

## Review of Frequency Response Analysis

### Objectives

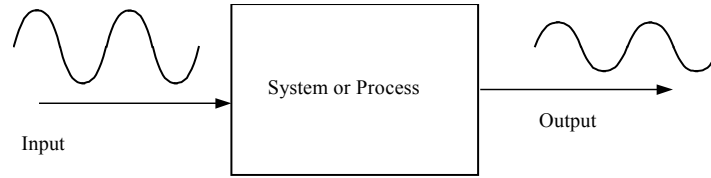
The objective of this appendix is to review the basic concepts behind frequency response analysis and its use in control system design. Topics discussed include

- Bode and Nyquist plots.
- Nyquist stability theorem.
- Closed-loop response characteristics.
- Controller performance and design criteria.

### B.1 INTRODUCTION

By *frequency response* we mean the response characteristics of the system when subject to sinusoidal inputs. The input frequency is varied, and the output characteristics are computed or represented as a function of the frequency. Frequency response analysis provides useful insights into stability and performance characteristics of the control system.

Figure B.1 shows the hypothetical experiment that is conducted.



**Figure B.1** How frequency response is defined.

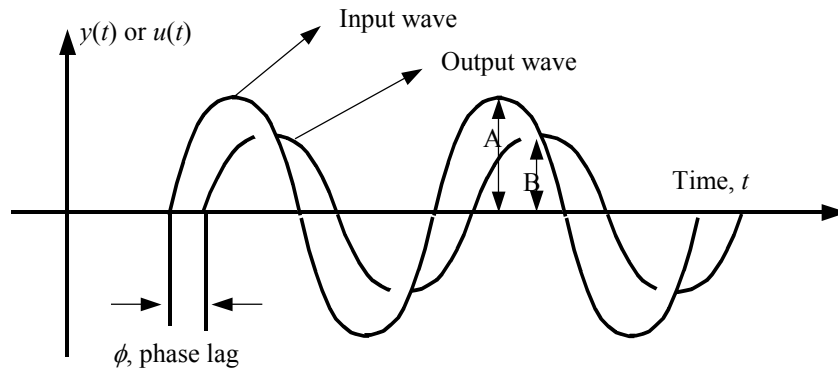
The system is subject to an input of the form

$$x(t) = A \sin(\omega t) \quad t > 0. \quad (\text{B.1})$$

After some initial transient period, the output settles down to a sine wave of the form

$$y(t) = B \sin(\omega t + \phi), t \gg 0. \quad (\text{B.2})$$

The amplitude and phase are changed by the system, but the frequency remains the same. This is shown in Figure B.2.

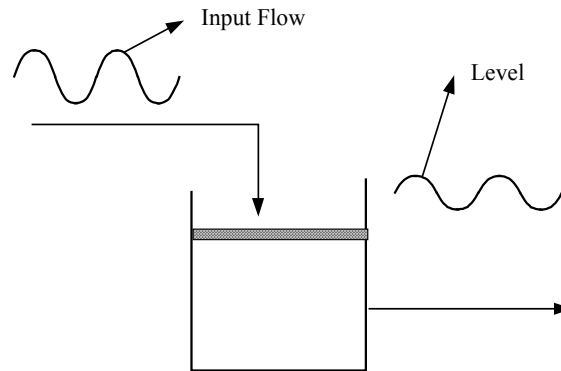


**Figure B.2** Frequency response.

Note that the output wave lags behind the input.  $\phi$  is defined as the *phase lag* (usually expressed in degrees or radians). The output amplitude is different from the input, and we can define a ratio:

$$\text{Amplitude Ratio}(AR) = B / A. \quad (\text{B.3})$$

Now let us examine the effect of changing the frequency of the input. Consider the response of the level in the tank (see Figure B.3) to sinusoidal changes in the input flow.



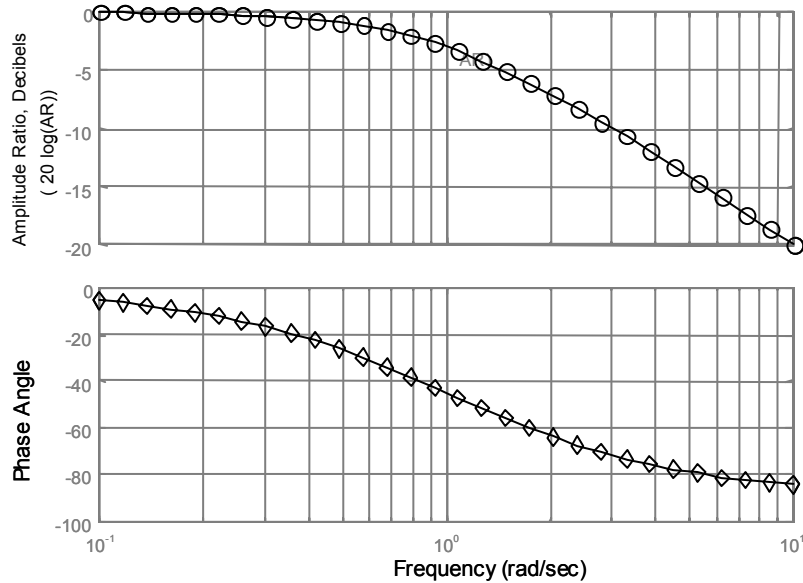
**Figure B.3** Self-regulating level response to inlet flow variations.

Let us say that the tank is self-regulating: as the level changes, the outlet flow changes due to the change in hydraulic pressure until, at steady-state, the time average of the inlet flow matches the outlet flow.

If the inlet flow changes sinusoidally, the level will respond likewise. At low frequencies, the level will have plenty of time to keep pace with the inlet flow changes. At very high frequencies, the tank will not have time to respond to the flow variations and the amplitude of level changes will be small. The tank, in effect, will average out the inlet flow fluctuations. The peak in level will occur sometime after the inlet flow has peaked; that is, the changes in level will lag behind the changes in inlet flow.

There are a number of ways to represent the frequency response of a process. We will use two of these representations: *Bode plots* and *Nyquist plots*.

