ACKNOWLEDGEMENTS

I have spent five enjoyable years working towards my PhD under the supervision of two wonderful advisors: Gadi Fibich and Shimshon Bar-Ad.

Gadi, being an Applied Mathematician, has taught me asymptotic analysis, perturbation analysis and a lot of numerics, but more importantly, he taught me how to conduct research in the highest level. Gadi is most likely as close as possible to being the perfect advisor: a genuine perfectionist who is involved in every detail, from how to phrase the shortest e-mail, through paper writing and presentation preparation, to the most important decisions in my career. Gadi is patient, generous and dedicated; he knows when to give freedom and when to press on the gas pedal and was always there to give a good advise.

Shimshon, being an experimental physicist, has taught me a lot about optics and condensed-matter physics, but also how to treat the problems at hand as a physicist. I am grateful to Shimshon for his endless patience, his willingness to address theoretical issues which are way aside from his expertise, for his constant encouragement and inspiration and for his personal advise.

I was fortunate to have such two advisors; I have enjoyed, and hope to enjoy in the future, their pleasant company and friendship. The possibility to benefit from the two worlds of Mathematics and Physics has made me a better scientist and it is through Gadi's and Shimshon's guidance that I have been able to complete, what I believe to be, an interdisciplinary and prolific PhD.

During these years, we have enjoyed fruitful collaborations with Michael Weinstein at Columbia University and with Boaz Ilan, Gadi's former student, currently at University of California at Merced. I am deeply grateful to Michael, who assisted me on many occasions and to Boaz for many long and fruitful discussions.

During my PhD I also had the opportunity to work on a related project in collaboration with the group of Arie Zigler from the Hebrew University. This collaboration has enriched me with new knowledge and with new friends.

I am indebted to Applied Materials, Inc. and to the Marejn family for the generous support I received from their foundations during my graduate studies.

Last but not least, it is through the love and support of Orit, my wife, that I have been able to dedicate so much to research. I thank her for her understanding, support and encouragement, and for adding some humor to the day-to-day work.

Yonatan Sivan,

Tel Aviv, Israel, 2008

Abstract

Solitons are a special type of nonlinear waves that maintain their shape along the propagation. They are a fundamental phenomenon in nonlinear dynamics and have attracted the attention of researchers from the physical and mathematical sciences over the last four decades. Solitons were found in water waves, solid-state physics, plasma physics, particle physics, biological systems, Bose-Einstein-condensation and nonlinear optics. Soliton research interests span from theoretical aspects such as soliton existence, computation of soliton profiles and soliton stability theory, through aspects such as soliton dynamics and soliton interactions, to applicative aspects.

Soliton research was particularly prolific in the field of nonlinear optics. Indeed, the number of nonlinear materials that are fully characterized by soliton-equations and the types of solitons discovered in them seems to be on a steady growth. Moreover, the ability to sample the waves directly as they propagate results in a field in which theory and experiments make rapid progress hand-in-hand.

A major part of the research on optical solitons was dedicated to the investigation of soliton formation in nonlinear inhomogeneous media, i.e., in media where the refractive properties vary in space. Indeed, since the 1980's, advancements in microfabrication technology make it possible to modulate the refractive properties of various materials. Such materials have a variety of industrial applications and are also very interesting from the theoretical point of view due to their analogy they provide to other fields of physics.

Although originally the research interests behind the investigation of light propagation in inhomogeneous media involved *linear* waves, the investigation of *nonlinear* effects, and solitons in particular, became equally important. Due to the variety of soliton-supporting media and the variety of available inhomogeneities, the number of different physical configurations in which solitons were studied became exceptionally large. So far, soliton stability and dynamics in each of these configurations was studied separately so that this class of studies has accumulated into a huge body of research. However, surprisingly, despite the multitude of papers on the problem, there is no *simple and complete* theory with which the soliton stability and instability dynamics can be predicted *qualitatively* and equally important, also *quantitatively*.

My PhD research provided exactly such theory for the fundamental, positive, bright solitons. This theory consists of a *qualitative* characterization of the type of instability, and a *quantitative* estimation of the instability rate and the strength of stability. It reveals the strong similarity between many different configurations which, *a priory*, look very different from each other and were thus studied separately until now. Our theory evolved through our first papers of solitons in media with nonlinear inhomogeneities, which were at the time, a novel type of materials. Our theory was extended in our later papers to the much better explored solitons in media with linear inhomogeneities and was finally presented in a mature and coherent way in a summary paper which was just submitted for publication. My Thesis is based on these papers.

Although the majority of the examples given in our papers were given for the standard cubic nonlinearity and for periodic or almost periodic inhomogeneities, the generality of our arguments and the multitude of the supporting numerical evidences implies that our qualitative and quantitative approaches apply to positive solitons in any dimension, any

type of nonlinearity and any inhomogeneity. As a final note, it should be said that we believe that a formulation of a qualitative and quantitative theory for "high-order" solitons may be possible, but requires further study. I personally hope that my Thesis will inspire such a study.

Contents

1 Introduction and outline

A general property of Electro-Magnetic wave-packets is that they tend to spread out as they propagate. A fundamental cause for this is that distinct frequency components, which are superposed to create the wave-packet, propagate with different velocities and/or in different directions. In particular, in a *temporal* wave-packet, a pulse, each Fourier component of the pulse has a different velocity due to the group-velocity *dispersion* (GVD); consequently, the Fourier components which initially overlap perfectly, get further away from each other and the pulse spreads in time. In a *spatial* wave-packet, a beam, a similar spreading of a laser beam occurs due to the different angles associated with each spatial frequency component; it is referred to as *diffraction*.

