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Innovative activity and sunk cost

Todd R. Kaplan[a], Israel Luski[b], David Wettstein[b,*]

[a]*Department of Economics, University of Exeter, Exeter, UK*
[b]*Department of Economics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel*

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Abstract

We analyze innovative activity in a general framework with time-dependent rewards and sunk costs. When firms are identical, innovation is delayed by an increase in the number of firms or a decrease in the size of the reward. When one firm has higher profit potential, it is more likely to innovate first. Our framework generalizes an all-pay auction; however, we show that under certain conditions there is qualitatively different equilibrium behavior.

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1. Introduction

Innovations are an important factor in the ongoing process of technological change. Innovations often emerge from R&D carried out by firms and agencies. By treating the firms as players in a game with payoffs given by the future profits, we can use game theoretic analysis to better understand the firms' decisions on R&D expenditures. Such understanding is a first step towards designing R&D policies that encourage and foster innovative activity.

Models of the R&D game can take a myriad of forms. The main paradigm

*Corresponding author.
E-mail address: wettstn@bgumail.bgu.ac.il (D. Wettstein).

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analyzed in the literature entails a stochastic setting. Loury (1979) and Lee and Wilde (1980) model the uncertain innovations dates as exponential processes. Firms compete by choosing an amount of R&D investment that increases their instantaneous chance of innovation. Harris and Vickers (1987) analyze a multi-stage environment using the Lee–Wilde framework. The less common, deterministic setting has the innovation time as a deterministic function of R&D expenditure. In Dasgupta and Stiglitz (1980), the strategic interaction is modelled as a first-price auction where only the winner invests and losing firms incur no costs. Likewise, in Katz and Shapiro (1987) only the winner incurs costs; however, the losing firm gets a payment contingent on the winner's decision.[1] In Dasgupta (1986), the strategic interaction assumes the form of a first-price all-pay auction where all firms incur costs. This sunk cost assumption is important since a significant amount of R&D expenditure of the losing firm occurs prior to the determination of the winning firm.

Due to Dasgupta's work, the literature on all-pay auctions now cites patent races as a primary economic application (Amann and Leininger, 1996; Baye et al., 1996). However, studying patent races as such ignores any effect that the cost incurred may have on the reward gained. In particular, while higher cost insures earlier innovation, earlier innovation does not increase revenue. The advantage of innovating early should not only be the gain from beating the competition, but also the gain from introducing the product earlier into a market. Although the aforementioned stochastic models consider such time-dependent rewards, they fail to shed light on environments where the relationship between expenditure and innovation time differs from that implied by an exponential process.[2] The deterministic Katz–Shapiro model also has time-dependent rewards, but lacks sunk costs. Both styles of models have the same stringent assumption on the time dependence—modelling it as a discounted constant reward. This excludes common situations where the size of the market may vary over time. Scherer (1967) and Reinganum (1981) employ both sunk costs and deterministic, time-dependent (albeit exponentially discounted) payoffs to innovation competition. However, the environments that they study are not a winner-take-all environment. In addition, the winner's payoff also depends upon the time chosen by the loser (rather than just the fact that he is the winner).

We analyze a deterministic framework where, as with the all-pay auction approach, all firms incur costs and the winner takes the whole reward. The novelty of our approach (in this environment) is to explicitly introduce time into the reward structure—the size of the reward is associated with the innovation times. Our model offers a better understanding of competitive environments with run-of-

[1] Katz and Shapiro (1987) have time pass and at each time period, if an innovation has not yet occurred, then each firm has the option of innovating. This is analogous to a Dutch auction.

[2] In an exponential process with parameter λ the variance is $1/\lambda^2$ while the expected time till innovation occurs is $1/\lambda$. Thus, when uncertainty goes to zero, the innovation must also occur instantly.

the-mill innovations such as software development or automobile design where the costs of development can be well approximated. In addition, we can transform our model to study a rich set of uncertain environments, due to the general nature of the time dependence.

We begin our analysis of such environments in Section 2 by studying a general class of environments with identical firms. We characterize the unique, symmetric, mixed-strategy equilibrium and find that all firms have zero expected profits and that the expected innovation date increases with the number of firms. In Section 3 we analyze standard economic environments by imposing several restrictive assumptions on the environment and having rewards equal the present value of a constant nominal reward. We find that expected innovation date is decreasing in the discount rate. We also show that both per-firm and aggregate R&D expenditures decline in the number of firms. In Section 4 we augment our model by considering firm-specific rewards and costs where one firm (the strong firm) always has higher net profits when winning than the other firm (the weak firm) has. We show there exists an equilibrium where the strong firm always enters the race while the weak firm sometimes stays out. With constant nominal rewards and equal costs, the strong firm is more likely to win but there is still a positive probability that the weak firm would be the winner. In Section 5 we compare our findings to the all-pay-auction literature. The equilibrium behavior in our environment is qualitatively different from that in the standard all-pay auction: there may be a strictly positive probability of a firm not participating, and equilibrium strategies may also involve disconnected supports. In Section 6 we report several empirical findings that agree with the theoretical predictions offered by our models. Finally, in Section 7, we conclude and discuss directions of future research.

2. A general symmetric model

The symmetric environment consists of n identical, risk-neutral firms competing to be the first to innovate and earn a reward (patent). The firms compete by simultaneously choosing a time $t \geqslant 0$ of innovation at a cost of $c(t)$. This cost is sunk and thus is expended regardless of which firm wins the race.

The firm that chooses the lowest time wins the patent race (ties are broken randomly) and receives the present-value-discounted reward of $R(t)$, while the losers receive nothing. Thus, the winning firm choosing t has a payoff of $R(t) - c(t)$. The cost and reward functions are assumed to be continuous and strictly positive. We also assume each firm has the option of staying out of the race altogether and making zero profits. We denote this option by $t = \infty$. We further assume that $c(\infty) = R(\infty) = 0$. In order for a non-trivial equilibrium to exist, we must have $R(0) < c(0)$ (so everyone would not choose $t = 0$) and there must be a finite time denoted by $t_{\min}$ where $t_{\min} = \mathrm{Inf}_{t \geq 0}\{t | R(t) - c(t) > 0\}$ (so everyone would not choose $t = \infty$).

As shown in Appendix A, there does not exist a pure-strategy Nash equilibrium. However, there do exist mixed-strategy equilibria and we now proceed to characterize them. A mixed strategy is given by a cumulative distribution function $F(t)$ yielding the probability that the actual innovation time chosen by the firm will be less than or equal to t. The intuition for why equilibria must entail mixed strategies is seen as follows. Take the case where there are two firms. If one firm chooses a time and wins with strictly positive profits, the other firm would like to choose an earlier time and win instead. Thus, the firms will have an incentive to undercut one another until a firm chooses a time where winning yields zero profits. At this point the other firm would do better to simply stay out of the race; however, if this happens, the winning firm would have an incentive to choose a later time. Hence, a pure strategy equilibrium cannot exist.

We now show (Proposition 1) that there exists a unique symmetric equilibrium for our model. Although, we restrict our analysis to symmetric equilibria, we will discuss the properties of asymmetric equilibria after the Proposition.