A Bode plot is a plot of the amplitude ratio (AR) and the phase lag as a function of the frequency of the input line wave (which is the same as the frequency of the output wave). Logarithmic scales are used for the frequency axis. The y-axis is often plotted using the units of decibels, which is  $20 \log(\text{AR})$ . Figure B.4. shows the Bode plot for a first-order process. In this example both the AR and  $\phi$  decrease as the frequency increases. At low frequencies, the output is able to respond to the slow varying input disturbances with only a small attenuation (AR close to 1). However, at higher frequencies, the AR decreases rapidly, approaching an asymptote with a slope of  $-1$  in the log-log graph shown for the first-order self-regulating process of the tank. Note that this system acts as a *low pass filter*, that removes the high-frequency inputs. A first-order system has decreasing phase angle, which approaches  $-90$  asymptotically at higher frequencies. This implies that the output will lag behind the input (hence the name first-order lag).



**Figure B.4** Bode plot of a first-order process with  $G(s) = 1/(s+1)$ .

The phase angle  $\phi$  is usually negative. In this text we use the convention that phase lag is the negative of the phase angle. As  $\omega$  increases,  $\phi$  becomes more negative (i.e., the phase lag increases). This again represents the fact that at higher frequencies, the output will peak later than the input.

Typically, most processes exhibit a low AR at high frequencies. Hence, any low frequencies present in the input signal is passed through the process, whereas high-frequency components of the input signal are reduced significantly in amplitude as they pass through the process. We can view such a process as a low pass filter, which allows low frequencies to pass through without attenuation.

Any periodic signal can be viewed as a composite sum of various frequency components (obtained via a Fourier transform of the signal). Likewise, a system can be viewed as a filter that attenuates the input signal according to the frequency contained in the signal.

A Nyquist plot is another way to show the frequency response. Here  $G(i\omega)$  is plotted in the complex plane. An example of a Nyquist plot is shown in Figure B.5. Note that there are two parts to the curve: one that shows the plot for  $\omega$  varying from 0 to  $\infty$  and another that shows the curve for  $\omega$  varying from  $-\infty$  to 0. (This is true only for transfer functions with the degree of the denominator polynomial higher than the degree of the numerator.)

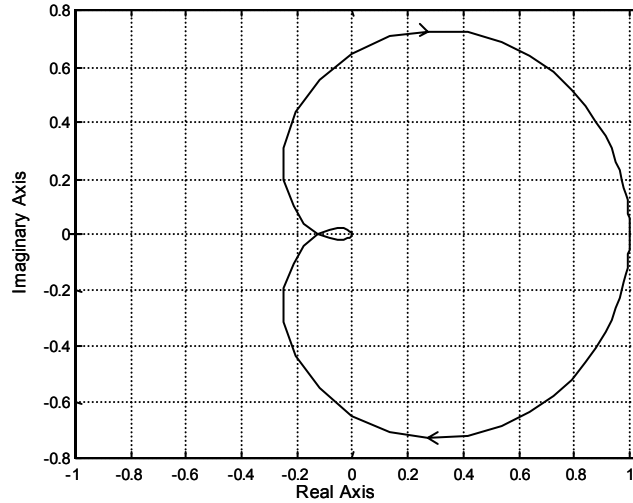


Figure B.5 Nyquist plot of  $G(s)=1/(s+1)^3$ .

## B.2 FREQUENCY RESPONSE FROM TRANSFER FUNCTIONS

The frequency response can be derived from the transfer function using the following theorem: *Consider a process with transfer function  $G(s)$ . Then the frequency response is given by*

$$\begin{aligned} AR &= |G(i\omega)|, \\ \phi &= \angle G(i\omega). \end{aligned} \tag{B.4}$$

The proof of the theorem can be found in many undergraduate texts (see Luyben 1989, for example).

### Example B.1

Consider

$$\begin{aligned} G(s) &= \frac{K}{\tau s + 1}, \\ G(i\omega) &= \frac{K}{\tau(i\omega) + 1}. \end{aligned}$$

Applying the theorem, we get

$$AR = |G(i\omega)| = \frac{K}{(1 + \omega^2 \tau^2)^{1/2}},$$

$$\phi = \angle G(i\omega) = \tan^{-1}(-\omega\tau).$$

This is plotted in Figure B.4 for  $\tau=1$  and  $K=1$ . As  $\omega$  increases, the AR gets smaller and smaller. At  $\omega_b = \tau$ , the  $AR = 1/\sqrt{2}$  and  $\phi = -\pi/4$  rad =  $-45^\circ$ . This is called the *break frequency*. (Break frequency is the frequency at which the low frequency and high frequency asymptotes intersect.) The maximum phase lag of  $-90^\circ$  is reached as  $\omega \rightarrow \infty$ . At low frequencies,  $AR \rightarrow K$ , which means that the output amplitude is the input amplitude multiplied by the process gain.

◆

The following corollary to the theorem in Example B.1 is useful in computing the frequency response of transfer functions in series.

*Corollary: The frequency response of two transfer functions in series is given by*

$$G(s) = G_1(s) G_2(s), \quad (\text{B.5a})$$

$$AR(G) = AR(G_1) AR(G_2), \quad (\text{B.5b})$$

$$\phi(G) = \phi(G_1) + \phi(G_2). \quad (\text{B.5c})$$

Thus ARs are multiplied together, whereas the phase angles are additive. For example, we can obtain the frequency response of a third-order transfer function using the result of equation series (B.5).

