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Solving Linear Rational Expectation Models with Lagged Expectations

1 The Model

These notes lay out a version of the solution method of Wang and Wen (2006) to linear rational expectation models in the presence of lagged expectations.

Consider the following linear rational expectations model:

$$\alpha_0 E_t\left(X_{t+1}\right) + \alpha_1 X_t + \alpha_2 X_{t-1} + \beta\varepsilon_t + \sum\nolimits_{i=1}^{N} \gamma_i E_{t-i}\left(X_t\right) = 0 \tag{1}$$

$$X_{-1} \text{ given}$$

where X is an $n \times 1$ vector of variables determined in date t or earlier, these include both endogenous (state and jumping) and exogenous variables, and ε is an $m \times 1$ vector of innovations. The matrices α_0, α_1, α_2, γ_1,..., γ_N are $n \times n$, and β is $n \times m$. Note that under flexible information all γ's are zero matrices.

The solution to the model is given by:

$$X_t = A_{n\times n} X_{t-1} + B_{n\times mN}\epsilon_t \tag{2}$$

where all eigenvalues of A must lie inside the unit circle, and:

$$\begin{aligned}
\epsilon_t &\equiv \left[\begin{array}{cccc} \varepsilon_t' & \varepsilon_{t-1}' & \dots & \varepsilon_{t-(N-1)}' \end{array}\right]_{mN\times 1}' \\
\epsilon_t &= \Theta\epsilon_{t-1} + \left[\begin{array}{c} I_m \\ 0_{m(N-1)\times m} \end{array}\right]\varepsilon_t \\
\Theta &\equiv \left[\begin{array}{cc} 0_{m\times m(N-1)} & 0_{m\times m} \\ I_{m(N-1)} & 0_{m(N-1)\times m} \end{array}\right]
\end{aligned}$$

1.1 Transforming the model into a standard RE model without past expectations

The AR structure of the solution suggests that it also has an $MA(\infty)$ representation:

$$X_t = \sum_{j=0}^{\infty} \Phi_j \varepsilon_{t-j}$$

where the Φ_j's are $n \times m$ undetermined matrices. Notice:

$$E_{t-i}(X_t) = X_t - [X_t - E_{t-i}(X_t)]$$

and:

$$X_t - E_{t-i}(X_t) = \sum_{j=0}^{i-1} \Phi_j \varepsilon_{t-j} \tag{3}$$

Using these results the model can be rewritten as:

$$\alpha_0 E_t(X_{t+1}) + \left(\alpha_1 + \sum_{i=1}^{N} \gamma_i\right) X_t + \alpha_2 X_{t-1} + \beta\varepsilon_t - \sum_{i=1}^{N} \gamma_i \sum_{j=0}^{i-1} \Phi_j \varepsilon_{t-j} = 0$$

Rearrange the last element:

$$\begin{aligned}
\sum_{i=1}^{N} \gamma_i \sum_{j=0}^{i-1} \Phi_j \varepsilon_{t-j} &= \left(\sum_{i=1}^{N} \gamma_i\right) \Phi_0 \varepsilon_t + \left(\sum_{i=2}^{N} \gamma_i\right) \Phi_1 \varepsilon_{t-1} + \dots \\
&\quad + \left(\sum_{i=N-1}^{N} \gamma_i\right) \Phi_{N-2} \varepsilon_{t-(N-2)} + \gamma_N \Phi_{N-1} \varepsilon_{t-(N-1)}
\end{aligned}$$

Define:

$$
\begin{aligned}
\Omega &\equiv \begin{bmatrix} \Gamma_1 & \Gamma_2 & \dots & \Gamma_N \end{bmatrix}_{n\times nN} \qquad \Gamma_j = \sum\nolimits_{i=j}^{N} \gamma_i \qquad j = 1\dots N \\
\Psi &\equiv \begin{bmatrix} \Phi_0 & & 0 \\ & \ddots & \\ 0 & & \Phi_{N-1} \end{bmatrix}_{nN\times mN} \\
\widetilde{\beta} &\equiv \begin{bmatrix} \beta & 0_{n\times m(N-1)} \end{bmatrix}_{n\times mN} \\
\widetilde{\alpha}_1 &\equiv \alpha_1 + \sum\nolimits_{i=1}^{N} \gamma_i
\end{aligned}
$$

Therefore the model becomes:

$$
\alpha_0 E_t\left(X_{t+1}\right) + \widetilde{\alpha}_1 X_t + \alpha_2 X_{t-1} + \left(\widetilde{\beta} - \Omega\Psi\right)\epsilon_t = 0 \tag{4}
$$

This equation has the standard linear rational expectation form except that the matrix Ψ is constructed of undetermined coefficients (under flexible information $\Omega = 0$, $\widetilde{\alpha}_1 = \alpha_1$).