Proposition 1. *A unique, symmetric, mixed-strategy equilibrium for the innovation race is given by an n-tuple of identical cumulative distribution functions* $F(t) = 1 - (Min_{t' \leq t}(c(t')/R(t')))^{1/n-1}$ *(for* $t \geq t_{\min}$*) with a probability* $p = Inf_{s \geq 0}(c(s)/R(s))^{1/n-1}$ *of not taking part in the race.*[3,4]

Proof. We prove the theorem by first assuming a symmetric equilibrium exists and proving the following properties (shown in Appendix A).

1. The support has no atoms.
2. All points in the support yield zero expected payoff.
3. The bounds of the support as well as the equilibrium cumulative distribution function are uniquely determined.

Hence, if a symmetric equilibrium exists it must be unique. We now complete the proof by showing that the above construction is indeed an equilibrium. A mixed-strategy equilibrium must satisfy the following properties: (i) a firm using a mixed-strategy must be indifferent among the points in the support of the mixed strategy (given the strategies of the other firms), (ii) a firm weakly prefers a

[3] This style of construction called increasing covers has been used to look at Bertrand–Edgeworth price competition by both Osborne and Pitchik (1986) and Kaplan and Wettstein (2000).

[4] In Appendix A, we provide numerical examples of symmetric equilibria with both connected and unconnected supports.

strategy in its support to any strategy outside its support, and (iii) the cumulative distribution functions must be legitimate (non-negative, increasing in t, and vary from zero to one minus the probability of staying out of the race on their supports).

First notice that the cumulative distribution function $F(t)$ on the support equals $F(t) = 1 - (c(t)/R(t))^{1/n-1}$. Therefore, the expected profit on the support given by $(1 - F(t))^{n-1}R(t) - c(t)$ equals zero. Second, we can see that choosing a t outside of the support does weakly worse by noticing that for this t there must exist a $t' < t$ in the support, where the probability of winning for choosing t must equal $(1 - F(t'))^{n-1}$. However, since t is outside of the support, $c(t)/R(t) \geqslant c(t')/R(t')$. Thus, expected profits $(c(t')/R(t'))R(t) - c(t)$ are less than zero. Finally, by construction, F satisfies the properties in (iii). □

It should also be mentioned that there exist asymmetric equilibria as well. For instance, there can be asymmetric equilibria with n firms where any m firms $(n > m \geqslant 2)$ replicate the m-firm symmetric equilibrium and the remaining firms stay out. The characterization of the symmetric equilibrium allows us to perform comparative statics to analyze effects on the expected innovation time. Following in Corollary 1, we analyze the comparative statics with respect to the number of firms.

Corollary 1. *For $n > 1$, as the number of firms increases, the probability of innovating by time t weakly decreases, the probability of no innovation weakly increases, the expected innovation time is delayed for each firm, and the overall expected time of innovation is delayed (weakly higher expected t).*

Proof. The probability of innovating by time t is given by $1-(c(t)/R(t))^{n/(n-1)}$, since $n/(n-1)$ decreases in n and $c(t)/R(t)$ is less than or equal to 1, this probability does not increase in n. Hence the overall expected time of innovation does not decrease in n. By symmetry, an individual firm's expected innovation time does not decrease with n as well. Furthermore, since $1/(n-1)$ decreases in n, the probability of no innovation, given by $\text{Inf}_{s\geqslant 0}(c(s)/R(s))^{1/(n-1)}$, does not decrease in n. □

The intuition in our model as to why an increase in the number of firms leads to a decrease in per firm expense and an overall delay in innovation can most easily be explained by examining the case when the reward is a constant R (as in the case of all-pay auctions). First we examine the relationship between the number of firms and the firms' expenses. Consider the symmetric equilibrium with n firms, for which we have shown there are zero expected profits. If there are $n+1$ firms, the probability of winning decreases from $1/n$ to $1/(n+1)$. This lowers the expected reward from R/n to $R/(n+1)$. If firms kept the same strategy, they would have negative profits. The only way expected profits could return to zero is if firms decreased their expected expenses to equal their expected rewards.

In order to explain the intuition underlying the overall delay in innovation caused by increasing the number of firms, we take the simple case of moving from two to three firms. Let us start with two firms. Assume that if a firm chooses a particular time t, it has a $1/3$ chance of winning. This means that the other firm chooses a time after t with probability $1/3$. Therefore (by symmetry), both firms choose a time later than t with probability $(1/3)^2$. Now if a firm also chooses this time t when there are three firms, its probability of winning must still be $1/3$ since it must still have zero profits. This means that each of the other firms must be choosing a time after t with probability $(1/3)^{1/2}$. Again by symmetry, the chance of all three firms choosing a time after t is $(1/3)^{3/2}$. This value is greater than $(1/3)^2$ implying innovation is delayed.

These findings while in agreement with empirical regularities (see Section 6) stand in contrast to those of Loury (1979) and Lee and Wilde (1980) who both find that as the number of firms increases innovations occur earlier rather than later and to Lee and Wilde (1980) who find that per firm expenditure increases as well.

We now provide some intuition for the results obtained by Lee and Wilde (1980) using the Poisson model of innovation. In this set-up a firm has to choose how much to invest today as well as tomorrow if necessary. If the firm does not innovate today, it could still realize some potential profit by continuing tomorrow. Let us call this potential profit that a firm gets if no one innovates today the 'option value'. Thus, spending more today on the one hand increases your profits since your chance of innovating today goes up, but on the other hand it reduces your profits since the chance of getting this option value goes down (since you might innovate today), the firm's decision takes into account this trade-off. Due to the Poisson process, the chance of innovating today and receiving the reward is not hurt by the investment of the other firms. However, the chance of receiving this option value is hurt since the other firms may innovate today. Furthermore, if the other firms invest more tomorrow the option value itself goes down. If the number of firms increased and investment per firm did not change, the option value would go down. This makes the cost of investing today go down. Thus, a firm would invest more today.

To shed further light on the conflicting findings, note that our environment could also be viewed as an environment where each firm chooses a distribution F (possibly from a class of distributions) where the cost of choosing this F is $\int c(t)\,\mathrm{d}F(t)$. This is then similar to the Loury (1979) and Lee and Wilde (1980) environment where the distributions chosen are of the Poisson type. These distributions could be taken as the object of choice in our setup as well (see the specific numerical example in Appendix A). In this case, each member of the class would be characterized by a λ where $F(t) = 1 - \mathrm{e}^{-\lambda t}$. We denote the cost of choosing a distribution characterized by λ as $h(\lambda) = \int c(t)\ \mathrm{d}F(t) = \int_0^\infty c(t)\lambda\ \mathrm{e}^{-\lambda t}$

dt. One can show that if (as is usually assumed) $c(t)$ is a decreasing function, then $h''(\lambda) < 0$.[5] This implies that there are increasing-returns-to-scale with respect to λ. This violates Loury's (1979) assumption that at some point onwards there is decreasing-returns-to-scale. This shows the our model differs from Loury's model: the mapping from Poisson distributions to the costs of generating them does not satisfy Loury's assumption.

