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Equivalence Between Bargaining Sets and the Core in Simple Games

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Abstract: We investigate the equivalence between several notions of bargaining sets which occur in the literature and the core of simple games.

1 Introduction

Recently, several new concepts of bargaining sets for cooperative games have been introduced (Dutta et al. (1989), Mas-Colell (1989), Greenberg (1990, 1992)) in addition to the classical Aumann-Maschler bargaining set. All of these sets contain the core of the game. However, there are important cases in which some of these sets coincide with the core. It is well known that for convex coalitional games these bargaining sets are equivalent to the core (see Maschler, Peleg and Shapley (1972) for the Aumann/Maschler bargaining set, Dutta et al. (1984) for the consistent bargaining set, and Greenberg (1992) for the stable bargaining set). Mas-Colell (1989) proved that in an atomless pure exchange economy the Mas-Colell bargaining set coincides with the set of competitive equilibria and by Aumann's equivalence theorem (Aumann (1964)) also coincides with the core. Shapley and Shubik (1984) showed that the Aumann/Maschler bargaining set is approximately competitive in replica sequences of transferable utility exchange economies with smooth preferences. This result was extended in Anderson (1994) to non-replica sequences of non-transferable utility exchange economies with smooth preferences. Anderson, Trockel and Zhou (1994) showed that the Mas-Colell bargaining set may fail to converge to competitive outcomes in large finite non-transferable utility exchange economies. In this work we investigate the equivalence between the core and these bargaining sets in finite simple games. It is shown that for simple games with a non-empty core the Aumann-Maschler, the Mas-Colell, and the individual stable bargaining set coincide with the core. Since for such games Greenberg's individual stable bargaining set coincides with the

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consistent bargaining set of Dutta et al. (1989), the result also holds for the consistent bargaining set. However, it does not hold for the modified and the stable bargaining sets of Greenberg. We show that for proper simple games the core coincides with the modified bargaining set iff the game is a unanimity game.

In Section 2 we give the basic definitions which are relevant to this work. In Section 3 we state and prove the results and discuss some examples and open problems.

2 Basic Definitions

A *coalitional game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v: 2^N \to \boldsymbol{R}$ is a function which satisfies $v(\emptyset) = 0$. In this paper we assume that v takes non-negative values. The members of 2^N are called *coalitions*. A coalitional game (N, v) is *simple* if $v(S) \in \{0, 1\}$ for each coalition S. We assume that all simple games are monotonic, i.e., if $v(S) = 1$ and $T \supset S$ then $v(T) = 1$. If $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$ and $S \subset N$, we denote $x(S) = \sum_{i \in S} x_i$. The *core* of a coalitional game (N, v) is the set

$$C(N, v) = \{x \in \boldsymbol{R}^n / x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for } S \subset N\}$$

We now define several notions of bargaining sets which appear in the literature. A *payoff vector* in a game (N, v) is a vector $x \in \boldsymbol{R}^n_+$ which satisfies $x(N) \leq v(N)$. The set of all payoff vectors in (N, v) will be denoted by $X(N, v)$ and by $X^*(N, v)$ we denote the efficient payoff vectors, i.e. $X^*(N, v) = \{x \in \boldsymbol{R}^n_+ \mid x(N) = v(N)\}$. A vector $x \in X(N, v)$ is *individually rational* if $x_i \geq v(\{i\})$ for each $i \in N$.

Let x be a payoff vector in (N, v), and let i, j be two players in N. An *objection* of i against j in the vector x is a pair (y, S), where $y \in \boldsymbol{R}^n_+$, $S \subset N$, $i \in S, j \notin S$, $y(S) = v(S)$ and $y_k > x_k$ for each $k \in S$. A *counter objection* of j against the pair (y, S) is a pair (z, T), where $z \in \boldsymbol{R}^n_+$, $T \subset N$, $j \in T$, $i \notin T$, $z(T) = v(T)$, $z_k \geq x_k$ for $k \in T \backslash S$ and $z_k \geq y_k$ for $k \in S \cap T$.

The *Aumann-Maschler bargaining set* (see Auman-Maschler (1964)) of (N, v) is the set $B(N, v)$ of all individually rational and efficient payoff vectors x in (N, v) such that if a player i has an objection against j in x, then j has a counter objection against this objection.

Let x be a payoff vector in a game (N, v). A pair (y, S) is an objection to x in the sense of Mas-Colell if $y \in R^n_+$, $y(S) \leq v(S)$ and $y_i \geq x_i$ for each $i \in S$ with at least one strict inequality.