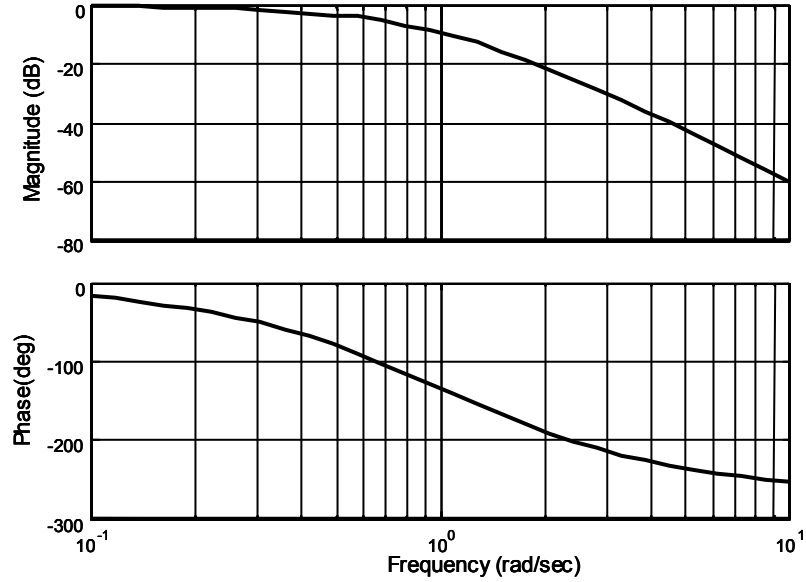
$$G(s) = \frac{1}{(s+1)(s+1)(s+1)},$$

$$AR(G) = AR\left(\frac{1}{s+1}\right) \cdot AR\left(\frac{1}{s+1}\right) \cdot AR\left(\frac{1}{s+1}\right),$$

$$= \frac{1}{(1 + \omega^2)^{3/2}},$$

$$\phi(G) = 3 \tan^{-1}(-\omega).$$

The result is plotted in Figure B.6.

Figure B.6 Bode plot of  $1/(s+1)^3$ .

### Example B.2 Time Delay

Consider

$$G(s) = e^{-Ds}.$$

Then

$$AR(G) = |e^{-Di\omega}| = 1,$$

$$\phi = -D\omega \text{ radians.}$$

The delay contributes to a phase lag, which increases with frequency.

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### Example B.3 Frequency Response of a PID Controller

Consider the PID controller

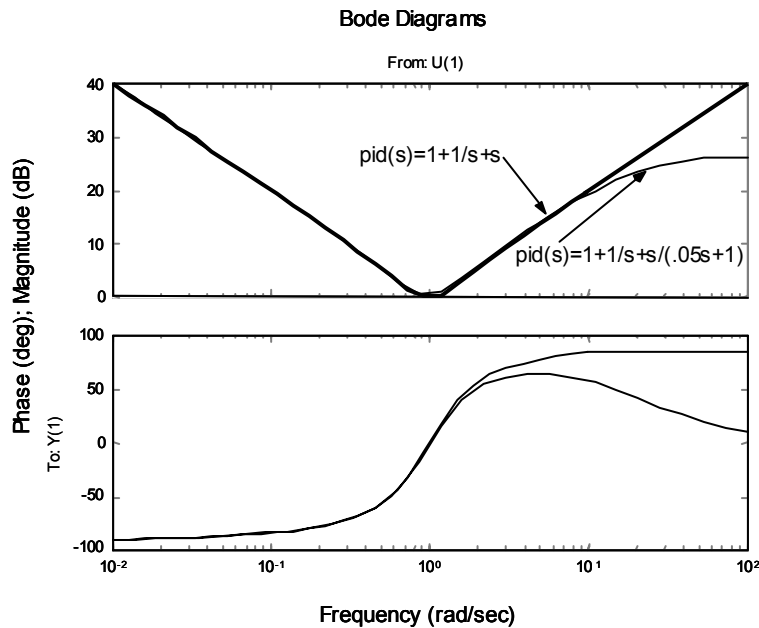
$$G_c(s) = \left(1 + \frac{1}{\tau_I s} + \tau_D s\right) K_c.$$

Figure B.7 shows its frequency response for  $K_c = 1, \tau_I = 1, \tau_D = 1$ . Note that this controller has a high gain at both low and high frequencies. Also shown in the diagram is the frequency response

$$G_c(s) = \left( +\frac{1}{\tau_I s} + \frac{\tau_D s}{0.05\tau_D s + 1} \right) K_c.$$

which includes a high-frequency noise filter to suppress the high-frequency gain. The addition of a first-order lag to the derivative term reduces the high-frequency gain to 20. Since measurement noise is usually of a high-frequency nature, this filter suppresses noise amplification.

Note the high gain of the controller for low frequencies. This is necessary to eliminate steady-state error when low-frequency disturbances are present. This high gain is caused by the integral term in the controller. Later in this section we justify the need for high gain at low frequencies.



**Figure B.7** Bode plots of ideal and real PID controllers.





### B.3 DISTURBANCE SUPPRESSION IN SISO SYSTEMS: EFFECT OF CONSTRAINTS

We can examine the capability of a system to suppress disturbances using frequency response methods. Selecting  $s = i\omega$  we get

$$y(i\omega) = p(i\omega)u(i\omega) + d(i\omega).$$

We can suppress the disturbance effect on  $y(s)$  only if perfectly only if  $p(s)$  is invertible. Even then the constraints of  $u$  usually prevent us from achieving perfect compensation. Hence at any frequency  $\omega$ , for perfect compensation,

$$p(i\omega)u(i\omega) = -d(i\omega)$$

or

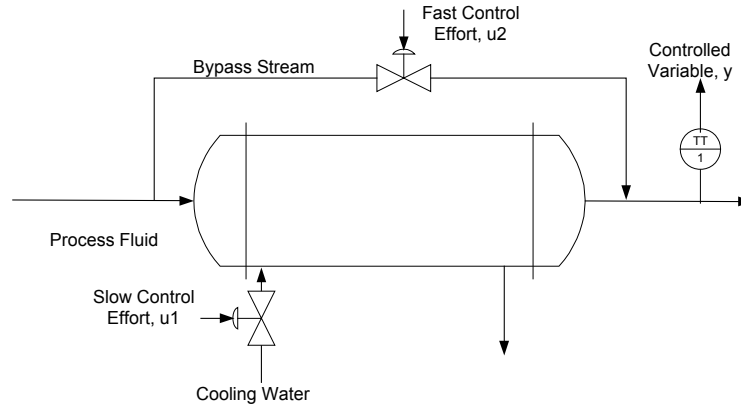
$$|p(i\omega)||u(i\omega)| = |d(i\omega)|.$$

The ability to suppress the disturbance depends on the value of  $|p(i\omega)|$  given the constraints on  $|u(i\omega)|$ .  $|p(i\omega)|$  is called the *dynamic gain* of the system. The steady-state analysis on the effect of input constraints discussed in Appendix A extends to the dynamic case, with the steady-state gain being replaced by the dynamic gain. Typically, the magnitude of the disturbances,  $|d(i\omega)|$  becomes small at high frequencies.  $|p(i\omega)|$  also tend to become small at high frequency. Hence the question is important in the intermediate frequency range, where  $|d(i\omega)|$  can be significant. If  $|p(i\omega)|$  is small at these ranges of frequency, then we will have a problem with disturbance suppression.