One may now ask why the intuition presented for explaining the Loury model with the Poisson distribution does not work in our model for at least in the case when the innovation process is also Poisson. Should not an increase in the number of firms lower the incentive to waiting and thus also spur innovation? The short answer is no since the equilibrium in our model is a mixed-strategy equilibrium. Let us demonstrate our model by looking at baseball. The recent surge of power hitters in the game has sparked interest with the likes of Mark McGuire, Sammy Sosa, and Barry Bonds. While this is good for the game, some feel that the need to ban steroids is of greater importance.[6] It is widely predicted that this ban would decrease the number of power hitters in the game. To see whether this is indeed the case we model the game between pitchers and hitters as a zero-sum two-by-two normal-form game. Let us say there are two types of hitters (power hitters and single hitters), and two types of pitchers (fastball pitchers and location pitchers). Let us say that the power hitters do well against the fastball pitchers and the single hitters do well against location pitchers. In a game that mirrors matching pennies, the hitter wants a hitter-favorable match while the pitcher does not want a hitter-favorable match. If we made the payoffs 1 for a favorable match and -1 for an unfavorable match, there is a unique mixed-strategy equilibrium with a population of 50:50 of each type of hitters and each type of pitchers. Now assume that a ban on steroids will lower the payoff for the power hitter against the fastball pitcher to $1/2$ (and the pitcher getting $-1/2$). An initial guess might be that since this hurts the power hitters, it will cause there to be less of them. The reverse is true. By simple calculation, in the new equilibrium $4/7$ of the pitchers will be fastball pitchers and $4/7$ of the hitters will be power hitters.[7] The reason that the number of power hitters goes up has nothing to do with the change in the payoff of

[5] The expression is $h''(\lambda) = \int_0^\infty c(t)t(t\lambda - 2)\cdot e^{-\lambda t}$ dt. Notice that $t(t\lambda - 2)\cdot e^{-\lambda t}$ is negative for $0 < t < 2/\lambda$ and positive for $t > 2/\lambda$. Let us look at the point $c(2/\lambda)$. Since c is a decreasing function, and this expression is negative for all $t < 2/\lambda$, we know that $\int_0^\infty c(t)t(t\lambda - 2)\cdot e^{-\lambda t}\,dt < \int_0^\infty c(2/\lambda)t(t\lambda - 2)\cdot e^{-\lambda t}\,dt = 0$. This is because $c(t) > c(2/\lambda)$ when $t(t\lambda - 2)\cdot e^{-\lambda t}$ is negative and $c(t) < c(2/\lambda)$ when $t(t\lambda - 2)\cdot e^{-\lambda t}$ is positive. Therefore, $h''(\lambda) < 0$.

[6] For reasons of simplicity, we make the assumption that a ban would affect the probability of success equally for all power hitters. By doing so, we do not wish to imply that all power hitters in baseball take steroids to improve their performance.

[7] Note that both hitters have an equilibrium payoff of $-1/7$ while the pitchers have a payoff of $1/7$. So in fact the hitters do lose from the ban but somewhat surprisingly both types of hitters lose.

the power hitters, on the contrary, it is because the payoff for the power pitcher goes up and a larger number of power hitters is needed to keep the pitchers indifferent. As in our model when the number of firms goes up the option value of waiting goes down. In order to keep the indifference condition satisfied, firms need to delay their innovations to increase this option value. Thus, as we see these results do not come from the sunk costs or certainty in our model but from the nature of the equilibrium.

Our other results are more in line with previous predictions, in Corollary 2, we analyze the comparative statics with respect to the size of the reward.

Corollary 2.

(i) The probability of innovating by time t is increasing in the reward $R(t)$.
(ii) The probability of no innovation is non-increasing in the reward $R(t)$.

Proof. This can be shown by differentiating the relevant expressions. □

The implication of these Corollaries is that a government (designer) interested in achieving innovations as soon as possible has two tools: increasing the rewards and limiting the number of firms entering (to two).

Although we see that disconnected supports exist in equilibrium when rewards are decreasing faster than costs, we may ask what economic conditions may lead to such behavior. One such possibility exists when market size varies over time. For instance, there may be a period where the market size rises sharply and then declines substantially. This can help explain the casual observation that the introduction of new products tends to be concentrated (such as new toys during Christmas season) by simply viewing our model as a race towards introducing a product.

3. Economic environments

We have characterized the equilibrium behavior for general environments with minimal restrictions on the cost and reward functions. Two peculiar characteristics of the equilibrium behavior are the possibility of a disconnected support for the equilibrium distribution function and the possibility of an atom on staying out.

Since we put few restrictions on the functions $R(t)$ and $c(t)$, it is important to see whether such behavior patterns could still prevail for standard environments where the structure of the cost and reward functions is economically 'plausible.' What are reasonable assumptions regarding the environment? Let us first assume that knowledge (an innovation) is not lost and has zero storage costs.[8] Given this knowledge assumption, the cost function must be decreasing, since a firm can always invent earlier and hold onto the technology. In addition the rewards must also be decreasing, since again one can keep an innovation from the market.[9] These assumptions are not sufficient to rule out the above-described behavior patterns.[10] The following two remarks are a direct consequence of the construction of equilibrium.

Remark 1. The support is connected if and only if $c(t)/R(t)$ is strictly declining for all t where $R(t) > c(t)$.

Remark 2. There is a zero probability of staying out if and only if $\text{Inf}_{t \geq 0}(c(t)/R(t)) = 0$.

Notice that assuming decreasing reward and cost functions does not necessarily lead to a declining $c(t)/R(t)$ or implies that $\text{Inf}_{t \geq 0}(c(t)/R(t)) = 0$. Hence both types of peculiar behavior are possible under economically reasonable restrictions imposed on the cost and reward functions.

The equilibrium behavior will entail a connected support for a more specialized environment where the reward is constant in nominal terms and is discounted at rate r, hence $R(t) = e^{-rt}R$ (we call this the constant nominal rewards case). For consistency, we must also introduce some notion of the discount rate into the cost structure. We start by assuming that starting a project at time t_1 and achieving an innovation at time $t_1 + t_2$ (duration of research is t_2) has cost given by the function $f(t_1, t_2)$ with $f_1 < 0$ and $f_2 < 0$. The negative f_1 captures an implicit technological improvement over time and the negative f_2 means that when starting a project it is always more expensive to finish it sooner rather than later. The time-zero discounted cost of achieving an innovation by time t is given by $c(t) = \text{Min}_x f(t - x, x) \cdot e^{-(t-x)r}$.

This structure implies that the cost of achieving an innovation at time t is

[8] It should be noted that this is not necessarily the case in a modern society, various technologies have been forgotten (a large part of the cost of the year 2000 bug was due to losing track of where and how the dates were stored). We see great efforts are made to keep certain technologies around. For instance, production lines are kept running, data are transfered to current systems, there is even a internet wide effort made to preserve early arcade games by abandoning preservation of the hardware, but preserving the software.

[9] If the reward were a patent of finite duration, then storage costs may be positive since obtaining a patent earlier will result in an earlier expiration time. Under circumstances such as a growing market, the reward may actually be increasing in time.