Let (y, S) be an objection to x. (z, T) is a counter objection to (y, S) if $z \in \boldsymbol{R}^n_+$, $z(T) \leq v(T)$, $z_i \geq y_i$ for $i \in S \cap T$, $z_i \geq x_i$ for $i \in T \backslash S$ and at least one of these inequalities is strict. An objection (y, S) to x is said to be justified in the sense of Mas-Colell if there does not exist any counter objection to (y, S). *The Mas-Colell bargaining set* (see Mas-Colell (1989)) of (N, v) is the set

$MB(N,v)=\{x\in X^*(N,v)|$ There does not exist (y,S) which is a justified objection to x in the sense of Mas-Colell$\}$.

We now define a modified version of $MB(N,v)$ which was introduced by Greenberg (see Greenberg (1990), p. 77).

Let (N,v) be a game. A pair (y,S) is an objection to $x\in X(N,v)$ in the sense of Greenberg if $S\subset N$, $y\in \boldsymbol{R}^n_+$, $y(S)=v(S)$, $y_i>x_i$ for all $i\in S$, and $y_i=x_i$ for all $i\in N\backslash S$. Let (y,S) be an objection to x. (z,T) is a counter objection to (y,S) if $T\subset N$, $z\in \boldsymbol{R}^n_+, z(T)=v(T)$, $z_i>y_i$ for all $i\in T$, and $z_i=y_i$ for all $i\in N\backslash T$. An objection (y,S) is justified in the sense of Greenberg if there does not exist any counter objection to (y,S). The *modified bargaining set*, $MBS(N,v)$ consists of all payoffs in $X(N,v)$ for which there exists no justified objection (in the sense of Greenberg). Finally, we define the stable bargaining set and the individual stable bargaining set which were introduced by Greenberg (see Greenberg (1992)).

Let $D=\boldsymbol{R}^n_+$. Define two binary relations $\prec$ and $\prec^*$ on D by

$x\prec y$ iff there exists $S\subset N$, $y(S)=v(S)$, $y_i>x_i$ for all $i\in S$ and $y_i=x_i$ for all $i\in N\backslash S$.

$x\prec^* y$ iff there exists $S\subset N$, $y(S)=v(S)$, $y_i\geq x_i$ for all $i\in S$ with at least one strict inequality and $y_i=x_i$ for all $i\in N\backslash S$.

Both $(D,\prec)$ and $(D,\prec^*)$ admit unique von-Neumann-Morgenstern abstract stable sets, A and A^*, respectively (see Theorem 2.2 in Greenberg (1992)).

The *stable bargaining* set of a game (N,v) is the set $SBS(N,v)=A\cap X(N,v)$, and the *individual stable bargaining set of* (N,v) is the set $ISBS(N,v)=A^*\cap X(N,v)$.

3 Equivalence Between Bargaining Sets and the Core in Simple Games

It is known that each one of the bargaining sets which were defined in Section 2 contains the core. In this section we investigate the equivalence between the core and those bargaining sets in simple games.

A coalition S in a simple game (N,v) is *winning* if $v(S)=1$. A player $i\in N$ is a *veto player* in (N,v) if he belongs to each winning coalition. A simple game is *weak* if it has at least one veto player. It is well known that a simple game has a non-empty core iff it is weak.

We are now ready to prove the following:

Theorem 1: Let (N,v) *be a weak simple game. Then*

$$B(N,v)=MB(N,v)=ISBS(N,v)=C(N,v).$$

Proof: Let V be the set of veto players in the game (N,v). Then $C(N,v)=\{x\in \boldsymbol{R}^n_+ \,|x(V)=1\}$.

We first show that $B(N,v)=C(N,v)$. Assume, on the contrary, that there is $x\in B(N,v)\backslash C(N,v)$. Then $x(V)<1$. Therefore, there is $i\in N\backslash V$ such that $x_i>0$. Let $j\in V$. Define $y=(y_1,\ldots,y_n)$ by

$$y_k=\begin{cases} x_k & k\neq i,j \\ 0 & k=i \\ x_i+x_j & k=j \end{cases}$$

Then $y(N\backslash\{i\})=x(N\backslash\{i\})+x_i=1=v(N\backslash\{i\})$ and $y_j>x_j$. Therefore, $(y,N\backslash\{i\})$ is an objection of j against i in x. Since $j\in V$, it is clear that $(y,N\backslash\{i\})$ does not have a counter objection, which contradicts the fact that $x\in B(N,v)$.

