact 2. Given $0 \leq m < M$ and a strategy profile $\sigma = (\sigma_i)_{i \in N} \in \Sigma(m)$ in a (constrained) generalized Tullock contest $G(m)$, if σ_i is not an almost pure strategy for some $i \in N$ then player i can strictly improve upon σ_i by using a pure strategy in $G(m)$.

Indeed, if σ_i is not almost pure for some $i \in N$, consider a pure strategy $s_i \in S_i(m)$ that is σ_i's expectation at every $t_i \in T_i$ (namely, $s_i(t_i) := \int_{[0,M]} x_i \sigma_i(t_i, dx_i)$).[10] But (**vi**) and (**vii**) immediately imply via (1) that the payoff function $u_i(t, (y_i, x_{-i}))$ is *strictly* concave in y_i, and so a comparison of the expressions in (2) and (3) leads to the conclusion that $U_i((s_i, \sigma_{-i})) > U_i(\sigma)$.

The following is an immediate corollary from Definition 2 and Fact 2:

Fact 3. For any $0 \leq m < M$, if a (constrained) generalized Tullock contest $G(m)$ possesses a BNE, then that BNE consists of almost pure strategies (which can w.l.o.g. be assumed to be pure).

3 BNE existence

Ever since Dasgupta and Maskin (1986), it has been understood that upper semicontinuity of the sum of payoffs in a game can play an important part in guaranteeing equilibrium existence. It was shown by Reny (1999) that upper semi-continuity of the payoffs' sum, together with the condition of payoff security, suffice for the existence of equilibrium in games where the payoffs may be discontinuous in any stronger sense. In the context of Bayesian games, Carbonell-Nicolau and McLean (2018) showed that a BNE exists when upper semi-continuity of the payoffs' sum and a uniform version of payoff security hold for any realization of the players' types.

The latter result immediately lends itself to establishing existence of BNE in a variety of all-pay actions (see Corollary 3 in Carbonell-Nicolau and McLean (2018)) and contests, due to the fact that they tend to be uniformly payoff-secure. However, using that result necessitates the assumption that the players' value for the prize is *common* and that the prize is awarded with certainty, because that is a natural

[10] The function s_i is $\mathcal{T}_i$-measurable by, e.g., Proposition 7.29 of Bertsekas and Shreve (2004).

way to obtain the continuity of the sum of payoffs. Indeed, the sum of the players' expected shares of the common value for the (always awarded) prize is equal to just the value and thus independent of the players' actions, despite that the expected shares themselves may depend on actions discontinuously;[11] see the Appendix for a simple example of the payoffs' sum not being upper semi-continuous without the common-value assumption.

We begin by stating the BNE existence result for common-value sure-prize contests, obtained as a fairly immediate corollary of Theorem 1 in Carbonell-Nicolau and McLean (2018).[12]

Proposition 1. If G is a sure-prize contest in which $V_1 = ... = V_n = V$ for a bounded common-value function V, and the cost functions $\{c_i(t, \cdot)\}_{i \in N, t \in T}$ are equicontinuous, then G possesses a BNE. If, moreover, G is a generalized Tullock contest, then it possesses a pure-strategy BNE.

We now dispose of the sure-prize and common-value assumptions. Our preliminary observation concerns the case of constrained contests $G(m)$ for $m > 0$. This case is trivial because (by Fact 1(**b**)) the payoff functions $(u_i)_{i \in N}$ are continuous in efforts, for every realization of the players 'types, when the lowest allowable effort is positive; thus, the payoffs in such $G(m)$ are fully continuous. The existence of BNE is then guaranteed by the classical result of Balder (1988). Although the forced commitment to a minimal positive effort may not be plausible in many contexts, the fact that BNE exists in constrained contests will be instrumental in showing BNE existence without constraints.

Proposition 2. If $0 < m < M$ then the constrained contest $G(m)$ possesses a BNE. If, moreover, $G(m)$ is a generalized Tullock contest, then it possesses a pure-strategy BNE.

[11] Even if the prize may be withheld with positive probability, BNE existence can sometimes be obtained by using another result of Carbonell-Nicolau and McLean (2018), which imposes the condition of *uniform diagonal security* (due to Prokopovych and Yannelis (2014)) without simultaneously requiring upper semi-continuity of the payoffs' sum (see their Corollary 6). However, without the common-value assumption, there are contests that are not uniformly diagonally secure; see the Appendix for an example.

[12] An alternative is to use Theorem 2 of of He and Yannelis (2016) that requires *random disjoint payoff matching* condition, but upper semi-continuity of the payoffs' sum is also needed alongside.

Our main result, Theorem 1 that follows, shows BNE existence in an unconstrained contest $G = G(0)$, assuming that it is pre-Tullock. The proof is rather heavy, but the underlying idea is simple. A natural approach is to consider "limits" of BNE strategy profiles that exist in constrained contests $G(m)$, as the positive lower bound m on efforts tends to 0, and to check whether such a limit is a genuine BNE in G, the contest without constraints, in which a deviating player's effort may be arbitrarily small, or zero. This direction has been fruitfully explored in some recent results on BNE existence in contests (see Einy et al. (2013), Ewerhart (2014), and Ewerhart and Quartieri (2020)). We will show that the limit approach works also in our very general framework.

In order for our strategy sets $\Sigma_i(0)$ to be amenable to the limit approach, they need to be compact (because we seek accumulation points of constrained strategies), and compactness of strategy sets is a natural feature in the weak topology of Balder (1988). The use of Balder's topology makes the proof technically demanding, however. For greater clarity, we chose to state and prove our theorem under the condition that, for each player i, $(T_i, \mathcal{T}_i)$ is *countably generated*, meaning that the σ-algebra $\mathcal{T}_i$ of events in T_i is generated by some countable subalgebra.[13] This condition is genuinely mild, as it is satisfied in applications near universally. Indeed, it clearly holds in all circumstances where the type sets are finite or countable,[14] but its main appeal is in that it admits all useful variants of a continuum of types: any $(T_i, \mathcal{T}_i)$ in which T_i is a (possibly uncountable) separable metric space, and $\mathcal{T}_i$ is the corresponding Borel σ-algebra, is countably generated.[15] In particular, any T_i that is a *Borel subset of a Euclidean space* (e.g., any open or closed set in some $\mathbb{R}^k$) with $\mathcal{T}_i = \mathcal{B}(T_i)$ is countably generated, thereby falling under the purview of our result.

Theorem 1. If G is a pre-Tullock contest in which the type-space $(T_i, \mathcal{T}_i)$ is countably generated for each $i \in N$, then G possesses a BNE. If, moreover, G is a generalized Tullock contest, then it possesses a pure-strategy BNE.

Various extensions of our framework and results will be considered in Section 5.

[13] This condition can be removed, however – see Section 5.1.