[10] We may further assume that the cost should be convex. The faster a firm innovates the more intensive it uses resources. With labor, it must first start paying overtime and then start hiring outside contractors, etc. However, this added assumption will not generate additional results.

decreasing at a rate greater than r. Consider the cost of achieving an innovation at time t as compared to that of achieving the innovation at time $t+\Delta t$. To achieve the innovation at $t+\Delta t$, the firm could delay the optimal choices made for t by Δt, and since f_1 is negative it must be that the cost to innovate at time $t+\Delta t$ is strictly less than $e^{-r\Delta t}c(t)$. Thus, the cost function should decline at a rate greater than r.[11]

By Remark 1, the equilibrium described in Proposition 2 can be written as $F(t)=1-(c(t)/R(t))^{1/(n-1)}$ with a probability $\lim_{t\to\infty}(c(t)/R(t))^{1/(n-1)}$ of staying out. Notice that the support of this cumulative distribution extends from $t_{\min}$ until ∞.

Comparative statics with respect to the nominal size of the reward R are given by Corollary 2; however, the introduction of a discount rate allows us to perform comparative statics with respect to r as well.

Corollary 3. *The probability of innovating by any time t is strictly decreasing in r.*

Proof. This probability is given by: $1-(c(t)/R(t))^{n/(n-1)}$. We will show that $c(t)/R(t)$ is increasing in r. We can write $c(t)/R(t)$ as $Min_x(f(t-x,x)\cdot e^{x\cdot r}/R)$ and by the envelope theorem, $d/dr(c(t)/R(t)) = f(t-x,x)\cdot e^{x\cdot r}/R\cdot x/R$ is positive. □

Corollary 4. *The probability of staying out of the race is non-decreasing in r.*

Proof. The probability of staying out of the race is 1 minus the probability of innovating at some finite time t. Since the probability of innovating by any time t is strictly decreasing in r, the probability of staying out of the race is non-decreasing in r. □

Whether or not the probability of staying out is strictly increasing in r depends on the functional form of f.[12]

It is also of economic interest to ask how the number of firms affects R&D expenditures. While answers were ambiguous in the general case, the current setting with a declining reward function provides clear cut relationships.

Corollary 5. *The expected aggregate expenditure on R&D is decreasing in n.*

Proof. The expected payoff of every firm in equilibrium is zero. Hence, the expected reward received in equilibrium should equal the expected aggregate

[11] The equation $c(t+\Delta t) < e^{-r\,\Delta t}c(t)$ implies that $(c(t+\Delta t)-c(t))/\Delta t < c(t)(e^{-r\,\Delta t}-1)/\Delta t$. The limit of this expression as $\Delta t\to 0$ gives us $c'(t) < -r\cdot c(t)$.

[12] When $f(t-x,x)=e^{1/x+((t-x)+1)/(2(t-x)+1)}$, the probability of staying out is $(e^{2\sqrt{r}}+0.5)/R$, which is increasing in r. On the other hand, when $f(t-x,x)=e^{(1/x)-(t-x)}$ there is a zero probability of staying out, which is independent of r.

expenditure of all the firms We will show that the expected reward is decreasing in n and conclude that the expected aggregate expenditure on R&D must be decreasing as well. The probability of innovating by time t is $1-(c(t)/R(t))^{n/(n-1)}$ which is decreasing in n. Hence an increase in n shifts probability from higher to lower values of $R(t)$ (recall that under our current assumptions $R(t)$ is decreasing in t), thus leading to a decrease in the expected reward received as n increases. □

Corollary 6. *The expected individual expenditure on R&D is decreasing in n.*

Proof. Follows directly from Corollary 5 by symmetry. □

Our expenditure findings while in agreement with empirical findings (see Section 6) stand once more in contrast to the results from the stochastic innovation literature. Loury (1979) finds that as the number of firms increases, the per-firm expenditures decrease and the total expenditures increase and Lee and Wilde (1980) under slightly different assumptions find that the per-firm expenditures increase as well. The intuition for these differences and the underlying differences in assumptions have already been elaborated upon in the discussions following Corollary 1. Corollaries 5 and 6 imply that if a government is interested in increasing R&D expenditures, it would be desirable to limit the number of firms entering to two and from Corollaries 1 and 2 in the previous section, this would speed up rather than delay innovation.

4. An asymmetric model

The cost or reward of an established and well diversified firm (such as IBM) is different than that of a smaller competitor. The larger firm already has the necessary infrastructure for the product introduction or the apparatus for obtaining a patent. The smaller firm does not have the same capabilities and would find it more costly to both innovate and put its innovation into use.[13] Martin (2001, pp. 93–94) summarizes the four main reasons mentioned in the literature for the presence of such an advantage.[14]

We can model this environment in a similar fashion to our symmetric model.

[13] This environment represents the case where there is no monopolist incumbent and both firms are competing for the first innovation, but on unequal terms. It is also possible to study an environment where an entrant firm competes against an incumbent firm (see Gilbert and Newberry, 1982; and Reinganum, 1983). While this would bear some resemblance to our asymmetric model, it would require the specification of two reward functions for the incumbent.

[14] The four reasons are: "1. Larger firms are able to spread the fixed costs of research over a larger sales base. 2. Large firms may have an advantage in financial markets. 3. Larger firms may be able to exploit economies of scale and scope in research. 4. A large diversified firm is more likely to be able to exploit an unexpected discovery."

There is a strong firm, called firm 1, which has reward $R_1(t)$ and cost $c_1(t)$ and there is a weak firm, called firm 2, which has reward $R_2(t)$ and cost $c_2(t)$. The reward and cost functions are assumed to satisfy the same conditions imposed on these functions in the symmetric case with the added (fitting with our description of strength) assumption that $R_1(t) - c_1(t) > R_2(t) - c_2(t)$ holds for all t. We define $t_{\min 2}$ by $t_{\min 2} = \text{Inf}_{t \geq 0}\{t | R_2(t) - c_2(t) > 0\}$.

In order to avoid the trivial situation where firm 1 finds it in its best interest to innovate before $t_{\min 2}$, we assume that $t_{\min 2}$ is smaller than the time chosen by firm 1 if it were a monopolist. We will now proceed to characterize an equilibrium.

Lemma 1. *In any equilibrium, firm* 2 *makes zero profits, while firm* 1 *makes positive profits.*

Proof. Firm 2 would never choose a time earlier than $t_{\min 2}$. Hence, firm 1 can guarantee itself positive profits by choosing an innovation time just before $t_{\min 2}$ (since $R_1(t) - c_1(t) > R_2(t) - c_2(t)$). Thus, in any equilibrium, firm 1 must earn positive expected profits.