We now show that $MB(N,v)=C(N,v)$. Assume not, and let $x\in MB(N,v)\backslash C(N,v)$. Then $x(V)<1$. Therefore, there is $i\in N\backslash V$ such that $x(N\backslash\{i\})<1$. Let $T\subset N$ such that $x(T)=\min\{x(S)|v(S)=1\}$. Then $x(T)<1$. Define $y=(y_1,\ldots,y_n)$ by

$$y_i=\begin{cases} x_i+\dfrac{1-x(T)}{|V|} & i\in V \\ x_i & i\in N\backslash V \end{cases}$$

Then $y_i>x_i$ for each $i\in V$ and $y(T)=x(V)+1-x(T)+x(T\backslash V)=1=v(T)$. Therefore (y,T) is an objection to x in the sense of Mas-Colell. We will show that (y,T) is justified. Assume, on the contrary, that (z,S) is a counter objection to (y,T). Then $z(S)=v(S)=1$ and $z_i\geq y_i$ for each $i\in S\cap T$ and $z_i\geq x_i$ for each $i\in S\backslash T$ with at least one strict inequality. Therefore, $z(S)=z(S\backslash T)+z(S\cap T)>x(S\backslash T)+y(S\cap T)=y(S\backslash T)+y(S\cap T)=y(S)=1+x(S)-x(T)\geq 1$, which is impossible. Thus, (y,T) is a justified objection to x, which contradicts the fact that $x\in MB(N,v)$.

It remains to show that $ISBS(N,v)=C(N,v)$. Assume not, and let $x\in ISBS(N,v)\backslash C(N,v)$. Define $T\subset N$ and $y=(y_1,\ldots,y_n)$ as above. Then $x\prec^* y$. We will show that y is not dominated with respect to $\prec^*$ and thus $y\in A^*$, which contradicts the internal stability of A^* with respect to $\prec^*$. Assume that $y\prec^* z$. Then there is $S\subset N$ such that $z(S)=v(S)=1$, $z_i\geq y_i$ for each $i\in S$ with at least one strict inequality, and $z_i=y_i$ for each $i\in N\backslash S$. Now

$$z(S)=z(S\backslash T)+z(S\cap T)>y(S\backslash T)+y(S\cap T)=y(S)=1+x(S)-x(T)\geq 1.$$

which is impossible. □

Theorem 1 is not true for $SBS(N,v)$. If we consider, for example, the weighted majority game $G=[3;2,1,1]$, then $x=(\frac{1}{2},\frac{1}{4},\frac{1}{4})\in SBS(G)\backslash C(G)$. For otherwise there is $y\in A$ such that $x\prec y$. Without loss of generality $y=(\frac{1}{2}+\varepsilon,\frac{1}{2}-\varepsilon,\frac{1}{4})$, where $0<\varepsilon<\frac{1}{4}$. Let $0<\delta$ such that $\varepsilon+\delta<\frac{1}{4}$ and let $z=(\frac{1}{2}+\varepsilon+\delta,\frac{1}{2}-\varepsilon,\frac{1}{2}-(\varepsilon+\delta))$. Then $y\prec z$ and $z\in A$, which contradicts the internal stability of A. Therefore, $x\in A$

and thus $x \in SBS(G) \setminus C(G)$. Note that since for each coalitional game (N, v), $SBS(N, v) \subset MBS(N, v)$, Theorem 1 also does not hold for MBS(N, v). However, we can characterize all simple games (N, v) for which $MBS(N, v) = C(N, v)$. For this we need some definitions.

Let (N, v) be a simple game. (N, v) is a *unanimity* game if there exists $T \subset N$, $T \neq \emptyset$ such that $v = u_T$ where

$$u_T(S) = \begin{cases} 1 & S \supset T \\ 0 & \text{otherwise} \end{cases}$$

A coalitional game (N, v) is *convex* if for each $S, T \subset N$ we have $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$. Note that a simple game is convex iff it is a unanimity game (see Lemma 5.7, page 267 in Rosenmuller (1981)). If π is a permutation of $N = \{1, 2, \ldots, n\}$ we denote $P_i^\pi = \{j \in N \mid \pi(j) < \pi(i)\}$. For a coalitional game (N, v) we define a vector $a^\pi(v) \in \boldsymbol{R}^n$ by $a_i^\pi(v) = v(P_i^\pi \cup \{i\}) - v(P_i^\pi)$. It is well known that (N, v) is convex iff $a^\pi(v) \in C(N, v)$ for each permutation π (see Ichiishi (1981) and Shapley (1971)). A simple game (N, v) is *proper* if $v(S) = 1$ implies $v(N \setminus S) = 0$. For a proper simple game (N, v) we have $SBS(N, v) \neq \emptyset$ (this is a direct consequence of Theorem 6.5.6 in Greenberg (1990)). We are now ready to state and prove the following result.