[14] In the latter case, under the assumption that $\mathcal{T}_i$ is the set of all subsets of T_i.

[15] Such $(T_i, \mathcal{T}_i)$ is generated by a countable algebra $\mathcal{T}_i'$ that consists of all finite unions and intersections of sets belonging to some countable basis for the metric topology on T_i. (A countable basis exists by separability of T_i.)

4 Proofs

4.1 Proof of Proposition 1

We begin by verifying that G is *uniformly payoff-secure* (with reference to Definition 9 in Carbonell-Nicolau and McLean (2018)). For any $i \in N$, $s_i \in S_i(0)$ and $\varepsilon > 0$, it must be shown that there exists $\overline{s}_i \in S_i(0)$ with the property that, for every $(t, x_{-i}) \in T \times [0, M]^{n-1}$ there is a (relatively) open neighborhood $W_{x_{-i}} \subset [0, M]^{n-1}$ of x_{-i} such that

$$u_i(t, (\overline{s}_i(t_i), z_{-i})) > u_i(t, (s_i(t_i), x_{-i})) - \varepsilon \tag{7}$$

whenever $z_{-i} \in W_{x_{-i}}$.

Since $(c_i(t, \cdot))_{t \in T}$ are equicontinuous at 0 by assumption, there exists $\delta > 0$ such that $|c_i(t, \delta) - c_i(t, 0)| < \frac{\varepsilon}{2}$ for any $t \in T$. Now define

$$\overline{s}_i(t_i) := \begin{cases} s_i(t_i), & \text{if } s_i(t_i) > 0; \\ \delta, & \text{if } s_i(t_i) = 0 \end{cases}$$

for any $t_i \in T_i$. By Fact 1(**b**), $u_i(t, (y_i, x_{-i}))$ is continuous in x_{-i} for any fixed $y_i > 0$ and $t \in T$. Thus, if t is such that $s_i(t_i) > 0$ then (7) holds for any x_{-i} and any z_{-i} in some open neighborhood $W_{x_{-i}}$ of x_{-i}. Similarly, if t is such that $s_i(t_i) = 0$ then

$$u_i(t, (\overline{s}_i(t_i), z_{-i})) > u_i(t, (\overline{s}_i(t_i), x_{-i})) - \frac{\varepsilon}{2} \tag{8}$$

for any x_{-i} and any z_{-i} in a neighborhood $W_{x_{-i}}$ of x_{-i}; also note that by increasing his effort from 0 to δ player i does not decrease his expected share of the prize (as follows from (**v**)), and increases his cost by at most $\frac{\varepsilon}{2}$, hence

$$u_i(t, (\overline{s}_i(t_i), x_{-i})) \geq u_i(t, (s_i(t_i), x_{-i})) - \frac{\varepsilon}{2}. \tag{9}$$

The combination of (8) and (9) establishes the claim in (7) when $s_i(t) = 0$. Therefore G is uniformly payoff-secure.

It follows from (**ix**) and the common-value assumption that $\sum_{i \in N} u_i(t, x) = V(t) - \sum_{i \in N} c_i(t, x_i)$, and this function is continuous in x for any $t \in T$ by condition (**iii**). Thus, G is uniformly payoff-secure, has a continuous (and in particular upper semi-continuous) sum of players' payoffs for each type profile, and (by assumption) p is absolutely continuous w.r.t. $\otimes_{i \in N} p_i$. The three conditions in the premise of Theorem 1 in Carbonell-Nicolau and McLean (2018) are thereby satisfied,[16] and that theorem guarantees existence of a BNE in behavioral strategies. Finally, if G is furthermore a generalized Tullock contest, then it possesses a pure-strategy BNE by Fact 3.

4.2 Proof of Proposition 2

Since the payoff function $u_i(t, x)$ of each player $i \in N$ has properties (**a**)–(**c**) listed in Fact 1, its restriction to $T \times [m, M]^n$ is: (a') $\mathcal{T} \otimes \mathcal{B}([m, M]^n)$-measurable; (b') continuous in $x \in [m, M]^n$ for any fixed $t \in T$; and (c') bounded in absolute value from above by an integrable φ_i. Furthermore, (e') p is absolutely continuous w.r.t. $\otimes_{i \in N} p_i$ by assumption. The contest $G(m)$ therefore satisfies the list of conditions in the premise of Theorem 3.1 in Balder (1988), which guarantees existence of a BNE in behavioral strategies. If $G(m)$ is furthermore a generalized Tullock contest, then it possesses a pure-strategy BNE by Fact 3.

4.3 Proof of Theorem 1

The proof is divided into five parts.

Part 1: Topological background

We will endow the behavioral strategy set $\Sigma_i(0)$ of each player i with the weak topology of Balder (1988). Using one of its equivalent definitions (see Theorem 2.2(c) in Balder (1988)), this is the coarsest topology in which for every $A \in \mathcal{T}_i$ and every continuous function $f : [0, M] \to \mathbb{R}$, the functional $I_{A,f} : \Sigma_i(0) \to \mathbb{R}$ that is given for any $\sigma_i \in \Sigma_i(0)$ by

$$I_{A,f}(\sigma_i) = \int_A \int_{[0,M]} f(x_i) \sigma_i(t_i, dx_i) p_i(dt_i) \tag{10}$$

is continuous. By Theorem 2.3(a) of Balder (1988), $\Sigma_i(0)$ is *compact* in the weak topology. Now denote by $\mathcal{T}_i'$ some countable subalgebra of $\mathcal{T}_i$ that generates $\mathcal{T}_i$. By Theorem 2.2(d) of Balder (1988), the weak topology on $\Sigma_i(0)$ is characterized by the continuity of $I_{A,f}$ only for sets $A \in \mathcal{T}_i'$. The weak topology is therefore *metrizable*. That is because there exists a countable set F of continuous functions on $[0, M]$

[16] An implicit condition in that theorem, of bounded payoffs, is also satisfied by our assumptions of a bounded common value and equicontinuity of costs.

(e.g., polynomials) that can uniformly approximate any continuous function on $[0, M]$; let us now arrange the elements of the countable Cartesian product $\mathcal{T}_i' \times F$ in a sequence $\{(A_k, f_k)\}_{k=1}^{\infty}$. The metric $d(\sigma_i', \sigma_i'') := \sum_{k=1}^{\infty} \frac{1}{2^k} |I_{A_k,f_k}(\sigma_i') - I_{A_k,f_k}(\sigma_i'')|$ on $\Sigma_i(0)$ induces the weak topology.

