

Phase shift ellipses for pulsating flows

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A pulsating laminar flow of a viscous, incompressible fluid through a pipe with an orifice has been studied at relatively low Reynolds numbers. The motion is caused by an imposed sinusoidally varying pressure difference, $\Delta p(t)$. The induced flow rate, $Q(t)$, has a phase shift, ϕ_Q , with respect to the imposed pressure oscillations. We have used that a phase (x,y) -plane instantaneous state plot of two phase-shifted trajectories: $x(t) = \sin(\omega t + \phi_x)$ and $y(t) = \sin(\omega t + \phi_y)$, is an ellipse. The ellipses of the instantaneous states $Q(t)$ vs $\Delta p(t)$ during a cycle allow readily computing the phase shift, ϕ_Q . © 2003 American Institute of Physics. [DOI: 10.1063/1.1580123]

An incompressible viscous fluid, which is forced to move under a pulsating pressure difference, has a number of characteristic properties. One of the features of such a flow that an oscillating fluid has a phase shift, ϕ , with respect to the imposed pressure. In the present study we consider a pulsating laminar flow of a viscous, incompressible fluid in a pipe with an orifice. The Wormersley number, $Ws = r_0(\omega/\nu)^{1/2}$, is a measure of oscillating effects in a flow. There are fundamental differences between a pulsating flow induced by low- or high-frequency pressure gradient oscillations. It follows for slow oscillations that there is no phase shift between the induced fluid motion and the imposed very slow pressure oscillations. In the fast oscillations case, the mean velocity oscillates with a phase shift of 90° with respect to the imposed pressure oscillations.

The first limiting case of low-frequency oscillations ($Ws \ll 1$) means that the velocity varies very slowly with time, and the acceleration term $\partial u/\partial t$ in the governing equation of motion can, therefore, be neglected. Thus, the viscous term in the governing equation is balanced by the imposed pressure gradient term. Consequently, in this limiting case, the velocity varies periodically in the same phase as the pressure gradient. In the opposite case of high-frequency oscillations, where the Wormersley number, Ws , is large, the viscous term can be neglected everywhere except in the very narrow layers near the walls. The width of these layers is of the order of magnitude of the depth of penetration of the viscous wave, $\delta \propto (\nu/\omega)^{1/2}$. This case is typical for boundary layers when at a certain distance from the wall the fluid moves as if it was frictionless. This implies that in this case ($Ws \gg 1$) the unsteady term $\partial u/\partial t$ in the governing equation is balanced (except in the narrow δ -layer) by the imposed oscillating pressure gradient term: i.e., the terms $\rho \partial u/\partial t$ and $-\partial p/\partial z$ are of the same order of magnitude. Therefore, at a large distance from the wall the fluid is forced to move with a phase shift of 90° with respect to the exciting pressure gradient. We consider an incompressible fluid forced by a

pulsating pressure difference to move through a pipe with periodically distributed orifices. A schematic drawing of the pipe is presented in Fig. 1, where the computational domain is marked by dashed lines. The governing Navier–Stokes equations are

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\Delta p}{\rho L} \mathbf{e}_z - \nabla \Pi - (\mathbf{V} \cdot \nabla) \mathbf{V} + \nu \nabla^2 \mathbf{V}, \quad (1)$$

subjected to the incompressibility constrain

$$\nabla \cdot \mathbf{V} = 0. \quad (2)$$

(We consider an axisymmetric two-dimensional flow, namely: The velocity field $\mathbf{V} = [u(r,z,t), 0, w(r,z,t)]$, $\nabla = (\partial/\partial r, 0, \partial/\partial z)$, $\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + \partial^2/\partial z^2$, $\mathbf{e}_z = (0,0,1)$. For the r -component of Eq. (1), the Laplacian operator ∇^2 should be replaced by $\nabla^2 - 1/r^2$.) In (1), we split the pressure gradient into two terms, when the pressure difference, $\Delta p(t)$, is prescribed by

$$\Delta p(t) = \Delta p_0 [1 + \gamma_p \sin(\omega t)], \quad (3)$$

where $p_0(z) = \Delta p_0 z/L$ is a periodic in z -direction function; Δp_0 is a prescribed pressure difference across a computational domain. The computational domain length, L , has been set large enough to exclude the influence of the periodicity conditions. In all computations, the fully developed Poiseuille velocity profile has been established at the outlet (details of the computations will be published elsewhere). Taking divergence of (1), Π is computed to impose an incompressibility: $\nabla^2 \Pi = -\text{div}[(\mathbf{V} \cdot \nabla) \mathbf{V}]$. Π is assumed to be periodic in z -direction, which means that it does not contribute to the pressure difference, $\Delta p(t)$. In Eq. (3), γ_p and ω are the amplitude and the frequency of the pressure difference oscillations, respectively.

