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Allocation of resources in a divisionalized firm

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Abstract

This paper addresses the problem of allocating resources in a divisionalized firm. The firm consists of n distinct divisions. Inputs have to be allocated across the divisions and the profit function of each division may in general depend on the whole allocation of inputs. The manager of the firm does not know the divisional profit functions and yet would like to allocate resources in a profit-maximizing way. We construct a mechanism whose Nash equilibria generate a profit-maximizing allocation of inputs as well as a system of (endogenously determined) transfer prices. Furthermore, the mechanism uses smaller strategy spaces and intuitively more acceptable evaluation measures than those proposed previously.

Keywords: Divisionalized firm; Profit maximizing allocations; Transfer prices

1. Introduction

This paper considers a large firm consisting of several distinct divisions. An excellent survey of this subject has been given by Radner (1986). Classical references are Knight (1921) and Coase (1937). The issues of incentives, externalities and asymmetric information play a dominant role in the analysis of the firm's decisions and actions.

A distinct line of literature, 'the principal–agent approach' (Grossman and Hart, 1983; and Holmstrom, 1982), mainly addresses the incentives and asymmetric information issues. The firm's manager is the principal who has to design a compensation scheme that would induce the divisional managers to take the desired course of action. This analysis relies heavily upon the Revelation Principle (Harris and Townsend, 1981; and Myerson, 1979).

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The Revelation Principle, however, focuses only on one equilibrium (the one where the agents truthfully report their characteristics) and does not preclude the existence of other, perhaps 'bad', equilibria. Examples and further elaborations can be found in Postlewaite and Schmeidler (1986) and Repullo (1985).

This paper adopts the implementation point of view by looking for a mechanism whose set of equilibria coincides with the set of optimal outcomes. We present a model of a divisionalized firm resembling the model in Groves and Loeb (1979) and construct a mechanism that will enable the manager to take profit-maximizing actions. A similar approach has been taken in the literature on implementing Walrasian allocations in economic environments (Hurwicz, 1979; Schmeidler, 1980; and Postlewaite and Wettstein, 1989).

Groves and Loeb (1979) constructed a mechanism that has a solution in dominant strategies. We examine the Nash equilibrium solution although it may be a less appealing solution concept. However, the mechanism uses smaller strategy spaces and intuitively more acceptable evaluation measures. We use finite-dimensional strategy spaces, whereas Groves and Loeb (1979) as well as the principal–agent literature used infinite-dimensional strategy spaces entailing the transmission of divisional profit functions. As regards the evaluation, individuals in our setup are judged basically on their own performance; in Groves and Loeb (1979) they were judged by the entire firm's performance (this point was raised in Radner, 1986).

Another distinct feature of our mechanism is that its Nash equilibria give rise to a system of 'transfer prices'. Thus the mechanism solves the problem of pricing resources within the firm when there is no outside market price to use.

The paper is organized as follows. In Section 2 we present the model. The mechanism and its properties are described and proved in Section 3, and in Section 4 we outline extensions and related questions.

2. The model

We consider a firm consisting of n distinct divisions. To simplify notation we assume it uses one input, but the results we obtain hold for any finite number of inputs. The firm starts with a positive amount of input and we assume the resource constraint implied by it is binding.[1] The input has to be allocated across the divisions, and each division has a profit function which may in general depend on the whole allocation of inputs (thus we allow for externalities). The firm's profits are the sum of the divisional profits.

The goal of the manager is to allocate the inputs in a profit–maximizing

[1] An alternative model where it buys the input at some given market price yields similar constructions and conclusions.

manner. Assuming the manager is completely informed, we have a standard maximization problem. The difficulties start once the manager does not know the divisional profit functions and has to extract that information from the divisional managers.

The precise informational assumptions we make are that each divisional manager knows his own profit function, whereas the firm's manager does not. With this information structure individuals are able to compute a best response given the strategies adopted by others. We examine the Nash equilibrium points of this mechanism, the motivation being twofold. First, those would be the only stable ones from which no individual would have an incentive to deviate (see Moulin, 1982, for related interpretations of the Nash equilibrium). Second, they could be viewed as the outcome of a learning process as in Jordan (1991). Thus the manager's problem is one of designing a mechanism whose Nash equilibria outcomes would be profit maximizing.

We let $\bar{x}$ denote the available amount of the input. $x = (x_1, \ldots, x_n)$ in R^n_+ denotes an allocation across divisions (x_i denoting the amount allocated to division i). An allocation is feasible if $\sum_{i=1}^n x_i \leq \bar{x}$. $f_i : R^N_+ \to R$ denotes the profit function of division i, mapping input allocations into profits realized.