For understanding the implications of convergence of profiles of individual strategies, the following concept will be useful. A *correlated* behavioral strategy in G is a mapping $\tau : T \times \mathcal{B}([0, M]^n) \to [0, 1]$, such that $\tau(t, \cdot)$ is a probability measure on $[0, M]^n$ for every $t \in T$ and $\tau(\cdot, A)$ is $\mathcal{T}$-measurable for every $A \in \mathcal{B}([0, M]^n)$. The weak topology on the set $\overline{\Sigma}$ of correlated strategies may be defined just as this was done above for each $\Sigma_i(0)$, with the obvious modifications, but it will be convenient to use the following alternative definition (see Theorem 2.2(b) in Balder (1988)). A $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable function $g : T \times [0, M]^n \to \mathbb{R}$ is called *Carathéodory integrand* if $g(t, \cdot)$ is continuous for every $t \in T$, and there exists a p-integrable function φ on T such that $|g(t, x)| \leq \varphi(t)$ for every $(t, x) \in T \times [0, M]^n$. The weak topology on $\overline{\Sigma}$ is the coarsest topology in which, for every Carathéodory integrand $g : T \times [0, M]^n \to \mathbb{R}$, the functional $I_g : \overline{\Sigma} \to \mathbb{R}$ that is given for any $\tau \in \overline{\Sigma}$ by

$$I_g(\tau) = \int_T \int_{[0,M]^n} g(t, x)\tau(t, dx)p(dt)$$

is continuous. Just as the individual strategy sets, $\overline{\Sigma}$ is metrizable and compact in the weak topology.

Any strategy profile $\sigma = (\sigma_1, ..., \sigma_n) \in \Sigma(0)$ can be viewed as a correlated strategy $\tau^{\sigma} \in \overline{\Sigma}$ in which $\tau^{\sigma}(dx, t) = \prod_{i=1}^{n} \sigma_i(dx_i, t_i)$ for every $t \in T$, i.e., given any realization t of types players make their choices independently, w.r.t. their individual strategies. In particular, the expected payoffs in the contest under σ, defined in (2), can be expressed in the following simpler form: for any $i \in N$,

$$U_i(\sigma) = \int_T \int_{[0,M]^n} u_i(t, x)\tau^{\sigma}(t, dx)p(dt).$$

By Theorem 2.5 of Balder (1988), if $\{\sigma^k\}_{k=1}^{\infty} \subset \Sigma(0)$ is a sequence of strategy profiles in which $\lim_{k\to\infty} \sigma_i^k = \sigma_i^*$ for every $i \in N$, then the correlated strategies τ^{σ^k} corresponding to σ^k weakly converge to the correlated strategy τ^{σ^*} corresponding to the profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*) \in \Sigma(0)$. In accordance with the above definition of the weak topology on $\overline{\Sigma}$, this means that for any Carathéodory integrand g on $T \times [0, M]^n$,

$$\lim_{k\to\infty} \int_T \int_{[0,M]^n} g(t, x)\tau^{\sigma^k}(t, dx)p(dt) = \int_T \int_{[0,M]^n} g(t, x)\tau^{\sigma^*}(t, dx)p(dt). \tag{11}$$

Part 2: Choosing a candidate σ^* for BNE

Consider a sequence $\{m_k\}_{k=1}^{\infty} \subset (0, M)$ with $\lim_{k\to\infty} m_k = 0$, and a sequence $\{\sigma^k\}_{k=1}^{\infty}$ where $\sigma^k \in \Sigma(m_k)$ is a BNE in $G(m_k)$ for each k (the existence of such BNEs is guaranteed by Proposition 2). Since $\Sigma_i(0)$ is metrizable and compact in the weak topology for each $i \in N$, $\{\sigma_i^k\}_{k=1}^{\infty}$ has a subsequence that converges to some $\sigma_i^* \in \Sigma_i(0)$; it can w.l.o.g. be assumed that the subsequence of indices is the same for every $i \in N$, and, moreover, that the subsequence is the sequence itself. Consequently, we will proceed under the assumption that, for every $i \in N$, $\lim_{k\to\infty} \sigma_i^k = \sigma_i^*$ in the weak topology on $\Sigma_i(0)$. Our aim will be to show that the limit strategy profile, $\sigma^* = (\sigma_1^*, ..., \sigma_n^*) \in \Sigma(0)$, is a BNE of the unconstrained contest G.

Part 3: The limit profile σ^* cannot jointly put a positive mass on $\mathbf{0} \in \mathbb{R}^n$, the discontinuity point of ρ

Here we claim that

$$p(\{t \in T \mid \forall i \in N : \sigma_i^*(t_i, \{0\}) > 0\}) = 0. \tag{12}$$

Indeed, suppose to the contrary that the above probability is positive. Then the integral

$$\alpha := \int_T \sigma_1^*(t_1, \{0\}) \cdot ... \cdot \sigma_n^*(t_n, \{0\}) p(dt) = \int_T \tau^{\sigma^*}(t, \{\mathbf{0}\}) p(dt)$$

is positive. For any $0 < \varepsilon < M$, let $f_\varepsilon : [0, M]^n \to [0, 1]$ be a continuous function that satisfies $f_\varepsilon(\mathbf{0}) = 1$ and $f_\varepsilon \mid_{[0,M]^n \setminus [0,\varepsilon]^n} \equiv 0$. Observe that[17]

$$\begin{aligned}
& \liminf_{k\to\infty} \int_T \tau^{\sigma^k}(t, [0, \varepsilon]^n) p(dt) \geq \lim_{k\to\infty} \int_T \int_{[0,M]^n} f_\varepsilon(x) \tau^{\sigma^k}(t, dx) p(dt) \\
\text{(by (11))} \quad = \; & \int_T \int_{[0,M]^n} f_\varepsilon(x) \tau^{\sigma^*}(t, dx) p(dt) \\
\geq \; & \alpha.
\end{aligned}$$

Hence,

$$\liminf_{k\to\infty} \int_T \tau^{\sigma^k}(t, [0, \varepsilon]^n) p(dt) \geq \alpha > 0 \tag{13}$$

for any $0 < \varepsilon < M$. Since $\tau^{\sigma^k}(\cdot, [0, \varepsilon]^n) \leq 1$ and $0 < \alpha \leq 1$,

$$\begin{aligned}
& \int_T \tau^{\sigma^k}(t, [0, \varepsilon]^n) p(dt) \\
\leq \; & p\left(\{t \in T \mid \tau^{\sigma^k}(t, [0, \varepsilon]^n) > \frac{\alpha}{2}\}\right) + \frac{\alpha}{2}\left(1 - p\left(\{t \in T \mid \tau^{\sigma^k}(t, [0, \varepsilon]^n) > \frac{\alpha}{2}\}\right)\right),
\end{aligned}$$

[17] In what follows, (11) is applied to f_ε, which is viewed as a type-independent Carathéodory integrand.