The pulsating flow in a pipe without an orifice is well-known since Sexl's paper¹ published in 1930. In this case, a flow is fully developed: $\mathbf{V} = (0,0,w)$; nonlinear term in (1) vanishes; $\Pi = 0$, and the solution for $w(r,t)$ reads²

$$w(r,t) = \frac{\Delta p_0}{4\mu L} (r_0^2 - r^2) + \text{Real}[\hat{w}(r)e^{-i\omega t}], \quad (4)$$

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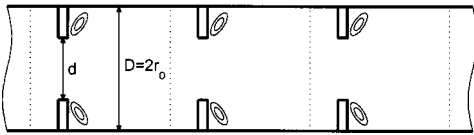


FIG. 1. Schematic of a pipe with an orifice.

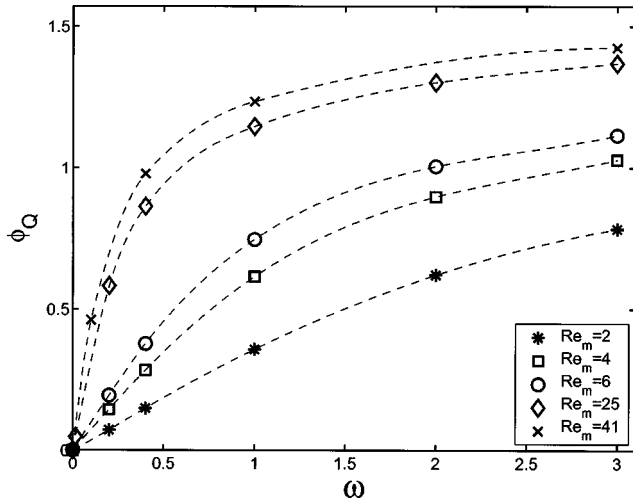


FIG. 2. ϕ_Q vs ω .

TABLE I. Phase shift, ϕ_Q , for different constriction ratios, d/D .

Ws	$d/D=0.5$	$d/D=0.75$	$d/D=1$
1	0.07	0.14	0.17
2.5	0.42	0.72	0.79
3.5	0.70	1.00	1.06
4.5	0.86	1.12	1.21
5.5	0.92	1.17	1.28

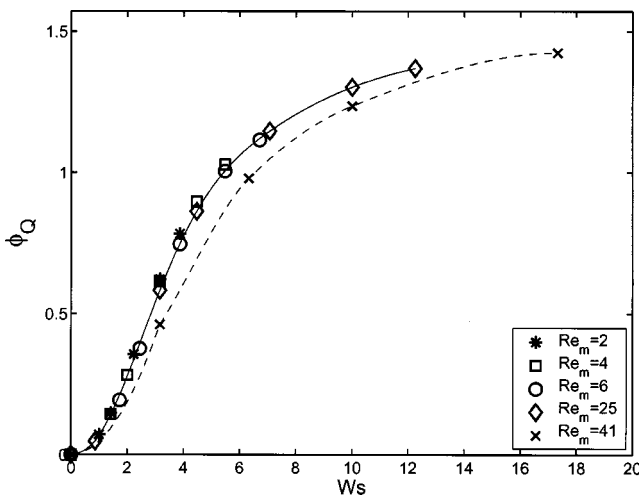


FIG. 3. ϕ_Q vs Ws .