#### Example B.4 Selection of Manipulated Variables in a Heat Exchange System

Consider the heat exchanger shown in Figure B.8. There are two possible manipulated variables: the bypass flow rate  $u_1$  and the cooling water flow rate  $u_2$ . The following transfer functions were obtained. All variables have been scaled and normalized using the procedure outlined in Appendix A.

$$\begin{aligned} y(s) &= p_1(s)u_1(s) + p_2(s)u_2(s) + d(s) & -0.5 < u_1 < 0.5 \\ &= \frac{10}{s+1}u_1(s) + \frac{2}{.05s+1}u_2(s) + d(s) & -0.5 < u_2 < 0.5, \end{aligned}$$



**Figure B.8** Instrumentation diagram for a heat exchanger.

where  $d(s)$  is the disturbance caused by feed flow variations.  $d(t)$  was determined to vary with an amplitude of 3.0 around a frequency of 1.0 rad with amplitude of 1.5 near a frequency of 10 rad/time. Which manipulated variable should be used in a single-loop feedback control of  $y$  to suppress the disturbances?

Figure B.9 shows the frequency response of the transfer functions. From this it is seen that at  $\omega = 1.0$  and the AR of  $p_1(s)$  and  $p_2(s)$  are 7.0 and 2.0 respectively. Given that  $u_2$  must lie between  $-0.5$  and  $0.5$ , the maximum amplitude that can be achieved in  $y$  using  $u_2$  at this frequency is only 1.0. Hence at this frequency, we can control  $y$  using  $u_1$  but not using  $u_2$ . At a frequency of  $\omega=10$  rad/time, the ARs of  $p_1(s)$  and  $p_2(s)$  are 1.0 and 1.78 respectively. In this case we can control  $y$  using  $u_2$  but not  $u_1$ . Hence both inputs should be used if control is desired over the frequency range of 1.0 to 10 rad/time.

Note that this result is intuitive. We must use fast-responding control effort to suppress fast changing disturbances. However, the slower control effort with a larger gain is useful in suppressing slow disturbances of higher magnitude. *Ideally, we want both large steady-state gain, fast response to control efforts, and large bounds on the control effort.* Chapter 18, “Advanced Model-Predictive Control,” discusses how to use both control efforts efficiently. See also the paper by Brosilow et al. (1986).

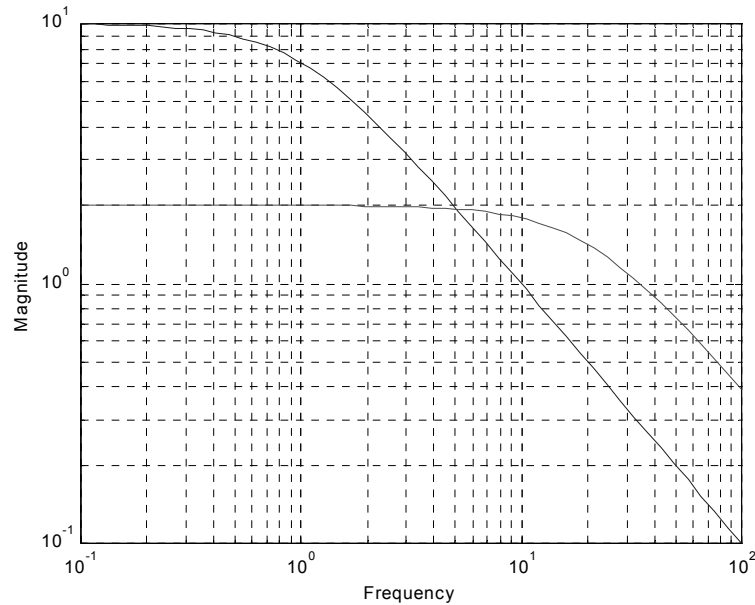


Figure B.9 Dynamic gain of the two control efforts for the heat exchanger.

## B.4 STABILITY IN THE FREQUENCY DOMAIN

Nyquist derived conditions for stability of a system from the frequency response characteristics. This result is known as the *Nyquist stability theorem*:

The number of zeros,  $Z$ , of the characteristic equation of the closed-loop system shown in Figure B.10, inside a closed contour,  $D$ , in the complex plane, is equal to the number of poles of the characteristic equation inside of the closed contour  $D$  plus the number of clockwise (counterclockwise) encirclements of the point  $(-1/k, 0)$  by the phasor  $g(s)$  as  $s$  moves in a clockwise (counterclockwise) direction around the closed contour  $D$ .

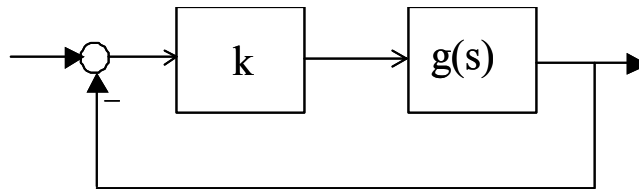
That is,  $Z = N + P$ ,

where  $Z = \#$  of zeros of  $(1 + kg(s))$  inside  $D$ .

$N = \#$  of encirclements of  $(-1/k, 0)$  point by  $g(s)$  as  $s$  moves once around  $D$ .

$P = \#$  of poles of  $(1 + kg(s))$  inside  $D$ .

