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Pattern Formation in Periodicaly Forced Oscillatory Systems

Abstract

Oscillatory systems are common features of our lives, one may find their realizations in simple clocks or in more complicated technological equipments like satellite-based navigation systems and pacemakers implanted to prevent cardiac arrhythmia. The ecological and biological worlds also reveal diverse oscillatory dynamical behaviors like seasonal flower blooms, predator-prey relations, or cell divisions. In the context of *large-scale* objects, it is easy to notice that the earth (like many other planets) is making oscillations around a sun with an approximate period of one year (365 days). It was also indicated that the galaxies perform rotational periodic motion around black holes, assumed to exist in their center. In *small-scale* systems, the Quantum mechanical descriptions of the elementary particles, Electromagnetism, crystals etc. involve oscillatory behaviors. The scientific significance of oscillatory dynamics have attracted both theoretical studies and studies of specific physical, chemical, biological, ecological and even economical systems.

When oscillatory systems are subjected to temporal forcing they may exhibit *entrainment* or *frequency locking* phenomena. A system is frequency locked when its oscillation frequency is adjusted to an irreducible fraction of the forcing frequency. This resonance condition admits a tongue-like domain in the plane spanned by the forcing amplitude and frequency. Outside the resonance tongue the system exhibits quasiperiodic oscillations. Frequency locking phenomena have been extensively studied for single oscillator type systems, however, the fundamental description of resonance phenomena for spatially extended systems is missing.

This thesis is concerned with frequency locking phenomena in spatially extended media and addresses the effects of pattern formation on resonance behavior. We study pattern formation mechanisms and parameters ranges where resonant and non-resonant patterns are developed. Among our results we show that in extended systems:

(a) Standing waves are the only patterns (besides uniform oscillations) that satisfy the frequency locking condition: each point in space can oscillate either in or out of resonance.

(b) Spatial structures and instabilities may reduce or extend the boundaries of frequency locking so that the resonance ranges for a single oscillator do not always coincide with resonance ranges in extended systems.

This research has been motivated by recent experiments on temporally driven Belousov-Zhabotinsky (BZ) reaction-diffusion systems. It was observed that standing–wave patterns may develop from spiral waves when the system is periodically forced at approximately twice its natural frequency (hereafter 2:1 resonance), and that the standing waves occupy only a part of the 2:1 resonance tongue. The experiments also indicate that labyrinthine standing–wave patterns may develop in two distinct ways. Labyrinths may develop by a fingering instability, and by nucleation of stripes from unlocked oscillations.

In both cases, large amplitude labyrinthine patterns form, involving adjacent domains of frequency locked oscillations differing in phase by π.

The study is based on the forced complex Ginzburg-Landau (FCGL) equation. This is the normal form equation for an extended system that goes through a Hopf bifurcation to uniform oscillations, and is subjected to uniform, time-periodic forcing at a frequency about twice as large as the frequency of unforced oscillations. The theoretical analyzes consist of linear stability studies of nonuniform solutions, weakly nonlinear multiple time-scale analyzes, and numerical calculations. The results are tested and supported by direct numerical solutions of the FCGL equation.

The FCGL equation has both resonant uniform solutions and resonant standing-wave solutions such as stripes or labyrinths. The former form a tongue-shaped region in the parameter plane of the forcing amplitude and frequency. On one side of the tongue, the boundary of resonant patterns is inside the tongue and is given by the Nonequilibrium Ising Bloch bifurcation to traveling waves. A traveling wave within the 2:1 resonance of the uniform system destroys the resonance as any individual point in the vicinity of the traveling front experiences a temporal phase change. On the other side of the tongue the appearance of a stationary Turing mode also changes, extends or reduces the resonance boundaries. Resonant standing waves may appear outside the 2:1 resonance tongue of uniform oscillations, whereas non-resonant standing waves may prevail inside the resonance tongue. A weakly nonlinear analysis of FCGL equation near a Hopf-Turing bifurcation gives the existence and stability regions of the standing–wave patterns.

Moreover, analysis of the FCGL equation captures both mechanisms of labyrinth formation observed the BZ experiments. The fingering instability is found to correspond to a transverse instability of front solutions inside the resonance tongue whereas the stripe nucleation is attributed to invasion of resonant Turing solutions to unlocked Hopf oscillations. These labyrinthine patterns are found in a similar location in the forcing amplitude and frequency parameter plane as the experimentally observed labyrinths - the high right edge of the resonance boundary. Additionally, we find a third mechanism of labyrinths formation: standing–wave labyrinths may also develop from a Turing instability of the uniform phase-locked states in a narrow range near one of the resonance tongue boundaries.

key-words:
oscillatory systems, nonlinear dynamics, pattern formation, entrainment, frequency locking, resonance, reaction–diffusion equations, partial differential equations, center manifold, normal forms, Amplitude equations, stability analysis, eigenvalues, bifurcation, secondary instability, complex Ginzburg-Landau equation, interfaces and fronts, standing waves, spiral waves, numerical analysis.