Now let us examine a possible support of an equilibrium strategy of firm 1. Positive expected profits requires that this support has a finite upper bound (denoted by) M with a strictly positive probability of winning at M. Let us also examine a possible equilibrium support for firm 2. Such a support cannot contain finite time points greater than M, since a choice of any such time would have a zero likelihood of winning. In order for firm 1 to have a positive probability of winning at M, one of two things must occur: either both firms have an atom at M or firm 2 has a positive probability of staying out of the race. In equilibrium, both firms cannot place an atom at the same finite point in time, since each could increase its expected payoff by innovating just prior to that point. Hence, firm 2 must be staying out of the race with positive probability and earning zero expected profits in equilibrium. □

Lemma 2. *In any equilibrium, the profit of firm* 1, *denoted by* π_1, *is*:

$$Max_{t \leq t_{\min 2}} R_1(t) - c_1(t).$$

Proof. Since in equilibrium firm 2 would never choose any time $t < t_{\min 2}$, equilibrium profits of firm 1 satisfy $\pi_1 \geq \text{Max}_{t \leq t_{\min 2}} R_1(t) - c_1(t)$. However, if $\pi_1 > \text{Max}_{t \leq t_{\min 2}} R_1(t) - c_1(t)$, we reach a contradiction as follows. Let t' be the infimum of all time points played by firm 1 in equilibrium. At such a point, winning profits of firm 1, $R_1(t') - c_1(t')$ are weakly greater than π_1. This implies $R_1(t') - c_1(t') \geq \pi_1 > R_1(t_{\min 2}) - c_1(t_{\min 2})$ (where the first inequality follows by continuity). Since $\pi_1 > \text{Max}_{t \leq t_{\min 2}} R_1(t) - c_1(t)$, we have $t' > t_{\min 2}$. Given this, firm 2 can generate positive profits by choosing a point between t' and $t_{\min 2}$ and thus must have positive equilibrium profits in contradiction to Lemma 1. □

We now proceed to characterize the equilibrium support and distribution functions. In order to do so, we define the two following functions: $q_1(t) = (c_1(t) + \pi_1)/R_1(t)$ and $q_2(t) = c_2(t)/R_2(t)$. We denote by $F_i(t)$ the distribution function used by firm i (by definition increasing in t).

Lemma 3. *The following three conditions on the distribution functions F_1 and F_2:*

(i) $F_i(t) \geqslant 1 - q_j(t) i \neq j$,
(ii) $F_i(t) = 1 - q_j(t)$ at any t chosen[15] by firm j, $i \neq j$,
(iii) There exists a finite A such that $F_1(A) = 1$,

are necessary and sufficient conditions for an equilibrium.

Proof. We first show that they are necessary. If there exists a time in equilibrium where the probability of winning for firm i is strictly higher than $q_i(t)$, then firm i's profits at this time would exceed the equilibrium profits derived in Lemmata 1 and 2. Furthermore, the probability of winning at any point chosen at equilibrium by firm i must equal $q_i(t)$. Hence, the F_i values must satisfy conditions (i) and (ii). Since firm 1 has positive expected profits in equilibrium, it cannot be staying out and thus there must exist a finite A with $F_1(A) = 1$.

To show sufficiency note that neither firm has an incentive to deviate. Any point in the support yields the same expected payoff (π_1 for firm 1 and 0 for firm 2) by (ii). Any finite point outside the support does not yield a higher expected payoff by (i). The staying out option yields firm 2 zero expected profits, whereas it would make firm 1 strictly worse off. □

We now define the functions $g_1(t) = \mathrm{Min}_{t' \leqslant t} q_1(t')$ and $g_2(t) = \mathrm{Min}_{t' \leqslant t} q_2(t')$ and let G_1 and G_2 denote the sets where g_1 and g_2 are strictly decreasing, respectively. By using these definitions, we are able to characterize an equilibrium in the following proposition.

Proposition 3. *The distribution functions defined by $F_i(t) = Max\{1 - g_j(\inf_{t' > t} G_i), 0\}$ whenever the Infimum is finite and when it is infinite (there is no point in G_i that is strictly greater than t), we let $F_1(t) = 1$ and $F_2(t) = 1 - lim_{t' \to \infty} g_1(t')$ with the probability of staying out (for firm 2) given by $lim_{t' \to \infty} g_1(t')$, constitute an equilibrium.*[16]

Proof. In the Appendix A.

[15] We say a time t is chosen by firm i if it belongs to the interior of the support of the distribution function employed by firm i, or if it is an atom of that distribution.

[16] In Appendix A, we provide a numerical example of an asymmetric equilibrium.

While there may not be a unique equilibrium, notice that by Lemma 3, the equilibrium distribution function for one firm is uniquely determined on the points chosen by the other firm. The only possibility of non-uniqueness occurs if a firm is indifferent between two points between which there is no point chosen by the other firm. Our characterization gives the equilibrium which yields the earliest innovation times.

It may be perplexing that the determination of the strong firm is determined by only the difference between rewards and costs. For instance, if $R_1(t) - c_1(t) = R_2(t) - c_2(t) + \epsilon$ and $c_2 \gg c_1$, one may ask how is the weak firm compensated for the risk of losing such a large amount.[17] This puzzle is solved by seeing that if $c_2 \gg c_1$, then $c_2(t)/R_2(t) \approx 1$. This ensures that the weak firm will almost always win when it enters.

From the above, we now see clearly that there is no guarantee that the strong firm always wins. However, we can provide conditions where the strong firm is more likely to innovate by any time t and win the race; that is, the distribution F_2 first-order stochastically dominates F_1.

Corollary 7. *With constant nominal rewards* (*as defined in Section* 3) *and identical costs* ($c_1(t) = c_2(t) \equiv c(t)$), *where costs are decreasing at a rate faster than rewards* (*as in Section* 3), *the equilibrium is unique and* F_2 *first-order stochastically dominates* F_1.[18]

Proof. With constant nominal rewards, if $(c(t) + \pi_1)/R_1(t)$ is decreasing, so must $c(t)/R_2(t)$. This implies $G_1 \subseteq G_2$, which then implies $\mathrm{Inf}_{t'>t}\ G_1 \geq \mathrm{Inf}_{t'>t}\ G_2$. By way of contradiction, assume there exists a point s where $F_2(s) > F_1(s)$. By definition of the cumulative distribution functions, $g_2(\mathrm{Inf}_{t'>s}\ G_1) > g_1(\mathrm{Inf}_{t'>s}\ G_2)$. Since g_2 is a decreasing function, $g_2(\mathrm{Inf}_{t'>s}\ G_2) \geq g_2(\mathrm{Inf}_{t'>s} G_1) > g_1(\mathrm{Inf}_{t'>s}\ G_2)$. Thus, there exists an s' such that $g_2(s') > g_1(s')$. We can rewrite this (using the definitions of g_1 and g_2) as $\mathrm{Min}_{t' \leq s'}\ c(t')/R_2(t') > \mathrm{Min}_{t' \leq s'}(c(t') + \pi_1)/R_1(t')$. Therefore, in particular, there must be an $\hat{s}$ such that $c(\hat{s})/R_2(\hat{s}) > (c(\hat{s}) + \pi_1)/R_1(\hat{s})$. By the assumed shape of the rewards, we can rewrite this inequality as $(R_1 - R_2) \cdot c(\hat{s}) > R_2 \cdot \pi_1$. Since $c(t)$ is decreasing, this can never occur (recall from the definition of $t_{\mathrm{min}2}$ that $(R_1 - R_2) \cdot c(t_{\mathrm{min}2}) = R_2 \cdot \pi_1$). Uniqueness stems from our assumptions on rewards and costs that imply at most one t can solve $p \cdot R_i(t) - c_i(t) = \alpha$ for any $p > 0$ and $\alpha \geq 0$. Hence, the indifference that may cause non-uniqueness cannot occur. □

While in the previous Corollary we saw that there are conditions for which reasonable behavior may occur, we see in the following Corollary that perplexing

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[17] We are indebted to Dan Sasaki for posing this question.