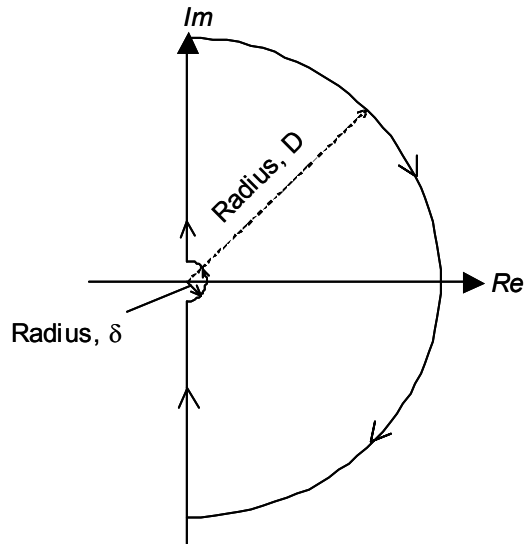
A control system is stable if, and only if, the contour  $D$  encloses the entire right half of the  $s$  plane and the number of zeros  $Z$ , as calculated above, is zero. (An intuitive proof of the Nyquist stability theorem can be found in Appendix IV of Vegte, 1986).



Characteristic equation =  $1 + kg(s)$

**Figure B.10** Feedback diagram for the Nyquist stability criterion.

If  $g(s)$  has a pole at  $s = 0$ , then the contour  $D$  is usually taken, as shown in Figure B.11, and the radius  $\delta$  is made to approach zero so that the contour encloses the entire right half of the  $s$ -plane.

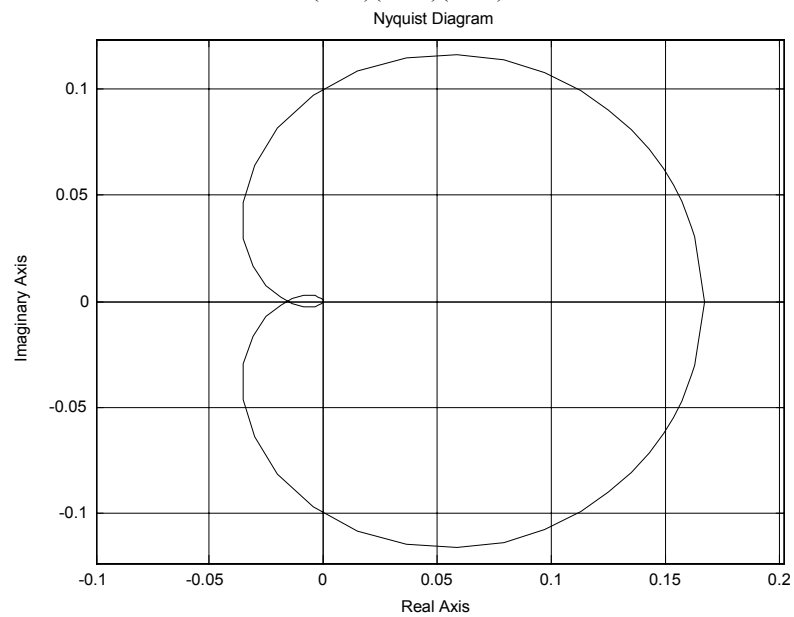


**Figure B.11** Nyquist  $D$  contour when there is a pole at the origin.

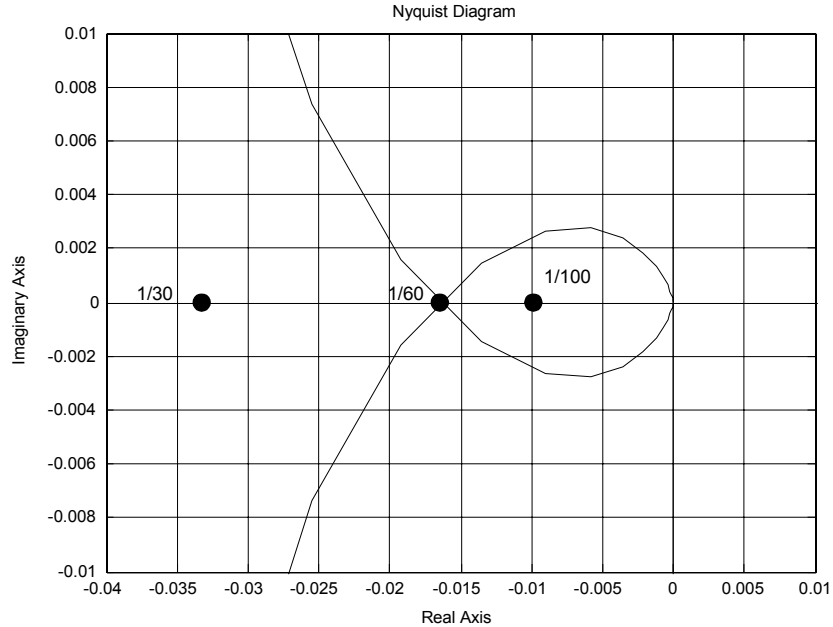
**Example B.5. Application of Nyquist Stability Theorem**

Figure B.12 shows the Nyquist plots of  $G(s)$  for

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}.$$



**Figure B.12a** Nyquist plot of  $1/(s+1)(s+2)(s+3)$  is shown. The graph below is a zoomed version near the origin.



**Figure B.12b** Nyquist plot of  $1/(s+1)(s+2)(s+3)$  near the origin.

Consider proportional feedback control of this system using  $K_c = 30$  and  $K_c = 100$ . For  $K_c = 30$ , the Nyquist plot of  $K_c G(i\omega)$  does not encircle the  $(-1/30, 0)$  point. Hence  $N = 0$ . Also,  $G(s)$  is open-loop stable, so that  $P = 0$ . Hence, according to the Nyquist stability theorem,  $Z = 0$  (i.e., the closed-loop system should have no unstable poles). This is verified by the root locus plot shown in Table B.1. For  $K_c = 100$ , there are two encirclements of the  $(-0.01, 0)$  point and hence  $Z = 2 + 0 = 2$ . Thus the closed-loop system should have two unstable poles. This is also verified by Table B.1.