which implies that

$$p\left(\{t \in T \mid \tau^{\sigma^k}(t, [0, \varepsilon]^n) > \frac{\alpha}{2}\}\right) \geq \frac{\int_T \tau^{\sigma^k}(t, [0, \varepsilon]^n) p(dt) - \frac{\alpha}{2}}{1 - \frac{\alpha}{2}};$$

thus, by (13),

$$\liminf_{k \to \infty} p\left(\{t \in T \mid \tau^{\sigma^k}(t, [0, \varepsilon]^n) > \frac{\alpha}{2}\}\right) \geq \frac{\frac{\alpha}{2}}{1 - \frac{\alpha}{2}} =: \beta > 0. \tag{14}$$

It follows from (14) that for any $0 < \varepsilon < M$ there exists $k(\varepsilon)$ such that

$$p\left(\{t \in T \mid \tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n) > \frac{\alpha}{2}\}\right) \geq \frac{\beta}{2}. \tag{15}$$

For any given $x \in [0, M]^n$, denote by $i(x) \in N$ the lowest-numbered player i for whom $\rho_i(x) \leq \frac{1}{2}$; since $\rho(x)$ is a sub-probability vector for each $x \in [0, M]^n$, the sets

$$E_i = \{x \in [0, M]^n \mid i(x) = i\} \in \mathcal{B}([0, M]^n) \tag{16}$$

(for $i = 1, 2$) constitute a partition of $[0, M]^n$, and so

$$\max_{i=1,2} \tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n \cap E_i) \geq \frac{1}{2} \tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n).$$

Now denote by $j(t, \varepsilon) \in \{1, 2\}$ the lowest-numbered player for whom the maximum in $\max_{i=1,2} \tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n \cap E_i)$ is attained. The function $j(\cdot, \varepsilon)$ is $\mathcal{T}$-measurable, and its definition implies that, whenever $\tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n) > \frac{\alpha}{2}$, we have $\tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n \cap E_{j(t,\varepsilon)}) > \frac{\alpha}{4}$. It therefore follows from (15) that

$$p\left(\{t \in T \mid \tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n \cap E_{j(t,\varepsilon)}) > \frac{\alpha}{4}\}\right) \geq \frac{\beta}{2}. \tag{17}$$

Since the events $F_i = \{t \in T \mid j(t, \varepsilon) = i\} \in \mathcal{T}$ (for $i = 1, 2$) constitute a partition of T, it follows from (17) that there exists a type-independent $\widehat{i}(\varepsilon) \in \{1, 2\}$ such that

$$p\left(\{t \in T \mid \tau^{\sigma^{k(\varepsilon)}}(t, [0, \varepsilon]^n \cap E_{\widehat{i}(\varepsilon)}) > \frac{\alpha}{4}\}\right) \geq \frac{\beta}{4} > 0. \tag{18}$$

Now choose a sequence $\{\varepsilon_l\}_{l=1}^{\infty} \subset (0, M)$ with $\lim_{l \to \infty} \varepsilon_l = 0$ for which $\widehat{i}(\varepsilon_l)$ is the *same* player for all l, and assume (w.l.o.g.) that this is player 1. Denote

$$T^l := \{t \in T \mid \tau^{\sigma^{k(\varepsilon_l)}}(t, [0, \varepsilon_l]^n \cap E_1) > \frac{\alpha}{4}\}. \tag{19}$$

By (18), we have $p(T^l) \geq \frac{\beta}{4} > 0$. Observe that $\liminf_{l \to \infty} \int_{T^l} V_1(t)\, p(dt) > 0$. That is because V_1 is positive by assumption, and hence $p(\{t \in T \mid V_1 > v\}) > 1 - \frac{\beta}{8}$ for some sufficiently small positive v, meaning that $p(T^l \cap \{t \in T \mid V_1 > v\}) > \frac{\beta}{8}$ for every l. Thus, there exists $\gamma > 0$ such that

$$\int_{T^l} V_1(t)\, p(dt) \geq \gamma \tag{20}$$

for every l.

For a fixed $0 < \eta < M$ and any l, consider a modification $\overline{\sigma}_1^{k(\varepsilon_l),\eta} \in \Sigma_1\left(m_{k(\varepsilon_l)}\right)$ of the BNE strategy $\sigma_1^{k(\varepsilon_l)}$ of player 1 in the constrained contest $G\left(m_{k(\varepsilon_l)}\right)$ that has a support in $[\eta, M]$ and satisfies $\overline{\sigma}_1^{k(\varepsilon_l),\eta}(t, [a, M]) = \sigma_1^{k(\varepsilon_l)}(t, [a-\eta, M])$ for any $a \in [\eta, M]$ and $t \in T$. (That is, if X_1 is a $\sigma_1^{k(\varepsilon_l)}(t, \cdot)$-distributed random variable on $[0, M]$, then $Y_1 := \min\{X_1 + \eta, M\}$ is $\overline{\sigma}_1^{k(\varepsilon_l),\eta}(t, \cdot)$-distributed; in other words, according to $\overline{\sigma}_1^{k(\varepsilon_l),\eta}$, player 1 increases his effort by η compared to what his original strategy $\sigma_1^{k(\varepsilon_l)}$ would instruct him to do, whenever possible.) Observe that $\rho_1(x) \leq \rho_1(\min\{x_1 + \eta, M\}, x_{-1})$ for any x by property (**v**) of ρ. Also, by properties (**iv**) and (**viii**) of ρ, we have

$$\rho_1(\min\{x_1 + \eta, M\}, x_{-1}) > \frac{1}{2}\left(\widehat{\rho}_1 + \frac{1}{2}\right) > \frac{1}{2}, \text{ and } \rho_1(x) \leq \frac{1}{2}, \tag{21}$$

as long as $x \in [0, \varepsilon_l]^n \cap E_1$ and l is sufficiently large (which ensures that ε_l is small). In particular, for all sufficiently large l, by deviating from strategy $\sigma_1^{k(\varepsilon_l)}$ to strategy $\overline{\sigma}_1^{k(\varepsilon_l),\eta}$ player 1 increases his probability of being the winner by at least $\frac{1}{2}\left(\widehat{\rho}_1 + \frac{1}{2}\right) - \frac{1}{2} = \frac{1}{2}\left(\widehat{\rho}_1 - \frac{1}{2}\right) > 0$ when $t \in T^l$ and the correlated strategy $\tau^{\sigma^{k(\varepsilon_l)}}$ chooses effort profiles in $[0, \varepsilon_l]^n \cap E_1$. It follows that, for all sufficiently large l,