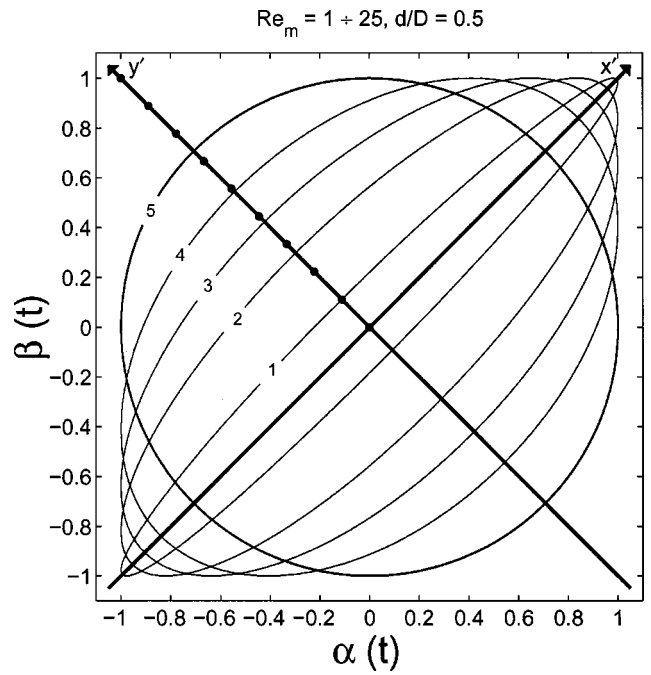


FIG. 4. $\alpha(t) = (\Delta p(t) - \Delta p_0) / \gamma_p \Delta p_0$ vs $\beta(t) = (Q(t) - Q_0) / \gamma_Q Q_0$ instantaneous states.

where

$$\hat{w}(r) = \frac{\Delta p_0 \gamma_p}{\rho L \omega} \left[\frac{J_0(r \sqrt{i \omega / \nu})}{J_0(r_0 \sqrt{i \omega / \nu})} - 1 \right]. \quad (5)$$

Numerical implementation of the expression (4), which includes a Bessel function with imaginary argument, is quite cumbersome. However, it obviously shows that the flow rate $Q(t) = 2 \pi \int_0^{r_0} w(r, t) r dr$ has a phase shift with respect to the imposed oscillating pressure and, therefore, can be written as

$$Q(t) = Q_0 [1 + \gamma_Q \sin(\omega t + \phi_Q)]. \quad (6)$$

A phase-shifting effect yields the following corollary:

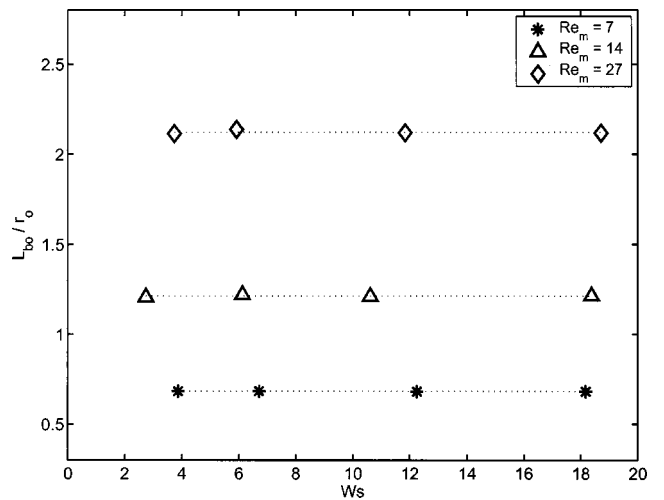


FIG. 5. L_{∞} vs Ws .

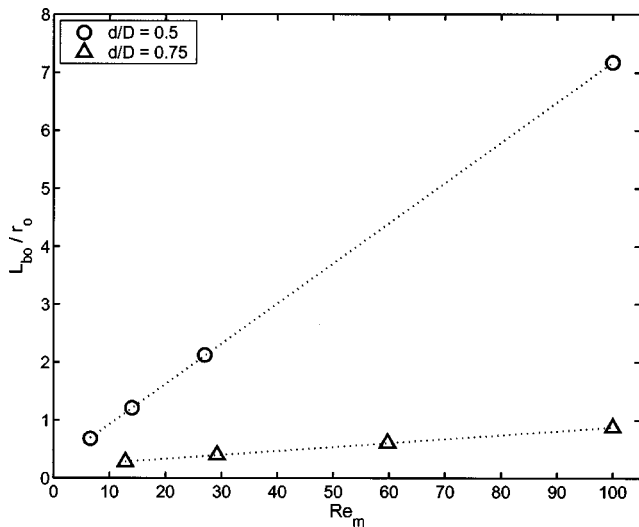


FIG. 6. L_{b0} vs Re_m .