The ultimate gain, the gain at which the system just becomes unstable, is also shown. This occurs for  $K_c = 1/0.0167 = 60$ . In this case, the system will be oscillatory with the ultimate frequency. The advantage of the Nyquist stability theorem is that we can graphically determine the closed-loop characteristics from the open-loop Nyquist plot. Table B.1 shows the frequency response of the process transfer function  $p(s)$ . Adding a proportional controller  $c(s) = K_c$  will not change the phase angle but will multiply the magnitude by  $K_c$ . The ultimate frequency with such a proportional controller is when the phase angle of the open-loop process becomes  $-180$ , which occurs at around  $\omega = \sqrt{11}$ . The magnitude of  $p(i\omega)$  at this frequency is given by  $1/60$ . Hence using a controller gain of 60 will make the magnitude of  $pc$

= 1, causing the Nyquist diagram to go through the (-1,0) point. This is the ultimate gain for this process.

Table B.1 Frequency Response of  $p(s) = 1/(s+1)(s+2)(s+3)$

Frequency rad/sec	Magnitude	Phase Angle, degrees
0.1000	0.1655	- 10.4822
0.1600	0.1638	- 16.7156
0.2560	0.1596	- 26.5264
0.4942	0.1431	- 49.5302
0.7906	0.1176	- 74.6628
1.2649	0.0805	-106.8415
2.0236	0.0430	-143.0395
3.2375	0.0176	-178.3086
3.5565	0.0143	-184.7952
6.8665	0.0027	-221.8740
10.000	0.0009	-236.2802



## B.5 CLOSED-LOOP FREQUENCY RESPONSE CHARACTERISTICS

Consider the control system shown in Figure B.13. For this closed-loop system, we can define two transfer functions:  $S(s)$  and  $CS(s)$ . We have, using block diagram algebra,

$$\frac{y(s)}{y_{set}(s) - n(s)} = \frac{p(s)c(s)}{1 + p(s)c(s)} = CS(s) \tag{B.8a}$$

$$\frac{y(s)}{d(s)} = \frac{1}{1 + p(s)c(s)} = S(s) \tag{B.8b}$$

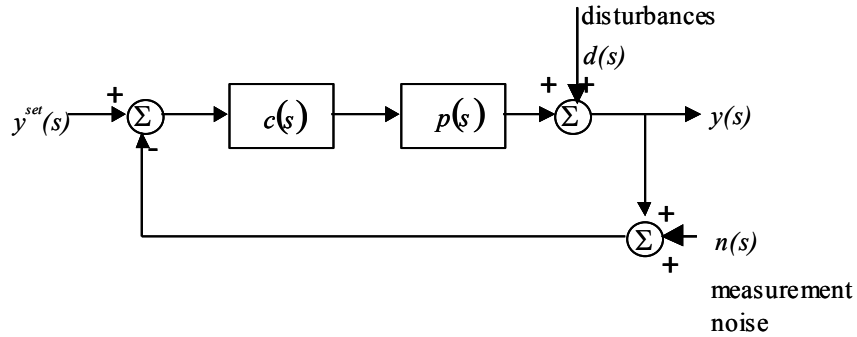
where

$S(s)$  = sensitivity function,

$CS(s)$  = complementary sensitivity function.

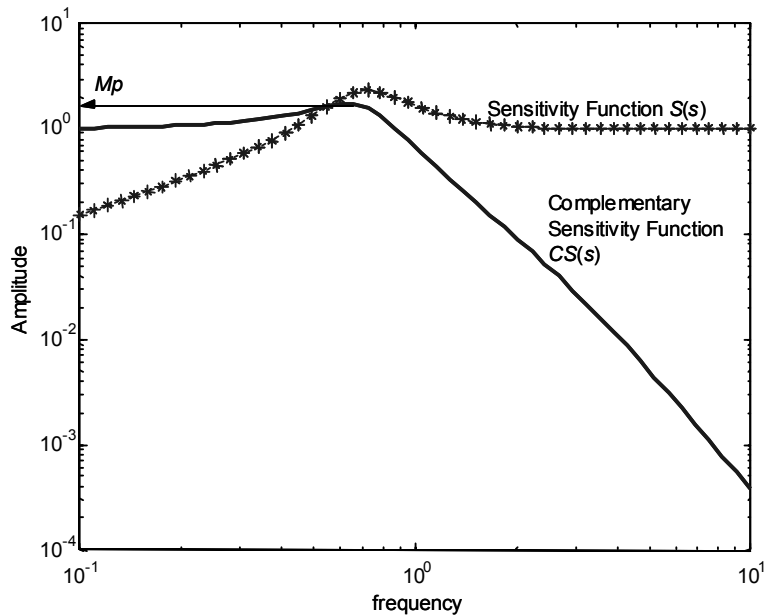
Note that

$$CS(s) + S(s) = 1$$



**Figure B.13** Block diagram of closed-loop system.

Recall that  $|p(i\omega)|$  generally decreases in value rapidly as  $\omega$  gets large. It shows a downward trend with increasing frequency. This implies that  $CS(s)$  will also decrease at large frequencies, assuming  $|c(s)|$  does not become extremely large as  $\omega$  increases. Figure B.14 shows the typical behavior of  $CS$  and  $S$ .



**Figure B.14** A typical sensitivity and complementary sensitivity function.



Ideally, we want  $S(s)$  to be as small as possible over a wide range of frequencies. Similarly, we want  $CS(s)$  to be close to 1 over a wide range of frequencies. The frequency at which  $CS(s)$  drops to  $1/\sqrt{2}$  is called the *bandwidth* of the system. Beyond that frequency, the controller will not be able to follow the setpoint well.  $CS(s)$  may exhibit a maximum.

Now let us consider how to choose the controller response characteristics. Our goal will be to pick  $c(s)$  such that the control objectives set in the time domain or frequency domain can be met. Some objectives and their consequences are discussed next.