1.2 Solving the model

Substitute (2) into (4):

$$
\begin{aligned}
\alpha_0 E_t\left(AX_t + B\epsilon_{t+1}\right) + \widetilde{\alpha}_1 X_t + \alpha_2 X_{t-1} + \left(\widetilde{\beta} - \Omega\Psi\right)\epsilon_t &= 0 \\
\left(\alpha_0 A + \widetilde{\alpha}_1\right) X_t + \alpha_2 X_{t-1} + \left(\alpha_0 B\Theta + \widetilde{\beta} - \Omega\Psi\right)\epsilon_t &= 0
\end{aligned}
$$

Substitute again:

$$
\left(\alpha_0 A + \widetilde{\alpha}_1\right)\left(AX_{t-1} + B\epsilon_t\right) + \alpha_2 X_{t-1} + \left(\alpha_0 B\Theta + \widetilde{\beta} - \Omega\Psi\right)\epsilon_t = 0
$$

and rearrange:

$$
\left(\alpha_0 A^2 + \widetilde{\alpha}_1 A + \alpha_2\right) X_{t-1} + \left[\left(\alpha_0 A + \widetilde{\alpha}_1\right) B + \alpha_0 B\Theta + \widetilde{\beta} - \Omega\Psi\right]\epsilon_t = 0
$$

This must hold for any realization of ϵ_t, and in particular for $\epsilon_t = 0$; similarly, it also must hold for any state of the economy, X_{t-1}, and in particular for $X_{t-1} = 0$. Therefore the matrices A and B must solve:

$$\begin{aligned} \alpha_0 A^2 + \widetilde{\alpha}_1 A + \alpha_2 &= 0 & (5) \\ \left(\alpha_0 A + \widetilde{\alpha}_1\right) B + \alpha_0 B \Theta + \widetilde{\beta} - \Omega \Psi &= 0 & (6) \end{aligned}$$

Notice that A is independent of undetermined matrices B and Ψ.

1.2.1 Solving for A

Method 1: Solving a quadratic matrix equation This method is straightforward to understand but it does not always work since it depends on a rank condition which may or may not hold even when the model is well specified and has a unique solution.

We need to solve the following quadratic matrix equation:

$$\alpha_0 A^2 + \widetilde{\alpha}_1 A + \alpha_2 = 0 \tag{7}$$

Define:

$$\begin{aligned} \Xi &\equiv \begin{bmatrix} -\widetilde{\alpha}_1 & -\alpha_2 \\ I_n & 0_{n\times n} \end{bmatrix} \\ \Delta &\equiv \begin{bmatrix} \alpha_0 & 0_{n\times n}0 \\ 0_{n\times n} & I_n \end{bmatrix} \end{aligned}$$

Consider the generalized eigenvalue problem:

$$\Xi s = \lambda \Delta s$$

Write s as $\begin{bmatrix} s_1' & s_2' \end{bmatrix}'$ where $s_1,\, s_2 \in \mathbb{R}^N$. If s is a generalized eigenvector that is associated with the eigenvalue λ, then:

$$\begin{bmatrix} -\widetilde{\alpha}_1 & -\alpha_2 \\ I_n & 0_{n\times n} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \lambda \begin{bmatrix} \alpha_0 & 0_{n\times n}0 \\ 0_{n\times n} & I_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{8}$$

which by the lower block suggests:

$$s_1 = \lambda s_2$$

That is, any eigenvector has the following form:

$$s = \begin{bmatrix} \lambda x \\ x \end{bmatrix}$$

Given this structure the upper block gives:

$$-\widetilde{\alpha}_1 \lambda x - \alpha_2 x = \lambda^2 \alpha_0 x$$

If there are n generalized eigenvalues $\lambda_1, \ldots, \lambda_n$ together with n linearly independent eigenvectors $s_1 = \begin{bmatrix} \lambda_1 x_1' & x_1' \end{bmatrix}', \ldots, s_n = \begin{bmatrix} \lambda_n x_n' & x_n' \end{bmatrix}'$, then by stacking the upper block of (8) we get:

$$-\widetilde{\alpha}_1 \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} - \alpha_2 \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} = \alpha_0 \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}$$

Given (7) this suggests:

$$A = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^{-1}$$

Solution is unique if there are exactly n λ_i's inside the unit circle. Notice, however, that this derivation depends on that the associated eigenvectors are linearly independent.