Phase Shift Ellipse. A phase (x,y) -plane plot of instantaneous states of two phase-shifted trajectories: $x(t) = \sin(\omega t + \phi_x)$, $y(t) = \sin(\omega t + \phi_y)$, is an ellipse:

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \tag{7}$$

where

$$a = \sqrt{2} \cos\left(\frac{\phi_y - \phi_x}{2}\right), \quad b = \sqrt{2} \sin\left(\frac{\phi_y - \phi_x}{2}\right), \tag{7a}$$

and

$$\begin{aligned} x' &= a \sin\left(\omega t + \frac{\phi_y - \phi_x}{2}\right) = \frac{\sqrt{2}}{2}(x+y), \\ y' &= b \cos\left(\omega t + \frac{\phi_y - \phi_x}{2}\right) = \frac{\sqrt{2}}{2}(-x+y). \end{aligned} \tag{7b}$$

For a flow through a pipe orifice, the velocity field is not fully developed, but $\mathbf{V} = [u(r,z,t), 0, w(r,z,t)]$, and the convective (nonlinear) term in Eq. (1) is not equal to zero. The latter means that a solution to (1) can not be written as $\mathbf{V} = \mathbf{V}_0(r,z) + \hat{\mathbf{V}}(r,z)e^{i\omega t}$, and, speaking generally, is not phase-shifted with respect to the imposed pressure. However, for flows with relatively low Reynolds numbers, the nonlinear term weakly contributes to the integral characteristics, such as a flow rate. It means that one can expect that a flow rate, $Q(t)$, behaves likewise Eq. (6). We have performed simulations of flows through a pipe orifice at the Reynolds number, $Re_m = U_m D / \nu$, based on the mean velocity and the pipe diameter, over the range of 1–40, and at the Womersley numbers, $Ws = r_0(\omega/\nu)^{1/2}$, of 0.5–25. In Fig. 2 we

present the phase shift ϕ_Q as a function of ω for different Reynolds numbers at the constriction ratio $d/D = 0.5$. In Table I we summarized the phase shifts for different orifice sizes. One can see that for the constriction ratio d/D approaching to one the phase shifts reduce to those obtained by Sexl¹ for a smooth pipe ($d/D = 1$). In Fig. 3 the phase shift is shown as the function of the Womersley number, Ws . From Fig. 3, one can see that for $Re_m = 1-25$ the data scattering is insufficient, which means that the phase shift is practically independent of the Reynolds number. In other words, for the considered range of the Reynolds numbers the nonlinear effects are negligible and in accordance to the corollary, $(\Delta p - Q)$ phase plane curve should be close to an ellipse. Using a proper nondimensionalizing,

$$\alpha(t) = \frac{\Delta p(t) - \Delta p_0}{\gamma_p \Delta p_0}, \quad \beta(t) = \frac{Q(t) - Q_0}{\gamma_Q Q_0}, \tag{8}$$

the $\alpha(t) - \beta(t)$ instantaneous states for $Re_m = 1-25$ could be described by a single phase shift ellipse

$$\alpha(t) = \sin(\omega t), \quad \beta(t) = \sin(\omega t + \phi_Q), \tag{9}$$

with the principal axis defined by Eq. (7a) (at $\phi_x = 0$, $\phi_y = \phi_Q$). It could be shown from Eqs. (9) that a phase shift, ϕ_Q , can be computed from

$$\cos(\phi_Q) = \alpha\beta + \sqrt{(1 - \alpha^2)(1 - \beta^2)}. \tag{10}$$

In Fig. 4, in $Ox'y'$ axis, we present the phase shift ellipses for different Ws : $Ws = 1.2$ (curve 1), 2.3(2), 3.2(3), 5.4(4). The high-frequency limiting case ($\phi_Q \rightarrow \pi/2$) corresponds to a circle $a = b = 1$ (curve 5).

For a flow through a pipe with a ring-type constriction (an orifice), a recirculating flow (a bubble) is developed behind an orifice. The recirculation length (L_b) oscillates in time, however, due to nonlinear effects, one cannot expect that $L_b(t)$ behaves likewise Eq. (6). Our calculations show that, for the range of the parameters considered in this study, L_b is of the form: $L_b(t) = L_{b0}[1 + \gamma_L \delta(t)]$, $\delta(t)$ is a periodic function. The averaged in time recirculating bubble length is independent of the Womersley number, Ws : $L_{b0} = L_{b0}(Re_m; d/D)$, as it is seen from Fig. 5. Our computations show (Fig. 6) that the length of this bubble, L_{b0} , grows linearly with the Reynolds number. At that, a small recirculating bubble exists for creeping flow ($Re \approx 0$) also, which agrees with previous calculations cited by White.²

ACKNOWLEDGMENT

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¹T. Sexl, "Über den von E. G. Richardson entdeckten annuläreffekt," Z. Phys. **61**, 349 (1930).
²F. M. White, *Viscous Fluid Flow* (McGraw-Hill, New York, 1991).