**1. No steady-state error.** Using final value theorem, steady-state corresponds to low-frequency behavior. If we have  $CS(i\omega) \rightarrow 1$  as  $\omega \rightarrow 0$ , then we will have no steady-state error. Since  $p(i\omega)$  approaches a finite value, we must have  $c(i\omega) \rightarrow \infty$  as  $\omega \rightarrow 0$ . The integral term ( $1/i\omega$ ) in a controller is used to ensure this criterion.

**2. Limit measurement noise amplification.** Generally, noise in measurement is significant at high frequencies. Noise transmission is determined by  $|CS(s)|$ , and this is ensured by keeping  $|CS(i\omega)| \approx 0$  at higher frequencies. We can limit high-frequency noise amplification by limiting the controller gain at high frequencies, as the process gain usually becomes very small at high frequencies. Typically, we try to keep

$$\lim_{\omega \rightarrow \infty} |c(i\omega)| < 20$$

This explains why ideal derivative action is never used in the real world. Ideal derivative action requires the following term in the controller.

$$c(s) = \tau_D s,$$

$$c(i\omega) = \tau_D i\omega.$$

As  $\omega$  gets large,  $|c(i\omega)|$  will get infinitely large. Typically, derivative action is implemented using a lead-lag instead:

$$c(s) = \frac{\tau_D s}{\alpha \tau_D s + 1},$$

with  $\alpha = 0.05$ . At high frequencies,

$$\lim_{\omega \rightarrow \infty} |c(i\omega)| \approx \frac{|\tau_D i\omega|}{|\alpha \tau_D i\omega|} = \frac{1}{\alpha} = 20.$$

**3. Provide good disturbance rejection at as wide a frequency range as possible.** Recall that disturbance rejection is given by the sensitivity function

$$S = \frac{1}{1 + pc}$$

To keep  $|S|$  small, we must keep  $|1 + pc|$  large. At low frequencies, this is accomplished by the integral action. In the medium range of frequencies, where disturbances usually dominate, we will have to keep  $|c(i\omega)|$  large. The controller gain  $|c(i\omega)|$  in the medium frequency range will be limited by stability considerations, as increasing the controller gain may cause encirclement of the  $(-1,0)$  point in the Nyquist diagram.

**4. Provide a high bandwidth.** The ultimate frequency,  $\omega_u$ , will determine the region around which  $|CS|$  will peak and start to decrease. The inverse of this frequency determines the dominant time constant and hence the speed of response of the control system. To increase bandwidth, we must make the frequency of which the phase lag approaches  $-180^\circ$  as large as possible. One way to do this is to add phase lead to the controller. This derivative term has the property of adding a phase lead near the intermediate frequency range. Adding the phase lead allows us to push the frequency to the right and hence get a higher bandwidth.

**5. Preserve stability in presence of model errors.** The maximum value of  $CS(i\omega)$  carries a special significance in controller design. According to the Nyquist stability theorem, in order for the closed-loop system to be stable, we must examine the encirclements of the  $(-1,0)$  point in the Nyquist plane. Consider a stable closed-loop system whose open-loop Nyquist diagram is shown in Figure B.15. According to this diagram, the function

$$CS = \frac{pc}{1 + pc}$$

is a measure of how close to  $(-1/K, 0)$  (with  $K = 1$ ) the Nyquist curve of  $pc$  lies. In actual applications, there will be a region of uncertainty surrounding  $p$ , as the process transfer function can be different from what was used in the controller design (due to modeling errors and process variations). For an inherently stable process, we can interpret the minimum distance  $|1 + pc|$  as shown in Figure B.15 as maximum change allowed in  $pc$  before an encirclement occurs, resulting in instability of the closed-loop. Hence the quantity

$$l = \left| \frac{1 + pc}{pc} \right| \tag{B.9}$$

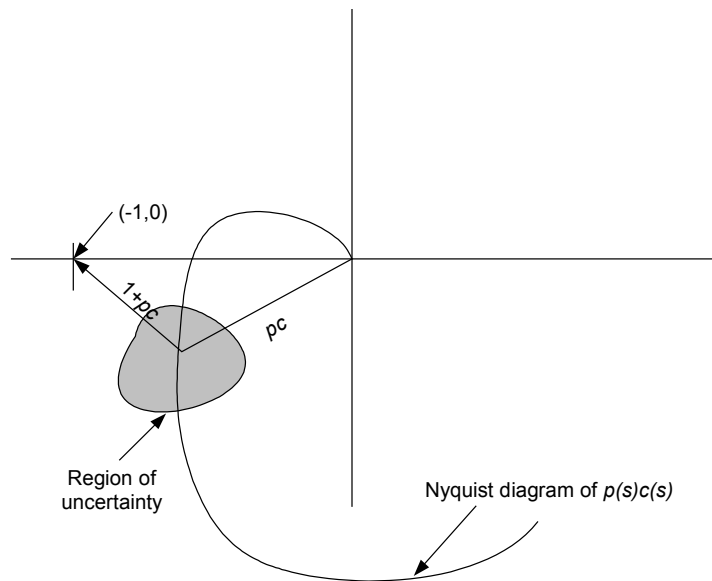
represents the maximum fractional change allowed in  $pc$  before instability occurs. We want to keep this as large as possible to get a large *stability margin*. This is achieved by keeping

$$M_p = \max_{\omega} \left| \frac{pc}{1 + pc} \right| = \max_{\omega} |CS| \tag{B.10}$$

as small as possible.  $M_p$  is called the *maximum closed-loop modulus*. If we let  $M_p < 1$  then this allows the process gain to increase by 100% before the closed-loop system becomes unstable. Since we know that  $|CS| \rightarrow 1$  as  $\omega \rightarrow 0$ , a slightly relaxed criteria,

$$M_p < 1.05, \quad (\text{B.12})$$

allows the  $CS$  to be close to 1 over a wider frequency range. This criterion is used in this text for tuning controllers. A large value of  $M_p$  implies that the closed-loop system is on the verge of instability.



**Figure B.15** Effect of uncertainty in process transfer function of stability.

This criterion conflicts with the desire to respond rapidly to setpoint changes and disturbances, which requires the high-frequency controller gain to be as large as possible. This is referred to as the tradeoff between robustness (insensitivity to model errors and plant variations) and performance (speed of response and disturbance suppression).