The definition of the mechanism will involve the strategy spaces and the evaluation measures according to which the divisional managers are judged. We assume the division manager wants to maximize the evaluation received. This assumption, together with the fact that evaluations will be an increasing function of profits, implies that once input has been allocated, each division will realize the level of profits given by its corresponding profit function.

A mechanism H will consist of the following:

M_i = the strategy space of the ith manager, and $M = \pi_{i=1}^n M_i$.

$g : M \to R^n_+$ = an allocation function, $g = (g_1, \ldots, g_n)$, where g_i denotes the input allocated to division i.

$e_i : R \times M \to R$ = an evaluation measure for the manager of the ith division. It maps its realized profits and the announced n-tuple of strategies into an evaluation given by a real number.

In defining the Nash equilibrium of the mechanism we use the following notation: $m = (m_1, \ldots, m_n)$ denotes an n-tuple of strategies; for $\overline{m}$ in M_i let $(m_{-i}, \overline{m}) = (m_1, \ldots, m_{i-1}, \overline{m}, m_{i+1}, \ldots, m_n)$.

A Nash equilibrium (NE) of the mechanism is an n-tuple of strategies $m = (m_1, \ldots, m_n)$ that satisfies for all $i = 1, \ldots, n$:

$$e_i(f_i(g(m)), m) \geq e_i(f_i(g(m_{-i}, \overline{m})), m_{-i}, \overline{m}) \quad \text{for all } \overline{m} \in M_i .$$

We let $N(H)$ denote the set of input allocations generated by the NE of the mechanism.

In the next section we construct a mechanism which realizes every profit-maximizing allocation as a NE and all of whose NE are profit maximizing. Furthermore, the mechanism generates a system of transfer prices through which the respective costs of the divisions are assessed.

3. The mechanism

The strategy space for the ith manager will be $M_i = P_i \times K_i \times N_i$.

$$P_i = \left\{(z_1, \ldots, z_n) \in R^{n^2} \,\middle|\, \sum_{j=1}^{n} z_{js} = a \text{ for some } a \geq 0 \text{ for all } s = 1, \ldots, n\right\},$$

$$K_i = \left\{(y_1, \ldots, y_n) \in R^n \,\middle|\, \sum_{j=1}^{n} y_j = \overline{X}\right\}$$

and

$$N_i = N \text{ (the natural numbers)}.$$

A generic element of the strategy space is denoted by (p^i, k^i, n^i).

The strategy of manager i can be interpreted as follows: p^i is a set of n price vectors $(p^i_1, \ldots, p^i_n)$ in R^n. p^i_j stands for the prices manager i would like to see manager j charged. Thus p^i_{js} is the price division j should pay for the input used by division s. k^i is the input allocation suggested by manager i, with k^i_j denoting the amount of input allocated to division j.

The allocation function g yielding the distribution of the input is constructed as follows: $g(m) = \Sigma_{i=1}^{n} (n^i/N)k^i$, where $N = \Sigma_{i=1}^{n} n^i$.

Prior to defining the evaluation measures, an average price $\overline{p}$ is constructed. Let $v_i = \Sigma_{j,j' \neq i} |p^j - p^{j'}|$ and $v = \Sigma_{i=1}^{n} v_i$, and define

$$w_i = \begin{cases} \dfrac{v_i}{v}, & \text{if } v > 0, \\ \dfrac{1}{n}, & \text{if } v = 0. \end{cases}$$

The average price is now defined as $\overline{p} = \Sigma_{i=1}^{n} w_i p^i$.

By its construction, $\overline{p}$ will be in P_i as well. We note that if all managers announce the same price, no single manager can change the $\overline{p}$ reached by deviating from it.

The evaluation measures are defined using $\overline{p}$ as follows:

$$e_i(f_i(g(m), m)) = f_i(g(m)) - \overline{p}_i g(m).$$

$\overline{p}_i$ is the price vector according to which division i is charged, with $g(m)$ being the allocation of input the mechanism generates.

Denote by $F(f, \overline{X})$ the set of profit-maximizing allocations of $\overline{X}$.

Theorem 1. *If the following assumptions are satisfied*: (i) $N \geq 3$, (ii) f_j *is concave and continuously differentiable for all* $j = 1, \ldots, n$, *then*[2] $F(f, \overline{X}) \subseteq N(H)$.