$$\begin{aligned}
& U_1\left(\overline{\sigma}_1^{k(\varepsilon_l),\eta}, \sigma_{-1}^{k(\varepsilon_l)}\right) - U_1\left(\sigma^{k(\varepsilon_l)}\right) && (22)\\
\geq\ & \int_{T^l} \frac{1}{2}\left(\widehat{\rho}_1 - \frac{1}{2}\right) V_1(t)\, \tau^{\sigma^{k(\varepsilon_l)}}(t, [0, \varepsilon_l]^n \cap E_1) p(dt) && (23)\\
& - \int_T \int_{[0,M]} \left(c_1(t, \min\{x_1 + \eta, M\}) - c_1(t, x_1)\right) d\sigma_1^{k(\varepsilon_l)}(t, dx_1)\, dp(t) && (24)\\
\geq\ & \frac{1}{2}\left(\widehat{\rho}_1 - \frac{1}{2}\right) \cdot \frac{\alpha}{4} \cdot \gamma && (25)\\
& - \int_T \int_{[0,M]} \left(c_1(t, \min\{x_1 + \eta, M\}) - c_1(t, x_1)\right) d\sigma_1^{k(\varepsilon_l)}(t, dx_1)\, dp(t), && (26)
\end{aligned}$$

where the last inequality is obtained by recalling the definition of T^l in (19), and (20).

It follows from conditions (**ii**), (**iii**) and the dominated convergence theorem that the term in (26) converges to 0 when $\eta \to 0+$, and so η can be chosen in such a way that this term does not exceed $\left(\widehat{\rho}_1 - \frac{1}{2}\right) \cdot \frac{\alpha\gamma}{16}$ in absolute value. Thus, (22)–(26) imply that, for all sufficiently large l,

$$U_1\left(\overline{\sigma}_1^{k(\varepsilon_l),\eta}, \sigma_{-1}^{k(\varepsilon_l)}\right) - U_1\left(\sigma^{k(\varepsilon_l)}\right) \geq \left(\widehat{\rho}_1 - \frac{1}{2}\right) \cdot \frac{\alpha\gamma}{16} > 0.$$

This contradicts the assumption that $\sigma^{k(\varepsilon_l)}$ is a BNE in $G\left(m_{k(\varepsilon_l)}\right)$, and the required equality (12) is established.

Part 4: BNE payoffs in constrained contests converge to payoffs in the limit profile σ^*

We now show that, for any $i \in N$,

$$\lim_{k\to\infty} U_i\left(\sigma^k\right) = U_i\left(\sigma^*\right). \tag{27}$$

Fix $i \in N$, and, for any positive integer l, consider two functions $g_l^-, g_l^+ : T \times [0, M]^n \to \mathbb{R}$ defined as follows:

$$g_l^-(t, x) := \inf\left\{u_i(t, y) + l\,\|y - x\| \mid y \in [0, M]^n\right\}$$

and

$$g_l^+(t, x) := \sup\left\{u_i(t, y) - l\,\|y - x\| \mid y \in [0, M]^n\right\}$$

for any $(t, x) \in T \times [0, M]^n$, where $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^n$. It is easy to see that, for each $t \in T$, the functions $g_l^-(t, \cdot)$, $g_l^+(t, \cdot)$ are l-Lipschitz on $[0, M]^n$, and hence continuous. Furthermore, since $u_i(t, \cdot)$ is continuous on $[0, M]^n \setminus \{\mathbf{0}\}$, the condition $y \in [0, M]^n$ in the definition of g_l^-, g_l^+ can be replaced by the requirement that $y \in [0, M]^n$ *and* that each component of y is a rational number; thus, g_l^- (respectively, g_l^+) is an infimum (respectively, supremum) of a countable number of $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable functions, and consequently g_l^-, g_l^+ are themselves $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable. Lastly, Fact 1(**c**) implies that both $g_l^-(t, x)$, $g_l^+(t, x)$ are bounded in absolute value by an integrable function on T, for any x. We conclude that g_l^-, g_l^+ are Carathéodory integrands, for all l. Since, clearly, $g_l^- \leq u_i \leq g_l^+$, it follows from (11) that, for any l,

$$\begin{aligned}
\int_T \int_{[0,M]^n} g_l^-(t,x)\tau^{\sigma^*}(t,dx)p(dt) &= \\
\lim_{k\to\infty} \int_T \int_{[0,M]^n} g_l^-(t,x)\tau^{\sigma^k}(t,dx)p(dt) &\le \liminf_{k\to\infty} \int_T \int_{[0,M]^n} u_i\,(t,x)\,\tau^{\sigma^k}(t,dx)p(dt) = \\
\liminf_{k\to\infty} U_i(\sigma^k) &\le \limsup_{k\to\infty} U_i(\sigma^k) = \\
\limsup_{k\to\infty} \int_T \int_{[0,M]^n} u_i\,(t,x)\,\tau^{\sigma^k}(t,dx)p(dt) &\le \lim_{k\to\infty} \int_T \int_{[0,M]^n} g_l^+(t,x)\tau^{\sigma^k}(t,dx)p(dt) \\
&= \int_T \int_{[0,M]^n} g_l^-(t,x)\tau^{\sigma^*}(t,dx)p(dt).
\end{aligned}$$

Thus, for any l,

$$\int_T \int_{[0,M]^n} g_l^-(t,x)\tau^{\sigma^*}(t,dx)p(dt) \le \liminf_{k\to\infty} U_i(\sigma^k) \tag{28}$$

and

$$\limsup_{k\to\infty} U_i(\sigma^k) \le \int_T \int_{[0,M]^n} g_l^+(t,x)\tau^{\sigma^*}(t,dx)p(dt). \tag{29}$$

Next, notice that the sequence $\{g_l^-\}_{l=1}^\infty$ is monotonically increasing and $\{g_l^+\}_{l=1}^\infty$ is monotonically decreasing pointwise; in particular, $\lim_{l\to\infty} g_l^-$ and $\lim_{l\to\infty} g_l^+$ are well-defined. Moreover, for any $t \in T$ and any x that is a continuity point of the bounded function $u_i(t,\cdot)$, it is easy to see that

$$\lim_{l\to\infty} g_l^-(t,x) = u_i(t,x) = \lim_{l\to\infty} g_l^+(t,x). \tag{30}$$

Thus, it follows from Fact 1(**b**) that (30) holds for every $x \in [0,M]^n \setminus \{\mathbf{0}\}$. By ((12)) in Part 3, the joint strategy τ^{σ^*}avoids $\mathbf{0}$ with probability 1 for p-almost every t, and so (30) holds almost surely w.r.t. the probability measure $\tau^{\sigma^*}(t,dx)p(dt)$. It therefore follows, by the monotone convergence theorem, that

$$\begin{aligned}
&\lim_{l\to\infty} \int_T \int_{[0,M]^n} g_l^-(t,x)\tau^{\sigma^*}(t,dx)p(dt) \\
=\ & \int_T \int_{[0,M]^n} u_i(t,x)\tau^{\sigma^*}(t,dx)p(dt) \quad (= U_i\,(\sigma^*)\,) \\
=\ & \lim_{l\to\infty} \int_T \int_{[0,M]^n} g_l^+(t,x)\tau^{\sigma^*}(t,dx)p(dt).
\end{aligned}$$

Combined with (28) and (29), this means that $\liminf_{k\to\infty} U_i(\sigma^k) = U_i\,(\sigma^*) = \limsup_{k\to\infty} U_i(\sigma^k)$, which establishes (27).