[18] This property also holds if $R_1'/R_1 \leq R_2'/R_2$ and costs are identical and decreasing.

behavior may occur such as the increase in the rewards may lead to a delay (increase) in innovation times.[19] This is counter to standard behavior and can occur if the rewards are increased for the strong firm increasing its expected profits in equilibrium.

Corollary 8.

(i) An increase of the rewards (or decrease in the costs) of the strong firm may increase or decrease probability of innovating by time t and may increase or decrease the probability of no innovation.
(ii) An increase of the rewards (or decrease in the costs) of the weak firm can only increase probability of innovating by time t and can only decrease the probability of no innovation.

Proof. Let us first look at part (i). An increase in the rewards or a decrease in the costs of the strong firm can either increase π_1 or not affect π_1. If π_1 is not affected, the fraction $(c_1(t) + \pi_1)/R_1(t)$ may only decrease leading to our result that innovations may occur earlier. Notice that π_1 can be increased without affecting $c_1(t)$ nor $R_1(t)$ for all t greater than $t_{\min}$. If such an increase occurs, $(c_1(t) + \pi_1)/R_1(t)$ will increase leading to the opposite result. Let us now look at part (ii). An increase in the rewards or a decrease in the costs of the weak firm cannot increase π_1. Thus, the fraction $(c_1(t) + \pi_1)/R_1(t)$ cannot increase. In addition, the fraction $c_2(t)/R_2(t)$, cannot increase. Together this leads to our result. □

The intuition behind Corollary 8 can be seen by looking at the firms' profits. If a firm's expected profit in equilibrium is unaffected by an increase in its rewards, then it must be the case that the probability of winning must decrease in order to keep its expected profit the same. This can only occur by the other firm choosing earlier innovation times. However, if winning profits of the strong firm remain the same while its expected profits in equilibrium increase, it must be the case that the weak firm delays innovation times, thus, increasing the strong firm's probability of winning.

It is interesting to note the possible implications of Corollary 8 for international R&D. Let us say that the strong firm represents a firm in the UK and the weak firm represents a firm in Israel. It may be possible that by the UK increasing the

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[19] Although the Corollary states that an increase in rewards for the strong firm can hurt innovation time, it can also easily be seen that an increase in rewards for both firms can also hurt innovation. For instance, if this increase occurs for times that are unprofitable for the weak firm, yet increase equilibrium expected profit for the strong firm.

rewards for the UK firm, it will only affect the Israeli firm's behavior. By doing so, it can increase the chances of the UK firm getting the patent without actually paying any of the increased rewards. This happens since the increase of rewards is for innovation times not chosen by the UK firms. However, the profitability for the UK firm will still go up with the innovation being delayed on average.

5. Relationship to all-pay auctions

The environments we considered are similar to an all-pay auction (a thorough analysis can be found in Baye et al., 1996); the difference being that our time-dependent reward would be the equivalent of a bid-dependent value (rather than a commonly assumed constant value). Comparing our symmetric case to the symmetric all-pay auction, the choice of $t_{\min}$ corresponds to bidding one's valuation in the all-pay auction. The optimal bid of a single buyer (usually assumed to be zero) equals the optimal time choice of a single firm acting on its own. Staying out in our framework corresponds to bidding zero. However, our results differ. First, in our case, an atom may be placed on staying out, whereas, in the all-pay-auction, bidders do not use atoms. The difference is due to the dual role of zero in the all-pay auction in providing both an optimal bid for a single buyer and the option of staying out for any bidder. This role in our set up is played by the optimal time for a single firm and ∞, respectively. Second, supports in our case may be disconnected, while in the all-pay auction supports are connected. However, the equilibrium behavior is qualitatively the same as is in the all-pay auction under the conditions specified in Remarks 1 and 2 in Section 3.

For all-pay auctions with asymmetric rewards, Baye et al. (1996) find that the weak bidder (the one with the low value) has zero expected profits, while the strong bidder (the one with the high value) earns the difference between the rewards. They also find that the equilibrium has mixed strategies on a support between zero and the lower valuation with the weak bidder placing an atom at zero.

Again, these results have both similarities and differences to our results. They are similar since the high value is comparable to $t_{\min 1}$ (the earliest innovation time where the profits of firm 1 turn positive), the low value is comparable to $t_{\min 2}$, and bidding zero is comparable to staying out. They are different since firm 2 has a finite limit which would be comparable to a minimum bid. This again stems from zero playing a double role in the all-pay auctions. They are also different in that equilibria in our model can have disconnected supports and atoms. We have a stronger similarity in the case of constant nominal rewards and identical costs. In this case, if $c(t)$ is decreasing at a faster rate than r with the property that c'/c is non-increasing, the only difference is the upper limit of the support of the strong firm.

6. Relationship to empirical literature

There is a vast amount of empirical literature on the relationship between innovative activity and variables such as firm size, industry concentration and market structure. The main empirical regularities reported agree with our theoretical findings with respect to both our symmetric and asymmetric models.

In the symmetric environment the main insight generated by our model is that the number of firms is inversely related to R&D expenditures and furthermore innovation is delayed with an increase in the number of firms. A smaller number of firms in our model reflects a larger degree of industrial concentration. Empirical findings have shown that industrial concentration indeed has a positive effect on R&D spending (see Vossen, 1999).

While in our model, a firm's expense affects only when it innovates and not how many innovations it has. The obvious inverse of this shows simply that an increase in spending reduces the 'efficiency' of the R&D activity. While this inverse relationship between spending and efficiency is assumed, it is important to note that this agrees with the empirical finding that the R&D activity in more concentrated markets tends to generate relatively fewer innovations per dollar spent.

In the asymmetric environment we have shown (Corollary 7) that with a standard structure of rewards and costs, large firms will invest more in R&D and are more likely to innovate first. Furthermore we have shown that while smaller firms might stay out of R&D activity altogether the larger firm will always enter the innovation race. This agrees with empirical findings as reported in Cohen and Klepper (1996) that larger firms are more likely to engage in R&D and that R&D expenditures increase with firm size.

7. Concluding remarks

We formulated and analyzed a deterministic innovation race where participants must incur costs and the reward structure is time dependent. We showed that mixed-strategy equilibria exist thus moving the uncertainty from the outcome of the race to the R&D expenditures themselves. As noted before, it is possible to maintain the uncertainty of the outcome by transforming our framework into a model where there is an uncertain innovation time for a given expenditure level. Each firm can choose a distribution F where the cost of choosing this F is $\int c(t)\,\mathrm{d}F(t)$. The set of possible distributions can be limited for various technological reasons; however, the equilibrium behavior will be identical to that of firms in our model as long as the equilibrium strategies that we characterized are available as choices.[20] In a different approach that has pure-strategy (Bayes–Nash) equilibria,

[20] We are grateful to Moshe Justman for suggesting this avenue of interpretation.