Method 2: Using Sims (2002) for solving linear RE models In absence of exogenous shocks the matrix A characterizes a solution to the model in equation (4). We can therefore use standard methods for solving linear RE models to solve for A. Here the model is transformed in order to fit Sims' method, although other methods can be applied as well.

Generally, Sims requires the model to be written in the following form:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Pi v_t + \Phi u_t$$

where y_t is a vector of endogenous variables that may include expectations of future variables, v_t is a vector of expectational errors, and u_t is a vector of exogenous shocks.

Hence, in the absence of exogenous shocks, (4) can be written as:

$$\begin{bmatrix} \alpha_0 & \widetilde{\alpha}_1 \\ 0_{n\times n} & I_n \end{bmatrix} y_t = \begin{bmatrix} 0_{n\times n} & -\alpha_2 \\ I_n & 0_{n\times n} \end{bmatrix} y_{t-1} + \begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix} v_t + \begin{bmatrix} 0_{n\times Nm} \\ 0_{n\times Nm} \end{bmatrix} u_t$$

where:

$$\begin{aligned} y_t &= \begin{bmatrix} E_t\left(X_{t+1}\right) \\ X_t \end{bmatrix} \\ v_t &= X_t - E_{t-1}\left(X_t\right) \\ u_t &= \epsilon_t \end{aligned}$$

Sims' code provides a solution to the model in the following form:

$$y_t = G_1 y_{t-1} + impact \cdot u_t$$

Since by construction we shut down the effect of the exogenous shocks it follows that $impact = 0$. Furthermore, the matrix G_1 has the following structure:

$$G_1 = \begin{bmatrix} BB & CC \\ 0_{n\times n} & A \end{bmatrix}$$

The matrix A that we look for is located at the bottom right $n \times n$ block of the matrix G_1. One can also verify that $A^2 = BB \cdot A + CC$.

1.2.2 Solving for B

Using (2) iterate backward to get:

$$\begin{aligned} X_t &= AX_{t-1} + B\epsilon_t \\ &= A^2 X_{t-2} + AB\epsilon_{t-1} + B\epsilon_t \\ &= A^3 X_{t-3} + A^2 B\epsilon_{t-2} + AB\epsilon_{t-1} + B\epsilon_t \\ \ldots &= A^i X_{t-i} + \sum\nolimits_{j=0}^{i-1} A^j B\epsilon_{t-j} \end{aligned}$$

Recall that B is $n \times mN$ and $\epsilon_t \equiv \begin{bmatrix} \varepsilon_t' & \varepsilon_{t-1}' & \ldots & \varepsilon_{t-(N-1)}' \end{bmatrix}_{mN\times 1}'$. Now partition B to N matrices, each size $n \times m$:

$$B = \begin{bmatrix} B_0 & B_1 & \ldots & B_{N-1} \end{bmatrix}$$

Therefore:

$$X_t = A^i X_{t-i} + \sum\nolimits_{j=0}^{i-1} A^j \sum\nolimits_{k=0}^{N-1} B_k \varepsilon_{t-j-k}$$

Since A is a stable matrix and X_t is stationary it follows that $\lim\limits_{i\to\infty} A^i X_{t-i} = 0$, therefore:

$$X_t = \sum\nolimits_{j=0}^{\infty} A^j \sum\nolimits_{k=0}^{N-1} B_k \varepsilon_{t-j-k}$$

Rearrange:

$$\begin{aligned}
X_t &= \sum\nolimits_{j=0}^{\infty} A^j \left[B_0 \varepsilon_{t-j} + B_1 \varepsilon_{t-j-1} + \ldots + B_{N-1} \varepsilon_{t-j-(N-1)}\right] \\
&= A^0 B_0 \varepsilon_t + \left[A^0 B_1 + A^1 B_0\right] \varepsilon_{t-1} + \ldots + \left[A^0 B_{N-1} + A^1 B_{N-2} + \ldots + A^{N-1} B_0\right] \varepsilon_{t-(N-1)} \\
&\quad + \sum\nolimits_{j=N}^{\infty} \left(\sum\nolimits_{k=0}^{N-1} A^{j-k} B_k\right) \varepsilon_{t-j}
\end{aligned}$$