Part 5. Payoffs in deviations from σ^* approximate (from below) payoffs in deviations from BNE in constrained contests

Now assume by way of contradiction that σ^* is not a BNE in G. Then there is a player (w.l.o.g., player 1) and $\sigma_1 \in \Sigma_1(0)$ such that $U_1(\sigma^*) < U_1(\sigma_1, \sigma^*_{-1})$. For any $0 < \varepsilon < M$, consider the strategy $\sigma_1^\varepsilon \in \Sigma_1(\varepsilon)$ ($\subset \Sigma_1(0)$) that satisfies $\sigma_1^\varepsilon(t, [\varepsilon, a]) = \sigma_1(t, [0, a])$ for any $a \in [\varepsilon, M]$ and $t \in T$. (That is, if X_1 is a $\sigma_1(t, \cdot)$-distributed random variable on $[0, M]$, then $Y_1 := \max\{X_1, \varepsilon\}$ is $\sigma_1^\varepsilon(t, \cdot)$-distributed.) By Fact 1(**d**),

$$\liminf_{\varepsilon \to 0+} u_1(t, (\max\{x_1, \varepsilon\}, x_{-1})) \geq u_1(t, x) \tag{31}$$

for every $t \in T$ and $x \in [0, M]^n$. Hence,

$$\begin{aligned}
\liminf_{\varepsilon \to 0+} U_1(\sigma_1^\varepsilon, \sigma^*_{-1}) &= \liminf_{\varepsilon \to 0+} \int_T \int_{[0,M]^n} u_1(t, (\max\{x_1, \varepsilon\}, x_{-1})) \, \tau^{(\sigma_1, \sigma^*_{-1})}(t, dx) \, p(dt) \\
\text{(by Fatou's lemma)} &\geq \int_T \int_{[0,M]^n} \liminf_{\varepsilon \to 0+} u_1(t, (\max\{x_1, \varepsilon\}, x_{-1})) \, \tau^{(\sigma_1, \sigma^*_{-1})}(t, dx) \, p(dt) \\
\text{(by (31))} &\geq \int_T \int_{[0,M]^n} u_1(t, x) \, \tau^{(\sigma_1, \sigma^*_{-1})}(t, dx) \, p(dt) = U_1(\sigma_1, \sigma^*_{-1}) \\
&> U_1(\sigma^*).
\end{aligned}$$

It can therefore by assumed (replacing σ_1 by σ_1^ε for some $\varepsilon > 0$ if necessary) that there exist $m > 0$ and $\sigma_1 \in \Sigma_1(m)$ such that $U_1(\sigma^*) < U_1(\sigma_1, \sigma^*_{-1})$.

Now define $g(t, x) := u_1(t, \max\{x_1, m\}, x_{-1})$ for every $(t, x) \in T \times [0, M]^n$. It follows from Fact 1(**a**)–(**c**) that g is a Carathéodory integrand on $T \times [0, M]^n$. Hence, by applying (11) to the sequence of strategy profiles $(\sigma_1, \sigma^k_{-1})$ that converges componentwise to $(\sigma_1, \sigma^*_{-1})$, we obtain

$$\lim_{k \to \infty} \int_T \int_{[0,M]^n} g(t, x) \tau^{(\sigma_1, \sigma^k_{-1})}(t, dx) p(dt) = \int_T \int_{[0,M]^n} g(t, x) \tau^{(\sigma_1, \sigma^*_{-1})}(t, dx) p(dt). \tag{32}$$

But, since $\sigma_1 \in \Sigma_1(m)$, $g(t, x) = u_1(t, x)$ for every $t \in T$ and $\tau^{(\sigma_1, \sigma^k_{-1})}(t, \cdot)$-almost every and also $\tau^{(\sigma_1, \sigma^*_{-1})}(t, \cdot)$-almost every $x \in [0, M]^n$. Therefore, the function g in (32) can be replaced by u_1, which yields

$$\lim_{k \to \infty} \int_T \int_{[0,M]^n} u_1(t, x) \tau^{(\sigma_1, \sigma^k_{-1})}(t, dx) p(dt) = \int_T \int_{[0,M]^n} u_1(t, x) \tau^{(\sigma_1, \sigma^*_{-1})}(t, dx) p(dt). \tag{33}$$

The left-hand side of the equality in (33) is precisely $\lim_{k \to \infty} U_1(\sigma_1, \sigma^k_{-1})$, and its right-hand side is precisely $U_1(\sigma_1, \sigma^*_{-1})$, which finally gives us

$$\lim_{k\to\infty} U_1(\sigma_1, \sigma_{-1}^k) = U_1(\sigma_1, \sigma_{-1}^*). \tag{34}$$

Since $U_1(\sigma_1, \sigma_{-1}^*) > U_1(\sigma^*)$ by assumption, and $\lim_{k\to\infty} U_1(\sigma^k) = U_1(\sigma^*)$ by (27), (34) implies that $U_1(\sigma_1, \sigma_{-1}^k) > U_1(\sigma^k)$ for any sufficiently large k. But since $\sigma_1 \in \Sigma_1(m)$ for $m > 0$, σ_1 also belongs to $\Sigma_1(m_k)$ for all sufficiently large k. Thus, for any such k, the strategy σ_1 constitutes a profitable unilateral deviation of player 1 from strategy profile σ^k in the constrained contest $G(m_k)$. This is a contradiction to σ^k being a BNE of $G(m_k)$. We conclude that σ^* is, after all, a BNE of G. Finally, if G is furthermore a generalized Tullock contest, then it possesses a pure-strategy BNE by Fact 3.

5 Extensions

In this section we discuss possible extensions of our framework and results.

5.1 General type-spaces

The assumption in Theorem 1 that type-spaces are countably generated is only used to ensure that Balder's (1988) weak topology on each strategy-set $\Sigma_i(0)$ is metrizable, thereby allowing us to work with simpler, sequential statements of compactness and continuity. However, inspection of the proof reveals that, with no appeal to metrizability, all the relevant statements can be formulated using a more general concept of *convergence of nets*, without affecting any of the arguments. Thus, the condition of countable generability is not, in fact, necessary for BNE existence.