Kaplan et al. (2002) introduce incomplete information into this environment. We could also introduce a small amount of incomplete information into our model, say by having a privately known ability that affects the cost (such as costs are $c(t)(1-a)$ where a is uniformly distributed on $[0, \epsilon]$). A pure-strategy equilibrium in such a model will mirror the mixed-strategy equilibrium in our model without this ability, that is, the distribution of time choices over types will be the same; however, higher-ability firms will choose earlier times.

The theoretical findings in our model are in agreement with most empirical findings. This lends further support to the use of our environments and techniques to model R&D activity. We thus provide both a theoretical basis for explaining empirical regularities and a framework for making policy recommendations.

Although the derivation of the equilibrium is not straightforward, we provide a simple, concise characterization of the equilibrium strategies. By using our techniques, it is possible to apply our analysis to other studies of R&D which used the previous models of innovation. One such case would be to study the possibility of both drastic and non-drastic innovations (Vickers, 1986; Reinganum, 1985). One can also perform welfare analysis in our model by explicitly introducing possible patent policies (see Denicolo, 1996). Such policy analysis may have uncommon implications, since as we saw (in Corollary 8) an increase in rewards may hurt innovation. Finally, another avenue of further research would be to analyze time-dependent rewards where firms can dynamically decide whether to continue innovation research where only some of the costs would be sunk.

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Appendix A

A.1. Nonexistence of a pure-strategy Nash equilibrium

A pure-strategy Nash equilibrium is a set of times $\{t_i'\}_{i=1}^n$ where no firm has an incentive to change its time given the other firms' choices. Let $\underline{t}$ be the minimum time chosen ($\underline{t}=\min_i t_i'$) and I be the set of firms choosing this minimum ($I=\{i|t_i'=\underline{t}\}$). If there is just one firm in I, there cannot be any firms outside I that choose a finite t. Such a firm (j) would have positive costs $(c(t_j))$ and would be able to reduce this cost by choosing $t=\infty$. So, if there is just one firm in I, then all firms outside I would choose $t=\infty$ and hence make zero profits. Now if the firm in I is making positive profits (and hence $\underline{t} > 0$ since we know that $R(0) < c(0)$), then any firm outside I could make positive profits by changing its innovation time to just before $\underline{t}$ (by continuity). If the firm in I is making zero profits, it could increase profits by choosing a finite t where $R(t) - c(t) > 0$. This t exists by our assumption on $t_{\min}$. If the firm in I is making strictly negative profits, it could make zero profits by choosing $t=\infty$. Hence, there cannot be a Nash equilibrium in pure strategies with just one firm in I. If there is more than one firm in I, then any firm in I could increase its profits by changing its innovation time to just before $\underline{t}$ (all firms in I are currently splitting the reward of $R(\underline{t}) \geqslant c(\underline{t}) > 0$). Thus, a Nash equilibrium in pure strategies cannot exist.

A.2. Proof of Proposition 1

Assume there is a symmetric equilibrium and denote the support of $F(t)$ by S. Denote the upper and lower bounds of S by $\underline{S}$ and $\overline{S}$, respectively.

A.2.1. The support has no atoms

The support (excluding ∞) cannot contain an atom. By contradiction, if there is an atom at a finite time t', then each firm could improve its expected payoff by innovating just before time t' rather than at time t' (by continuity).

A.2.2. All points in the support yield zero expected payoff

All the points in the support must, since it is an equilibrium, yield the same expected profit. Therefore in order to prove our claim it is enough to find one point in the support that yields zero profits. In the case where $\overline{S}$ is infinite, this point occurs at infinity by our assumptions on $R(\infty)$ and $C(\infty)$. When $\overline{S}$ is finite, the probability of winning at $\overline{S}$ must be strictly positive since revenue is bounded [21](by our assumptions on continuity and endpoints) and the cost is strictly positive $c(t) > 0$. Since there are no atoms in the support, for this to happen there must be a positive probability of staying out in a symmetric equilibrium. Again, since the profit from staying out is zero and all points in the support must have the same expected payoff, the expected profit at each point in the support must also equal zero.

[21] See Baye and Morgan (1999) and Kaplan and Wettstein (2000) for an example where unbounded revenue may lead to a positive profit equilibrium that would otherwise not have existed.

A.2.3. The bounds of the support and the equilibrium distribution function are uniquely determined

We will first show that $t_{\min} \leqslant \underline{S}$. Assume by way of contradiction that $\underline{S} < t_{\min}$. Each firm will then choose a time strictly less than $t_{\min}$ with strictly positive probability. Thus, there exists a time $t' < t_{\min}$ that will be in the support and where a firm choosing t' will have a strictly positive probability of losing. Since all $t < t_{\min}$ have $R(t) - c(t) \leqslant 0$, the expected profits of choosing t' must be strictly negative. Since a firm can always make zero expected profits by choosing $t = \infty$, this is in contradiction to t' being chosen in equilibrium. Therefore, $t_{\min} \leqslant \underline{S}$. If $t_{\min} < \underline{S}$, then by the definition of $t_{\min}$, there exists a $\hat{t}$ where $t_{\min} < \hat{t} < \underline{S}$ and $R(\hat{t}) - c(\hat{t}) > 0$. Choosing such a $\hat{t}$ wins with certainty and thus yields strictly positive expected profits. Since all points in the support yield zero expected payoffs this is a profitable deviation and thus the equilibrium strategy cannot have such a support. Therefore, $\underline{S} = t_{\min}$. Thus, the support of any symmetric equilibrium is $[t_{\min}, \bar{S}]$.

Finally, we show that both $\bar{S}$ and the equilibrium distribution F on the resulting support are uniquely determined.

Let the support of the equilibrium candidate $F(t)$ be $[t_{\min}, \bar{S}]$ (with $\bar{S}$ possibly infinite) and assign the probability p to the option of staying out of the race ($t = \infty$). When all firms use the $F(t)$ distribution, the probability of winning the race when choosing time t denoted by $q(t)$ equals $(1 - F(t))^{n-1}$. Thus, the expected payoff from choosing a particular t is: $q(t)R(t) - c(t)$. Since this is part of a Nash equilibrium and the equilibrium expected payoff is zero, $q(t)R(t) - c(t) = 0$ for all points in the support. We further characterize the support by noticing that if there is a point t_1 in the support, there cannot be a point t_2 in the support with $t_2 > t_1$ and $c(t_2)/R(t_2) > c(t_1)/R(t_1)$. Otherwise, the probability of winning must be increasing between t_1 and t_2 in order for expected profits to remain zero at both points. Hence, the support must be a subset of the set: $\{t | c(t)/R(t) \leqslant c(t')/R(t')$ for all $t' \leqslant t\}$. In order to ensure that there are no atoms, the support must equal this set (otherwise there would be a jump in the probability of winning which equals $c(t)/R(t)$ on the support). Therefore,

$$q(t) = \operatorname*{Min}_{t' \leqslant t} \frac{c(t')}{R(t')} \quad \text{and} \quad F(t) = 1 - \left(\operatorname*{Min}_{t' \leqslant t} \frac{c(t')}{R(t')} \right)^{1/n-1} \quad \text{for} \quad t \in [t_{\min}, \bar{S}].$$

In the case where $\bar{S}$ is finite, assume by way of contradiction that there exist two symmetric equilibria one with $\bar{S}_1$ and the other with $\bar{S}_2$, where $\bar{S}_1 < \bar{S}_2$. The probability of winning, $q(t)$, is uniquely determined up to $\bar{S}_1$ and coincides in both equilibria. The expected payoff for any choice of t in the equilibrium with $\bar{S}_2$ is zero. Hence, the expected payoff at $t = \bar{S}_2$, in the $\bar{S}_2$ equilibrium is zero. The probability of winning when choosing $\bar{S}_2$ at the $\bar{S}_1$ equilibrium is strictly larger than the probability of winning when choosing $\bar{S}_2$ at the $\bar{S}_2$ equilibrium. Thus, the expected payoff for choosing $\bar{S}_2$ at the $\bar{S}_1$ equilibrium must be strictly positive.

