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Business Cycle Fluctuations and the Hodrick-Prescott Filter

1 Introduction

Business cycles theory studies the fluctuations of economic aggregates around a trend. Typically, we think of business cycles as fluctuations that occur with a frequency of three to five years. In theoretical models the trend, or "growth path", is often well defined, but unfortunately this is not the case for the data. The data contain shocks in different frequencies, some die out quickly and others may affect the economy for a long period of time. The problem is then how to filter the raw data such that we remove the low frequency variation and get a new series that represents business cycle fluctuations.

These notes start by describing the Hodrick-Prescott (HP) filter. This filter is designed to remove low frequency variations from time series statistics. We will then follow Cooley and Prescott (1995) and apply the filter to post-war US data, this will generate some stylized facts on business cycles fluctuations.

2 The Hodrick-Prescott Filter

The HP filter estimates an unobservable time trend for time series variables. The procedure was first introduced by Hodrick and Prescott in 1980 in the context of estimating business

cycles; notably their paper was published only 17 years later in 1997 after the filter has already been widely used in macroeconomics.

Let y_t denote an observable (macroeconomic) time series. The HP filter decomposes y_t into a nonstationary trend, g_t , and a stationary residual component, c_t , that is:

$$y_t = g_t + c_t \tag{1}$$

Note that both g_t and c_t are unobservables. Since c_t is a stationary process we can think of y_t as a noisy signal for the nonstationary trend g_t . Hence, the problem boils down to how to extract an estimate for g_t from data on y_t .

The HP filter solves this problem by allocating some weight to a linear time trend against the signal y_t . Let λ represent that weight. If there is no noise then the signal is fully informative and therefore it is best to set λ to zero. As λ increases more weight is allocated to the linear trend, and for $\lambda \rightarrow \infty$, g_t approaches the ordinary least squares estimate of y_t against a linear time trend.

Hodrick and Prescott find that if c_t and the second difference of g_t , $\Delta\Delta g_t$, are identically and independently distributed, normal variables with mean zero and variances σ_c^2 and $\sigma_{\Delta\Delta g}^2$, then the best choice (in the MSE sense) of λ is $\sigma_c^2/\sigma_{\Delta\Delta g}^2$.¹ Generally, data in high frequency is noisier relative to low frequency time series, i.e. σ_c^2 is larger; therefore, high frequency data require a higher value of λ . In Eviews, for example, the default values of λ are 100 for annual data, 1600 for quarterly data, and 14400 for monthly data.

2.1 Derivation

The HP filter solves the following problem:

$$\underset{\{g_t\}_{t=1}^T}{Min} \sum_{t=1}^T (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2 \tag{2}$$

¹Typically, this condition does not hold in macroeconomic time series; however, Hodrick and Prescott show that in their sample the implications for the business cycles are insensitive to a wide range of values.

The FOCs of this problem are given by:

$$\begin{aligned}
g_1 & : & c_1 & = \lambda(g_1 - 2g_2 + g_3) \\
g_2 & : & c_2 & = \lambda(-2g_1 + 5g_2 - 4g_3 + g_4) \\
g_t & : & c_t & = \lambda(g_{t-2} - 4g_{t-1} + 6g_t - 4g_{t+1} + g_{t+2}) \quad \text{for } t = 3, 4, \dots, T-2 \\
g_{T-1} & : & c_{T-1} & = \lambda(g_{T-3} - 4g_{T-2} + 5g_{T-1} - 2g_T) \\
g_T & : & c_T & = \lambda(g_{T-2} - 2g_{T-1} + g_T)
\end{aligned}$$

In matrix notation this can be written as:

$$\hat{c} = \lambda F \hat{g} \quad (3)$$

Where the $\hat{\cdot}$ notation denotes estimates of the unobservables, and F is a $T \times T$ coefficients matrix that is given by:

$$F_{T \times T} = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & & & & & 0 \\ -2 & 5 & -4 & 1 & 0 & \dots & & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots & & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \dots & & & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \dots & & & & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \dots & & & & & 0 & 1 & -4 & 5 & -2 \\ 0 & \dots & & & & & & 0 & 1 & -2 & 1 \end{bmatrix} \quad (4)$$

Using (1) and (3) we get $y - \hat{g} = \lambda F \hat{g}$, and by rearranging we get the filtered time trend:

$$\hat{g} = (\lambda F + I)^{-1} y \quad (5)$$

Notice also that by the structure of the matrix F , the sum each column is zero, and therefore by construction the sum of the estimated noises is zero, that is:

$$\sum_{t=1}^T \hat{c}_t = 0 \quad (6)$$

3 Business Cycles: Stylized Facts

We use national accounts data from the Bureau of Economic Analysis and data on hours work and employment from the establishment survey of the Bureau of Labor statistics. All series are in quarterly frequency for the period 1964.1-2007.2. Series were logged before filtering so deviations from trend are expressed in percentage terms. The calculation uses a smoothing parameter of 1600.

**Table 1: Business Cycle Properties of the US Economy
(Quarterly data, 1964.1-2007.2)**

Variable	STD %	Cyclicalities	
		(Corr. w/ GDP)	Auto-Correlation
GDP	1.52	1.00	0.87
Nondurable Consumption	0.84	0.83	0.87
Durable Consumption	4.37	0.79	0.75
Private Investment	6.96	0.90	0.82
Exports	4.21	0.36	0.66
Imports	4.83	0.76	0.76
Aggregate hours	1.85	0.87	0.92
Average weekly hours	0.41	0.73	0.76
Employment	1.26	0.80	0.94

Table 1 presents the main time series properties of the filtered data. Notably the data in the sample display similar business cycle features as in Cooley and Prescott (1995). Here is a brief summary of the findings:

- Consumption of nondurables and services is much smoother than output.
- Investment and consumption of durable goods are much more volatile than output.
- Output and aggregate hours of work have a similar magnitude of fluctuation and have a strong positive correlation, suggesting that the business cycle is clearly manifested in the labor market.

- Employment fluctuates much more than average weekly hours, suggesting that most fluctuations in aggregate hours are due to movements into and out of the labor force, rather than adjustment of the workweek.
- Imports are more procyclical than exports.
- All series are quite persistent.

References

- [1] Cooley, T. F., Prescott, E. C., 1995. "Economic Growth and Business Cycles." in *Frontiers of Business Cycle Research*, Cooley, T. F. Editor, Princeton, New Jersey: Princeton University Press.
- [2] Hodrick, R. J., Prescott, E. C., 1980. "Post-war U.S. business cycles: An empirical investigation." Discussion paper 451, Carnegie-Mellon University.
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- [4] Reeves, J. J., Blyth, C. A., Triggs, C. M., Small, J. P., 2000. "The Hodrick-Prescott Filter, a Generalization, and a New Procedure for Extracting an Empirical Cycle from a Series." *Studies in Nonlinear Dynamics and Econometrics* 4 (1), pp. 1-16.