Recall that $X_t = \sum\nolimits_{j=0}^{\infty} \Phi_j \varepsilon_{t-j}$, and that for the solution we only need to pin down Φ_j's for $j = 0, 1, \ldots, N-1$. Therefore:

$$\Phi_j = \sum\nolimits_{k=0}^{j} A^{j-k} B_k \qquad j = 0, 1, \ldots, N-1$$

Now recall equation (6):

$$(\alpha_0 A + \widetilde{\alpha}_1) \begin{bmatrix} B_0 & B_1 & \ldots & B_{N-1} \end{bmatrix} + \alpha_0 \begin{bmatrix} B_0 & B_1 & \ldots & B_{N-1} \end{bmatrix} \Theta + \widetilde{\beta} - \Omega \begin{bmatrix} \Phi_0 & & 0 \\ & \ddots & \\ 0 & & \Phi_{N-1} \end{bmatrix} = 0$$

Substituting for the Φ's:

$$\begin{aligned}
& (\alpha_0 A + \widetilde{\alpha}_1) \begin{bmatrix} B_0 & B_1 & \ldots & B_{N-1} \end{bmatrix} + \alpha_0 \begin{bmatrix} B_0 & B_1 & \ldots & B_{N-1} \end{bmatrix} \Theta + \widetilde{\beta} \\
= \;& \Omega \begin{bmatrix} A^0 B_0 & & & & 0 \\ & \ddots & & & \\ & & \sum\nolimits_{k=0}^{j} A^{j-k} B_k & & \\ & & & \ddots & \\ 0 & & & & \sum\nolimits_{k=0}^{N-1} A^{N-1-k} B_k \end{bmatrix}
\end{aligned}$$

Recall that $\Theta \equiv \begin{bmatrix} 0_{m\times m(N-1)} & 0_{m\times m} \\ I_{m(N-1)} & 0_{m(N-1)\times m} \end{bmatrix}$, and $\Omega \equiv \begin{bmatrix} \Gamma_1 & \Gamma_2 & \dots & \Gamma_N \end{bmatrix}_{n\times nN}$, hence the system can be rewritten as:

$$\begin{aligned} & (\alpha_0 A + \widetilde{\alpha}_1) \begin{bmatrix} B_0 & B_1 & \dots & B_{N-1} \end{bmatrix} + \alpha_0 \begin{bmatrix} B_1 & \dots & B_{N-1} & 0_{n\times m} \end{bmatrix} + \widetilde{\beta} \\ = & \begin{bmatrix} \Gamma_1 A^0 B_0 & \Gamma_2 \left[A^1 B_0 + A^0 B_1\right] & \dots & \Gamma_N \sum_{k=0}^{N-1} A^{N-1-k} B_k \end{bmatrix} \end{aligned}$$

This can be written as N linear matrix equations (after using the definition of $\widetilde{\beta}$):

$$\begin{aligned} \left(\alpha_0 A + \widetilde{\alpha}_1 - \Gamma_1 A^0\right) B_0 + \alpha_0 B_1 + \beta &= 0 \\ \left(\alpha_0 A + \widetilde{\alpha}_1 - \Gamma_j A^0\right) B_{j-1} + \alpha_0 B_j &= \Gamma_j \sum_{k=0}^{j-2} A^{j-1-k} B_k \qquad j = 2 \dots N-1 \\ \left(\alpha_0 A + \widetilde{\alpha}_1 - \Gamma_N A^0\right) B_{N-1} &= \Gamma_N \sum_{k=0}^{N-2} A^{N-1-k} B_k \end{aligned}$$

The system has N unknown matrices: B_0, B_1,..., B_{N-1}.

Solution Method Although it is possible to find a closed form solution, it is easier to solve the system numerically. First start with a guess (for $N-2$ matrices):

$$B_j = 0 \qquad j = 1 \dots N-2$$

and then solve recursively:

$$\begin{aligned} B_0 &= (\alpha_0 A + \widetilde{\alpha}_1 - \Gamma_1)^{-1} (-\alpha_0 B_1 - \beta) \\ B_{j-1} &= (\alpha_0 A + \widetilde{\alpha}_1 - \Gamma_j)^{-1} \left(-\alpha_0 B_j + \Gamma_j \sum_{k=0}^{j-2} A^{j-1-k} B_k\right) \qquad j = 2 \dots N-1 \\ B_{N-1} &= (\alpha_0 A + \widetilde{\alpha}_1 - \Gamma_N)^{-1} \Gamma_N \sum_{k=0}^{N-2} A^{N-1-k} B_k \end{aligned}$$

Continue until convergence.

References

[1] Sims, C. A., 2002. Solving linear rational expectations models. Computational Economics, 20(1), pp. 1-20.

[2] Wang, P., Wen, Y., 2006. Solving Linear Difference Systems with Lagged Expectations by a Method of Undetermined Coefficients. Federal Reserve Bank of St. Louis Working Paper No. 2006-003C.

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