5.2 Player- and type-dependent effort caps

Our framework can accommodate the case where, in addition to M being the uniform upper bound on effort expenditure, each player i has a type-dependent personal cap $M_i(t_i) \leq M$ on admissible efforts; accordingly, i chooses his effort from the interval $[0, M_i(t_i)]$, given any $t_i \in T_i$. Assume that the graph of the correspondence $t_i \longmapsto [0, M_i(t_i)]$ is $\mathcal{T}_i \times \mathcal{B}([0, M])$-measurable. For any $0 \leq m \leq M$, the set of pure (and, respectively, behavioral) strategies of i can be redefined as the set of those $s_i \in S_i(m)$ for which $s_i(t_i) \in [0, M_i(t_i)]$ for p_i-almost every $t_i \in T_i$ (respectively, those $\sigma_i \in \Sigma_i(m)$ for which $\sigma_i(t_i, [0, M_i(t_i)]) = 1$ for p_i-almost every $t_i \in T_i$), which we denote by $S_i(m, M_i)$ (respectively, $\Sigma_i(m, M_i)$). Furthermore, assume that the cap function M_i of every player i is bounded from below by a positive constant $\lambda > 0$; then $S_i(m, M_i)$ (and $\Sigma_i(m, M_i)$) is non-empty for any $0 \leq m \leq \lambda$. Denote the corresponding contest by $G\left(m, (M_i)_{i\in N}\right)$.

As in the no-caps framework, although the payoff functions in $G\left(m, (M_i)_{i\in N}\right)$ are fully continuous in efforts only when $m > 0$, a BNE exists also when $m = 0$ if the contest is pre-Tullock. It can be shown that Proposition 2 and Theorem 1 can be extended to $G\left(m, (M_i)_{i\in N}\right)$: this contest possesses a BNE for every $0 < m \leq \lambda$, and for $m = 0$ under the pre-Tullock assumption (the proof appears in the Appendix).

When $\lambda = 0$, i.e., with zero efforts being allowed, we do not know whether BNE exists if the condition that all caps exceed a positive uniform constant is relaxed to admit caps that are merely positive for every type. What is known, however, is that the possibility of *zero* caps for some types may lead to equilibrium non-existence (see, e.g., Example 1 in Ewerhart and Quartieri (2020)).

5.3 Unbounded effort sets

Our proofs utilize results that require either compactness of action sets in the game (theorems 2.2, 2,3 and 3.1 of Balder (1988)) or (integrable) boundedness of payoffs across all actions if the action sets are not compact (Theorem 3.3 of Balder (1988)). Thus, our method of proof relies on the assumption that the players' admissible effort levels belong to a closed bounded interval $[0, M]$. While the assumption that all efforts are uniformly bounded by an (arbitrarily large) cap seems plausible in most contexts, this explicit assumption can often be omitted. For instance, consider the scenario where all cost functions are differentiable in effort, and there exist $a, b > 0$ such that $V_i(t) \leq a$ and $\partial_{x_i} c_i(t, x_i) \geq b$ for every $t \in T$, $i \in N$, and $x_i \in \mathbb{R}_+$. Then $u_i(t, x_i) < 0$ whenever $x_i > \frac{a}{b}$, meaning that players will choose efforts in the interval $\left[0, \frac{a}{b}\right]$ in their best responses with probability 1. Thus, assuming that admissible efforts belong to $\left[0, \frac{a}{b}\right]$ entails no loss of generality in this case.

5.4 Type-dependent success functions

We can admit $\mathcal{T} \times \mathcal{B}([0, M]^n)$-measurable type-dependent success functions $\rho : T \times [0, M]^n \to [0, 1]^n$ in which, for any $t \in T$, $\rho(t, \cdot)$ satisfies the conditions that we imposed on a type-independent success function in the various categories of contests. Propositions 1 and 2 will hold without any change, by identical arguments. However, in order for the proof of Theorem 1 to remain valid, there will be a need to assume that the functions $\{\rho_i(t,\cdot)\}_{t\in T}$ are pointwise equicontinuous, for any $i \in N$. This assumption will ensure that the first inequality in (21) in Part 3 of the proof of Theorem 1 is satisfied uniformly (as long as $x \in [0,\varepsilon_l]^n$ and l is sufficiently large) for all type-profiles (that now may affect ρ). Also note that the sets E_i defined in (16) that partition $[0,M]^n$ may now be state-dependent, as the underlying function $i(x)$ may be state-dependent. This does not affect the rest of the proof because the functions $\tau^{\sigma^{k(\varepsilon)}}(\cdot,[0,\varepsilon]^n \cap E_i)$ remain $\mathcal{T}$-measurable (by, e.g., Proposition 7.29 of Bertsekas and Shreve (2004)).

5.5 Productive efforts

If one is interested in the existence of BNE in behavioral strategies only, an inspection of our proofs reveals that the value functions $(V_i)_{i\in N}$ may be allowed to depend on effort profile x, in a way that each V_i is continuous and non-decreasing in x_i. However, there will then be a need to also assume that $(V_i)_{i\in N}$ are uniformly bounded from below by a positive constant $\kappa > 0$, as inequalities of the form (20) that are used in Part 3 of the proof of Theorem 1 need not necessarily hold the case of effort-dependent values.

A Appendix

A.1 Non-common-value contest that is not uniformly diagonally secure: an example

Here we present an example of a contest which does not have the property of uniform diagonal security and in which the sums of payoffs are not upper semi-continuous in efforts. Thus, Theorems 1 and 2 in Carbonell-Nicolau and McLean (2018) and Theorem 2 in He and Yannelis (2016) on BNE existence do not apply to this contest.

Consider a complete-information Tullock lottery with player set $N = \{1,2\}$, effort set $[0,1]$, and costs that are equal to efforts. Also assume that $V_1 = 1$ while $V_2 = \delta$ for a fixed $\delta \in (0,\frac{1}{6})$. According to (5), $\rho^T(x) = (\frac{x_1}{x_1+x_2},\frac{x_2}{x_1+x_2})$ whenever $\mathbf{0} \neq x \in [0,1]^2$, and assume that $\rho^T(\mathbf{0}) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Thus, the payoff function of each player $i = 1, 2$ is given by

$$u_i(x) = \begin{cases} \frac{x_1}{x_1+x_2} V_i - x_i & \text{if } x \neq \mathbf{0}, \\ \frac{1}{2} V_i & \text{if } x = \mathbf{0} \end{cases}$$

for any $x \in [0,1]^2$.