Proof. Let $(\hat{x}_1, \ldots, \hat{x}_n)$ be a profit-maximizing allocation of $\overline{X}$ across the n divisions. Then $(\hat{x}_1, \ldots, \hat{x}_n)$ solves:

$$\max_{x_1, \ldots, x_n} f_1(x_1, \ldots, x_n) + \cdots + f_n(x_1, \ldots, x_n)$$
$$\text{s.t. } x_1 + \cdots + x_n = \overline{X} ,$$
$$x_1, \ldots, x_n \geq 0 .$$

Hence, it satisfies the first-order conditions ($\hat{\lambda}$ is the Lagrange multiplier and f_{js} the partial derivative of f_j with respect to x_s):

$$\sum_{j=1}^{n} f_{js} - \hat{\lambda} \leq 0 ; \qquad \hat{x}_s \left(\sum_{j=1}^{n} f_{js} - \hat{\lambda} \right) = 0 \quad \text{for all } s = 1, \ldots, n ,$$

$$\hat{x}_1 + \cdots + \hat{x}_n = \overline{X} ; \qquad \hat{\lambda} \geq 0 .$$

Define n^2 prices by

$$p_{js} = f_{js} + \frac{\hat{\lambda} - \sum_{j=1}^{n} f_{js}}{n} .$$

These n^2 prices satisfy

$$f_{js}(\hat{x}_1, \ldots, \hat{x}_n) \leq p_{js} \quad \text{and} \quad \hat{x}_s(f_{js} - p_{js}) = 0 \quad \text{for all } j, s = 1, \ldots, n , \tag{3.1}$$

and

$$\sum_{j=1}^{n} p_{js} = \hat{\lambda} \quad \text{for all } s = 1, \ldots, n . \tag{3.2}$$

Using these facts, we will show that the following n-tuple of strategies constitutes a NE yielding the profit-maximizing allocation $(\hat{x}_1, \ldots, \hat{x}_n)$:

$$k^i = (\hat{x}_1, \ldots, \hat{x}_n) ,$$
$$p^i = (p_1, \ldots, p_n) ,$$
$$\text{where } p_j = (p_{j1}, \ldots, p_{jn}) \quad \text{for } i = 1, \ldots, n, \; j = 1, \ldots, n ,$$
$$n^i = 1 .$$

We denote this n-tuple of strategies of m^*. By (3.2), $p^i \in P_i$.

These strategies clearly yield $(\hat{x}_1, \ldots, \hat{x}_n)$ as the input allocation. It remains to be shown that they form a NE.

Manager j's evaluation reduces with this choice of strategies to

[2] If $N < 3$, the average price cannot be defined and the mechanism fails.

$$e_j(m^*) = f_j(\hat{x}_1, \ldots, \hat{x}_n) - \sum_{s=1}^{n} p_{js}\hat{x}_s$$

By the way $\overline{p}$ is constructed, manager j cannot change the prices determined by the mechanism. We will show that $(\hat{x}_1, \ldots, \hat{x}_n)$ maximizes the manager's evaluation given the prices p_{js}, and thus the manager cannot gain by inducing a change in the input allocation. Therefore the strategy choice m_j^* maximizes manager j's evaluation, given the strategies adopted by the others.

To show that the evaluation is maximized, note that manager j would prefer the input allocation solving

$$\max f_j(y_1, \ldots, y_n) - \sum_{s=1}^{n} p_{js} y_s$$

$$\text{s.t.} \sum_{s=1}^{n} y_s = \overline{X},$$

$$y_1, \ldots, y_n \geq 0 .$$

Since f_j is concave and continuously differentiable, a solution to this problem is completely characterized by the following first-order conditions:

$$f_{js} - p_{js} - \lambda \leq 0 ; \qquad y_s(f_{js} - p_{js} - \lambda) = 0 \quad \text{for } s = 1, \ldots, n \text{ and} \sum_{j=1}^{n} y_s = \overline{X} .$$

These conditions are satisfied by $y_s = \hat{x}_s$ and $\lambda = 0$, as can be seen from (3.1).

Therefore $(\hat{x}_1, \ldots, \hat{x}_n)$ maximizes the evaluation of manager j, given the strategies of others. Since this holds for all j we have shown that m^* forms a NE.

Thus, any profit-maximizing allocation can be realized as a NE of the mechanism. Q.E.D.

The assumption that the resource constraint is binding is now used to prove that any NE of the mechanism is profit maximizing.

Theorem 2. *If the following are satisfied*: (i) $N \geq 3$, (ii) f_j *is continuous for all* $j = 1, \ldots, n$, *then* $N(H) \subseteq F(f, \overline{X})$.