This is in contradiction to the fact all points in the equilibrium support of the $\bar{S}_1$ equilibrium yield zero expected profits.

Similarly it can be shown that if one symmetric equilibrium entails a finite $\bar{S}$ there does not exist an equilibrium with no finite upper bound.

A.3. Proof of proposition 3

We will show these distribution functions satisfy the conditions specified in Lemma 3.

To show that $F_1(t) \geqslant 1 - q_2(t)$ we consider the three cases: $\text{Inf}_{t'>t}\ G_1 = t$, $\text{Inf}_{t'>t}\ G_1 > t$, and $\text{Inf}_{t'>t}\ G_1 = \infty$. When $\text{Inf}_{t'>t}\ G_1 = t$, we have $F_1(t) = \text{Max}\{1 - g_2(\text{Inf}_{t'>t} G_1), 0\} \geqslant 1 - g_2(\text{Inf}_{t'>t} G_1) \geqslant 1 - q_2(t)$. When $\text{Inf}_{t'>t}\ G_1 = s > t$, we have $F_1(t) = 1 - g_2(s) > 1 - g_2(t)$ since $s > t$ and g_2 is decreasing. In the last case ($\inf_{t'>t}\ G_1 = \infty$), we are at a t for which no larger t' exists in G_1. In this case, $F_1(t) = 1$and the claim holds as well. Similarly, it can be shown that $F_2(t) \geqslant 1 - q_1(t)$ for the first two cases and in the third case, the claim holds since g_1 is decreasing.

To show there exists a finite A such that $F_1(A) = 1$, we will show that there exists a time t'' such that $g_1(t)$ is constant for any $t \geqslant t''$. Let t' be an arbitrary finite time and let $t'' > t'$ satisfy $\pi_1/R_1(t) > (c_1(t') + \pi_1)/R_1(t')$ for any $t > t''$. Such a t'' must exist since π_1 is strictly positive and R_1 tends to zero as time goes to infinity. Since $c_1(t)$ is nonnegative, $(c_1(t) + \pi_1)/R_1(t) > (c_1(t') + \pi_1)/R_1(t')$ for all $t > t''$. This implies that g_1 cannot be decreasing after t''. Hence, there is no point in G_1 greater than t'' and therefore $F_1(t'') = 1$.

Now we show that $F_i(t) = 1 - q_j(t)$ at any t chosen by firm j, $i \neq j$. If t is chosen by firm 1, it must be that $F_1(t)$ is not constant on any neighborhood of t. This can happen in one of two ways: either there is a right neighborhood of t that is in G_1 and g_2 is strictly decreasing or there is no right neighborhood of t in G_1 and $T_1 > T_2$, where $T_i = \text{Inf}_{t'>t} G_i$. In the first case, $F_2(t) = 1 - g_1(t) = 1 - q_1(t)$ since t is in the interior of G_2 and g_1 is strictly decreasing when moving to the right from t. In the second case, there is no right neighborhood of t in G_1. Thus, $F_2(t) = 1 - g_1(T_2) = 1 - q_1(t)$ since $T_1 > T_2$ implies that g_2 starts to increase before g_1. Likewise, we can show this property if t is chosen by firm 2. □

A.4. Numerical examples

We provide the following numerical examples to demonstrate possible equilibrium behavior.

A.4.1. The symmetric case: Poisson distribution case

Assume there are n firms with identical reward and cost functions given by: $R(t) = e^{-at}$ and $c(t) = e^{-bt}$ where $b > a$. The equilibrium distribution function employed by each firm is: $F(t) = 1 - e^{(a-b)t/(n-1)}$. This is a Poisson distribution with parameter $(b-a)/(n-1)$. The expected innovation time for an individual firm is $(n-1)/(b-a)$. The expected innovation time for the winner of the patent race is $(n-1)/(n \cdot (b-a))$. Both are increasing in n. The expected costs for a firm choosing with a distribution function $F(t) = 1 - e^{-\lambda t}$ is $\lambda/(\lambda + 2(b-a))$. This implies increasing returns with respect to λ.

A.4.2. The symmetric case: connected support

Assume there are two firms with identical reward and cost functions given by: $R(t) = 120 - t$ and $c(t) = 2000/t$. The earliest innovation time with non-negative profits is $t_{\min} = 20$. The equilibrium distribution function for $t \geqslant 20$ is then given by: $F(t) = 1 - \mathrm{Min}_{t' \leqslant t}(2000/(t'(120 - t')))$. Hence, the equilibrium distribution function employed by each firm is:

$$F(t) = \begin{cases} 1 - \dfrac{2000}{t(120-t)} & 20 \leqslant t \leqslant 60 \\ 4/9 & 60 < t \end{cases}$$

and each firm places a probability of 5/9 on staying out.

A.4.3. The symmetric case: disconnected support

To show the possibility of a disconnected support assume the reward function is as before but the cost function is given by:

$$c(t) = \begin{cases} 140 - 3t & 0 \leqslant t < 15 \\ 100 - t/3 & 15 \leqslant t < 45 \\ 355 - 6t & 45 \leqslant t < 355/6 \\ 0 & 355/6 \leqslant t \end{cases}$$

When there are two firms the equilibrium distribution function is given by:

$$F(t) = \begin{cases} 1 - \dfrac{140 - 3t}{120 - t} & 0 \leqslant t < 15 \\ 2/21 & 15 \leqslant t < 48.364 \\ 1 - \dfrac{355 - 6t}{120 - t} & 48.364 \leqslant t < 355/6 \end{cases}$$

A.4.4. The asymmetric case

Assume there are two firms with an identical cost function $c_1(t) = c_2(t) = 2000/t$. The reward functions of firm 1 and 2 are given by: $R_1(t) = 120 - t$ and $R_2(t) = 120 - 1.5t$. The equilibrium distribution functions are then given by:

$$F_1(t) = \begin{cases} 1 - \dfrac{2000}{t(120 - 1.5t)} & 23.67 \leqslant t < 40 \\ 1 & t = 52.004 \end{cases} \quad \text{and}$$

$$F_2(t) = \begin{cases} 1 - \dfrac{2000}{t(120-t)} - \dfrac{11.8263}{120-t} & 23.67 \leqslant t < 40 \\ 0.26 & t = 40 \end{cases}$$

with firm 2 placing a probability of 0.74 on staying out.

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