The notion of uniform diagonal security in a (complete-information) topological game, due to Prokopovych and Yannelis (2014), requires that for any $\varepsilon > 0$ and $x \in [0,1]^2$ there exist $x^* \in [0,1]^2$ with the following property: for any $y \in [0,1]^2$ there is a (relatively) open neighborhood $W_y \subset [0,1]^2$ of y such that, for every $z \in W_y$,

$$\sum_{i=1}^{2} u_i(x_i^*, z_{-i}) - \sum_{i=1}^{2} u_i(z) \geq \sum_{i=1}^{2} u_i(x_i, y_{-i}) - \sum_{i=1}^{2} u_i(y) - \varepsilon. \tag{35}$$

Now take $\varepsilon = \delta$, $x = (\delta, 0)$ and $y = \mathbf{0}$, and suppose that (35) holds for some x^* and W_y. Notice that

$$\sum_{i=1}^{2} u_i(x_i, y_{-i}) - \sum_{i=1}^{2} u_i(y) - \varepsilon = \frac{1}{2} - 2\delta. \tag{36}$$

On the other hand, $z := \frac{1}{k}x$ belongs to W_y for all sufficiently large k, with

$$u_1(x_1^*, z_2) - u_1(z) \leq 1 - u_1(z) = 1 - (1 - \frac{\delta}{k}) = \frac{\delta}{k}$$

and

$$u_2(z_1, x_2^*) - u_2(z) \leq \delta,$$

implying that

$$(1 + \frac{1}{k})\delta \geq \sum_{i=1}^{2} u_i(x_i^*, z_{-i}) - \sum_{i=1}^{2} u_i(z). \tag{37}$$

It follows from (35), (36) and (37) that $(1 + \frac{1}{k})\delta \geq \frac{1}{2} - 2\delta$ for all sufficiently large k, and therefore $3\delta \geq \frac{1}{2}$. But this contradicts the choice of $\delta \in (0, \frac{1}{6})$, and we conclude that the contest does not have the property of uniform diagonal security.

Notice also that the sum of payoff in the contest is not an upper semi-continuous function on $[0,1]^2$, since

$$\lim_{x_1 \to 0+} \sum_{i=1}^{2} u_i(x_1, 0) = 1 > \sum_{i=1}^{2} u_i(\mathbf{0}) = \frac{1+\delta}{2}.$$

Finally, observe that the above contest can be embedded in any incomplete information scenario, as being played at some (or all) realizations of the players' information types, and hence the corresponding Bayesian game would not be uniformly diagonally secure (according to Definition 12 in Carbonell-Nicolau and McLean (2018) of that property for Bayesian games).

A.2 Proof of the claim in Section 5.2

Claim. The contest $G\left(m, (M_i)_{i\in N}\right)$ possesses a BNE for every $0 < m \leq \lambda$, and for $m = 0$ when it is pre-Tullock.

Proof. When $0 < m \leq \lambda$, one should proceed exactly as in the proof of Proposition 2, but use Theorem 3.3 of Balder (1988) to deduce BNE existence (instead of Theorem 3.1). Theorem 3.3 allows state-dependent action spaces, but at a cost of some additional conditions on the game. These extra conditions are satisfied by $G\left(m, (M_i)_{i\in N}\right)$: for each $t \in T$, the admissible action sets $\{[0, M_i(t_i)]\}_{i\in N}$ and the action space $[0, M]$ are compact, and the graph of the admissible actions correspondence $t_i \longmapsto [0, M_i(t_i)]$ is $\mathcal{T}_i \times \mathcal{B}([0, M])$-measurable for each $i \in N$. Hence, according to Theorem 3.3 of Balder (1988), $G\left(m, (M_i)_{i\in N}\right)$ has a BNE in behavioral strategies.

As for the case of $m = 0$, we will first show that, for every $i \in N$, the capped behavioral strategy set $\Sigma_i(0, M_i)$ is a *compact* subset of $\Sigma_i(0)$ in the weak topology. This will be done as in the proof of Theorem 3.3 in Balder (1988). Consider the function $h_i : T_i \times [0, M] \to [0, \infty]$ that is given by

$$h_i(t_i, x_i) = \begin{cases} 0, & \text{if } x_i \in [0, M_i(t_i)]\,, \\ \infty, & \text{otherwise} \end{cases}.$$

Clearly, h_i is $\mathcal{T}_i \times \mathcal{B}([0, M])$-measurable because so is the graph of $t_i \longmapsto [0, M_i(t_i)]\,,$ and it is also inf-compact (that is, for any $t_i \in T_i$ and $\beta \in \mathbb{R}$, $\{x_i \in [0, M] \mid h_i(t_i, x_i) \leq \beta\}$ is compact). By Theorem 2.3(b) of Balder (1988), the functional $I_{h_i} : \Sigma_i(0) \to \mathbb{R}$ that is given by

$$I_{h_i}(\sigma_i) = \int_T \int_{[0,M]} h_i(t_i, x_i)\sigma_i(t_i, dx_i)p_i(dt_i),$$

for any $\sigma_i \in \Sigma_i(0)\,,$ is weakly inf-compact. The latter means that for any $\beta \in \mathbb{R}$, $\{\sigma_i \in \Sigma_i(0) \mid I_{h_i}(\sigma_i) \leq \beta\}$ is weakly compact. Notice now that $I_{h_i}(\sigma_i) \leq 0$ if and only if $\sigma_i(t_i, [0, M_i(t_i)]) = 1$ for p_i-almost every $t_i \in T_i$, i.e., if and only if $\sigma_i \in \Sigma_i(0, M_i)\,.$ Thus $\Sigma_i(0, M_i)$ is weakly compact.

The proof of our claim for $m = 0$ can now be obtained by repeating parts 2–5 of the proof of Theorem 1, with the following modifications: (1) all strategy sets and strategy-profile sets used in parts 2–5 should be replaced by their capped versions (e.g., each $\Sigma_i(0)$ should be replaced by $\Sigma_i(0, M_i)$)[18]; (2) in Part 2, the sequence $\{m_k\}_{k=1}^{\infty}$ should be chosen from the interval $(0, \lambda)$, and a sequence $\{\sigma^k\}_{k=1}^{\infty}$ should consist of BNE $\sigma^k \in \Sigma\left(m_k, (M_i)_{i\in N}\right)$ in constrained capped contests $G\left(m_k, (M_i)_{i\in N}\right)$; (3) in Part 3, the strategy $\overline{\sigma}_1^{k(\varepsilon_l),\eta}$ used in (22) should only be considered for $0 < \eta < \lambda$, and be redefined as $\overline{\sigma}_1^{k(\varepsilon_l),\eta}(t, [a, M_1(t)]) = \sigma_1^{k(\varepsilon_l)}(t, [a - \eta, M_1(t)])$ for any $a \in [\eta, M_1(t)]$ and $t \in T$, in order to be a capped strategy in $G\left(m_{k(\varepsilon_l)}, (M_i)_{i\in N}\right)$; (4) in Part 5, the strategy $\sigma_1^{\varepsilon} \in \Sigma_1(\varepsilon, M_1)$ $(\subset \Sigma_1(0, M_1))$ should only be defined for $0 < \varepsilon < \lambda$. $\square$

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