Proof. Let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ and $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ be an input allocation and price system induced by a NE. From the construction of $\hat{x}$ it is obvious that any manager j can get, by announcing a large enough n_j, arbitrarily close to any feasible input allocation (regardless of the strategies used by others). Since f_j is continuous, the evaluation of manager j is continuous. $\hat{x}$ being induced by a NE, must then yield manager j the highest possible evaluation given the prevailing prices $\hat{p}$. If that were not so, the manager could, by announcing a large enough n_j

and, if necessary, changing the input demand requested, improve the evaluation. Hence $\hat{x}$ and $\hat{p}$ satisfy

$$f_j(\hat{x}_1, \ldots, \hat{x}_n) - \sum_{s=1}^{n} \hat{p}_{js}\hat{x}_s \geq f_j(y_1, \ldots, y_n) - \sum_{s=1}^{n} \hat{p}_{js}y_s$$

$$\text{for all } y_1, \ldots, y_n \text{ satisfying } \sum_{s=1}^{n} y_s = \overline{X}\,.$$

That holds for all $j = 1, \ldots, n$ and summing over j yields:

$$\sum_{j=1}^{n} f_j(\hat{x}_1, \ldots, \hat{x}_n) - \sum_{j=1}^{n}\sum_{s=1}^{n} \hat{p}_{js}\hat{x}_s \geq \sum_{j=1}^{n} f_j(y_1, \ldots, y_n) - \sum_{j=1}^{n}\sum_{s=1}^{n} \hat{p}_{js}y_s$$

$$\text{for all } y_1, \ldots, y_n \text{ satisfying } \sum_{s=1}^{n} y_s = \overline{X}\,.$$

Since $\hat{p} \in P_i$ and $\sum_{s=1}^{n} y_s = \sum_{s=1}^{n} \hat{x}_s$, the above reduces to

$$\sum_{j=1}^{n} f_j(\hat{x}_1, \ldots, \hat{x}_n) \geq \sum_{j=1}^{n} f_j(y_1, \ldots, y_n)$$

$$\text{for all } y_1, \ldots, y_n \text{ satisfying } \sum_{s=1}^{n} y_s = \overline{X}$$

or, in other words, $\hat{x}$ is a profit-maximizing allocation. Q.E.D.

4. Concluding remarks

We have constructed a mechanism that yields profit-maximizing allocations in a divisionalized firm. The mechanism does not entail sending in profit functions. The divisions send in input requests, price suggestions and an additional parameter that plays a role in 'averaging' the different requests into a final input allocation. Furthermore, the prices sent in yield in equilibrium a system of transfer prices used to impute costs to the various divisions.

The mechanism also allows for externalities. If externalities are ruled out, the strategy space can be drastically simplified. Each division will send in a single price (one dimensional), and an input request for itself only, as well as an 'averaging number'.

The outcome function of the mechanism is continuous and feasible. Continuity is of importance since it implies the mechanism will yield nearly optimal results even in the presence of small perturbations and mistakes in the behavior and parameters of the participating individuals. Feasibility, in and out of equilibrium, guarantees that if for some reason, the mechanism is stopped short of reaching an equilibrium, it still prescribes a well-defined and feasible input allocation.

It remains to be seen whether the implementation approach used in this paper can be extended to more general environments. One direction of further research would introduce effort and effort aversion on the part of managers.

A further extension would be to consider environments with privately observed correlated shocks to the profit functions. This would involve the techniques appearing in Palfrey and Srivastava (1989) and Postlewaite and Schmeidler (1986).

References

R.H. Coase, The nature of the firm, Economica 4 (1937) 386–405.

S.J. Grossman and O.D. Hart, An analysis of the principal–agent problem, Econometrica 51 (1983) 7–45.

T. Groves and M. Loeb, Incentives in a divisionalized firm, Management Sci. 25 (1979) 221–230.

M. Harris and R. Townsend, Resource allocation under asymmetric information, Econometrica 49 (1981) 33–64.

B. Holmstrom, Moral hazard in teams, Bell J. Econom. 13 (1982) 324–340.

L. Hurwicz, Outcome functions yielding Walrasian and Lindahl allocations at Nash Equilibrium points, Rev. Econom. Studies 46 (1979) 217–225.

J.S. Jordan, Bayesian learning in normal form games, Games and Economic Behavior 3 (1991) 60–81.

F.H. Knight, Risk, Uncertainty and Profit (Houghton Mifflin Company, Cambridge, MA, 1921, reprinted by Kelley and Millman, New York, 1957).

H. Moulin, Game Theory for the Social Sciences (New York University Press, 1982).

R. Myerson, Incentive compatibility and the bargaining problem, Econometrica 47 (1979) 61–73.

T. Palfrey and S. Srivastava, Implementation with incomplete information in exchange economies, Econometrica 57 (1989) 115–134.

A. Postlewaite and D. Schmeidler, Implementation in differential information economies, J. Economic Theory 39 (1986) 14–33.

A. Postlewaite and D. Wettstein, Continuous and feasible implementation, Rev. Econom. Studies 56 (1989) 603–611.

R. Radner, The internal economy of large firms, Econom. J. Supplement 96 (1986) 1–22.

R. Repullo, Implementation in dominant strategies under complete and incomplete information, Rev. Econom. Studies 52 (1985) 223–229.

D. Schmeidler, Walrasian analysis via strategic outcome functions, Econometrica 48 (1980) 1585–1593.