### Problems

**B.1** Consider the following FOPDT system

$$G_p(s) = \frac{3e^{-2s}}{5s+10}.$$

- Using frequency response, find the ultimate gain and ultimate period of the process with transfer function (Hint: Assume a proportional controller is used. Obtain the phase angle and AR of  $G_p$ . Determine the value of  $K_c$  that causes the Nyquist plot of  $K_c G_p$  to pass through the  $(-1, 0)$  point which corresponds to a phase angle of  $-\pi$  rad and AR of 1.)
- What are the recommended settings for a PID controller using the Ziegler-Nichols controller tuning method?
- One suggested approximation for the time delay is the (1,1) Pade approximation

$$e^{-Ds} \approx \frac{1 - \frac{D}{2}s}{1 + \frac{D}{2}s}.$$

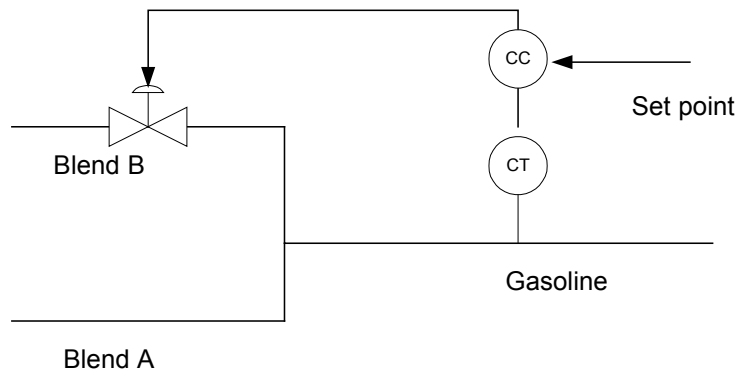
Using this approximation in the transfer function, determine the characteristic equation of the system under proportional feedback control. Use the root locus method to determine the ultimate gain. Compare with Part a.

**B.2** Gasoline is blended in-line by mixing two streams. The octane number of the gasoline is controlled by adjusting the flow rate of stream A. The process transfer function can be represented by

$$y(s) = 5e^{-2s}x(s),$$

where  $y$  is the octane number and  $x$  is the flow of stream B.

- What is the ultimate controller gain if proportional action alone is used?
- If a controller gain equal to one half of the ultimate gain is used, plot the response of  $y$  to a unit step change in setpoint. Show scale of  $x$  and  $y$  axes and the steady-state value reached by  $y$ .



**B.3** Consider the the process with transfer function

$$G(s) = \frac{1}{(s+1)(2s+1)(3s+1)} = \frac{1}{6s^3 + 11s^2 + 6s + 1}.$$

- Sketch the Bode and Nyquist plot of the system.
- Using the frequency domain stability criteria, determine the ultimate gain and ultimate frequency when a proportional feedback controller is used with this process. (Hint: MATLAB has some functions to facilitate the construction of Bode diagrams. Use the help command to obtain additional information about these commands.)

```

» num=[1]; % defines the numerator of the transfer function
» den=[6 11 6 1]; % defines the denominator coefficients
» system=tf(num,den) % defines the system
» bode(system) % creates the bode plot
» nyquist(system) % creates the nyquist plot
» grid on

```

- Suppose a proportional controller with a gain of  $K_c = 5$  is used to control this process. Derive the sensitivity and complementary sensitivity functions for the closed-loop system. Plot these functions use MATLAB. What is  $M_p$  for the closed-loop system?

**B.4** Distillation processes have a large number of time constants. Consider a distillation process response, which can be approximated by 10 first-order transfer functions in series, each with a time constant of 5 seconds. Estimate the ultimate period of this process.

**B.5** The temperature in a reactor is measured using a thermocouple with a time constant of 5 seconds. The gain is 1. The normal reactor temperature is 300°F. The temperature high alarm is set at 325°F in the control computer.

Occasionally, the reactor temperature goes into oscillations with a period of 100 seconds and amplitude of 30°F above and below the normal steady-state. We can model these deviations as a sine wave with amplitude of 30°F. Will these oscillations cause the temperature alarm to sound? Explain how you arrived at your conclusion.

**B.6** The objective of this exercise is to look at the design characteristics of PID controllers from a frequency response point of view. Use MATLAB and/or SIMULINK to do the calculations.

- Consider the process with transfer function

$$p(s) = \frac{1}{(10s+1)^3}.$$

What is the ultimate,  $K_u$ ? Ultimate frequency  $P_u$ ?

- Compute tuning constants for a PID controller using the Ziegler-Nichols tuning method. Use a real PID controller with the equation

$$c(s) = K_c \left( 1 + \frac{1}{\tau_I s} + \frac{\tau_D s}{\alpha \tau_D s + 1} \right).$$

- Graph the frequency response of this controller.
- Plot step response to setpoint changes. Discuss your result. Is the response satisfactory?

- e. Plot the sensitivity and complementary sensitivity function. What is the  $M_p$  for this closed-loop system?

**B.7** Consider a first-order plus delay process with transfer function

$$p(s) = \frac{10e^{-Ts}}{s+1}.$$

- Derive the AR and phase angle as a function of frequency.
- Write an equation to compute the ultimate frequency. Solve for the case  $T = .5$ .
- Show that the ultimate frequency decreases as  $T$  increases.
- Write an equation to compute the ultimate gain. Solve for the case  $T = .5$ . For the case  $T = .5$ , compute the gain of a proportional feedback controller so that the  $M_p$  is 1.05.

- B.8** a. Derive the equations for the sensitivity and complementary sensitivity functions for a process with transfer function  $p(s) = \frac{1}{s+1}$  and controller  $c(s) = 1 + 1/s$ .
- Sketch the amplitude ratio of both functions as a function of frequency.
  - What is the  $M_p$  for this system? At what frequency is the closed-loop system most sensitive to disturbances?
  - Is the system stable? Show using Nyquist plot.

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