Theorem 2: Let (N, v) be a proper simple game. Then $MBS(N, v) = C(N, v)$ iff v is a unanimity game.

Proof: Assume first that there is $T \subset N$, $T \neq \emptyset$ such that $v = u_T$. Let $x \in MBS(N, v)$. Assume, on the contrary, that $x \notin C(N, v)$. Then $x(T) < 1$. Define $y = (y_1, \ldots, y_n)$ by

$$y_i = \begin{cases} x_i + \dfrac{1 - x(T)}{|T|} & i \in T \\ x_i & i \notin T \end{cases}$$

Then (y, T) is an objection to x in the sense of Greenberg. We will show that it is justified. Indeed, if (z, S) is an objection against (y, T), then $v(S) = 1$ and thus $S \supset T$. Therefore, $1 = z(S) > y(S) \geq y(T) = 1$, which is impossible.

In order to prove the converse part of Theorem 2, we will show that $MBS(N, v) = C(N, v)$ implies that (N, v) is a convex game. Assume not, then there is a permutation π of N such that $a^\pi(v) \notin C(N, v)$. Let V be the set of veto players in (N, v). Then there is $i \in N \setminus V$ such that

$$a_j^\pi(v) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

As $a^\pi(v) \notin MBS(N, v)$, there is a justified objection (x, S) to $a^\pi(v)$ (in the sense of Greenberg). As $x(S) = v(S) = 1$, $i \notin S$. We distinguish two possibilities.

(a) There is $j \in S$ such that $v(N \backslash \{i, j\}) = 1$.

In this case we have $x(N \backslash \{i, j\}) = x(S) - x_j = 1 - x_j < 1$. Define $y = (y_1, \ldots, y_n)$ by

$$y_k = \begin{cases} x_k + \dfrac{x_j}{n-2} & k \neq i, j \\ x_k & k = i, j \end{cases}$$

Then $(y, N \backslash \{i, j\})$ is a counter objection to (x, S), which contradicts the fact that (x, S) is justified.

(b) $v(N \backslash \{i, j\}) = 0$ for each $j \in S$.

In this case $v(T) = 1$ implies $i \in T$ or $S \subset T$. Since (N, v) is not convex, $v(V) = 0$. Let $j \in S \backslash V$. Define $y = (y_1, \ldots, y_n)$ by

$$y_k = \begin{cases} \dfrac{1}{2|V|} & k \in V \\ \dfrac{1}{4} & k = i, j \\ 0 & \text{otherwise} \end{cases}$$

Then $y \notin C(N, v)$. We show that $y \in MBS(N, v)$ and will get a contradiction. Let (z, T) be an objection against y. Then $V \subset T$. Assume first that $i \in T$. Then since $z(T) = 1, j \notin T$. Therefore $z(N \backslash T) = y(N \backslash T) = \frac{1}{4}$ and thus $z(N) = \frac{5}{4}$. Now $z_i > y_i = \frac{1}{4}$. Therefore $z(N \backslash \{i\}) < 1$, and thus one can construct a counter objection $(z', N \backslash \{i\})$ against (z, T). Thus any objection to y has a counter objection. Therefore $y \in MBS(N, v)$. Now if $i \notin T$, then $S \subset T$ and thus $j \in T$. Therefore $z_j > y_j > \frac{1}{4}$ and a counter objection $(z', N \backslash \{j\})$ to (z, T) can be constructed. In any case we have $y \in MBS(N, v)$, which contradicts the assumption that $MBS(N, v) = C(N, v)$. □

Since $SBS(N, v) \subset MBS(N, v)$, Theorem 2 implies that for a unanimity game (N, v), $SBS(N, v) = C(N, v)$. We do not know if the converse is true, i.e. if $SBS(N, v) = C(N, v)$, then (N, v) is a unanimity game. However, we note that for a weak simple game (N, v) we may have $SBS(N, v) \subsetneq MBS(N, v)$. For example, we consider the weighted majority game $G = [3; 2, 1, 1, 1]$. Then it is easy to check that $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \in MBS(N, v) \backslash SBS(N, v)$.

It is shown in Greenberg (1992) (see Theorem 4.2 there) that for a convex coalitional game, $SBS(N, v) = C(N, v)$. A question which was raised in Waisman (1992) is if the converse is true, i.e. does $SBS(N, v) = C(N, v)$ imply that (N, v) is a convex game? In Waisman (1992) it is shown that this is true for 3 person games and for 4 person symmetric games.

Finally, we note that since the consistent bargaining set $CB(N, v)$ of Dutta et al. (1989) coincides with $ISBS(N, v)$ (see Greenberg (1992)), the result in this work on $ISBS$ also holds for CB.

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