This description pertains to linear propagation of beams or pulses. Nonlinear effects generally accelerate the disintegration of a wave-packet. However, under special conditions, the nonlinearity may compensate the linear effects of dispersion/diffraction. The resulting balanced localized pulse or beam propagates without any change of its profile, and is generally known as a *soliton*.

Frequently, additional properties are attributed to a soliton, e.g., being stable, belonging to an integrable nonlinear-wave model, and most importantly, having particle-like interactions such as elastic collisions etc.. Solitons which lack some of these properties are usually called *solitary waves* in the mathematical literature. However, the more recent nomenclature is less strict, and it is now customary to refer to all self-trapped waves as solitons, see e.g., [125].

The study of solitons is important from various aspects. First, solitons are, a fundamental nonlinear wave phenomenon which has been demonstrated to exist in many physical systems: surface waves in shallow water [102], plasma waves [83], high-energy physics, sound waves in ^{3}He [98], localized vibrational modes in biological systems [136], matter waves in Bose-Einstein Condensates [2, 74] and nonlinear waves in optics. Despite the diversity of the mechanisms under which they can be created, many of the soliton properties are universal. Second, the soliton serves as a *nonlinear attractor*, i.e., any initial beam with arbitrary shape reshapes itself into a soliton (or several solitons) and radiates off excess power. This property was proved rigorously for exactly integrable models such as the one-dimensional Nonlinear Schrödinger equation, see below. However, it is believed that this property is a general characteristic of most, if not all, nonlinear systems [107]. Therefore, understanding soliton dynamics is crucial for understanding the dynamics of many non-solitonic waves. Third, the availability of numerous material systems that are fully characterized by soliton equations, and the ability to sample the waves directly as they propagate result in a field in which theory and experiments make rapid progress hand-in-hand. Finally, solitons have several obvious applications in Communications systems. For example, a commercial fibreoptic telecommunications link using solitons (actually, "dispersion-managed" solitons, for which the local dispersion coefficient varies periodically between positive and negative values, see below), about 3000 km long, was launched in Australia in 2003.

In my Thesis, I discuss solitons in the context of nonlinear optics. Accordingly, I adopt the optics terminology for the soliton dynamics. However, the results described below are relevant to solitons in other branches of physics.

The outline of the Thesis is as follows. In Section 2, the model of the Nonlinear

Schrödinger Equation (NLSE) is introduced and the various types of soliton-supporting media are reviewed. It should be noted that Section 2.2.2 includes an elaborate discussion of the nonlinearity due the process of Stimulated-Raman-Scattering (SRS). Although this subject is not directly related to the main results of the Thesis, this discussion is included here since it describes results that were obtained as a side project, but were not published in the papers on which my Thesis is based.

Section 3 includes a description of the types of optical solitons, their theoretical prediction and experimental realization. In Section 4, a brief review on the research on soliton propagation in inhomogeneous media is given.

Section 5 presents the major aspects in soliton theoretical research that were addresses in my research. In particular, Section 5.1 discusses the analytical and numerical methods for the computation of soliton profiles with emphasize on the contribution of our research. Our contribution to the development of numerical methods for soliton computation is given in much more detail in Appendix B since it was a crucial part in all our papers but was not published in a coherent and complete way in any of them. Our contribution to the development of analytical methods for soliton computation can be found in [44]. Section 5.2 gives a historical overview of the evolution of soliton stability theory. This overview is concluded with Theorem 5.1 which is the starting point of our research.

Section 6 gives a brief review of the current status of knowledge on soliton stability with emphasize on inhomogeneous media. It poses the research problem addressed in my Thesis - to provide a qualitative and quantitative theory for the stability and instability dynamics of positive bright solitons. Section 6.2 describes our qualitative and quantitative approachs in detail. Section 7 contains a detailed description of the motivation and results behind each of our papers; the description of each paper is followed by paper itself. My Thesis results are summarized in Section 8.

2 Light propagation in nonlinear materials

2.1 Nonlinear Schrödinger model

The propagation of paraxial, linearly polarized, time-harmonic beams is modeled by the Nonlinear Schrödinger Equation (NLSE) which in the MKS/SI unit system is given by [22]

$$2ik_0 A_z(z,\vec{x}) + \nabla^2 A + \frac{k_0^2}{n_0^2}\Delta n\left(|A|^2\right)A = 0. \tag{2.1}$$

Here, the field $A(z,\vec{x})$ corresponds to the complex amplitude of the electric field E

$$E(z,\vec{x},t) = A(z,\vec{x})e^{-i\omega_0 t} + c.c., \tag{2.2}$$

where ω_0 is the wave frequency, $k_0 = k(\omega_0) = \omega_0 n_0(\omega_0)/c$ denotes the wavenumber in the medium, c is the speed of light in vacuum and n_0 is the linear refractive index of the medium. The coordinate z is the direction of propagation, $\vec{x} = (x_1,\ldots,x_D)$ is the transverse D-dimensional space and $\nabla^2 = \partial_{x_1x_1} + \cdots + \partial_{x_Dx_D}$ is the D-dimensional Laplacian operator. For example, in a planar (slab) geometry, light is confined in the vertical direction and no dynamics occurs in this direction. Hence, in this case, $D = 1$. In bulk medium, transverse dynamics is possible in both dimensions so that $D = 2$. The refractive index change Δn is a function of the intensity $|A|^2$. In my Thesis only focusing (positive) nonlinearities were considered. Several functional dependencies of $\Delta n\left(|A|^2\right)$ are described in Section 2.2. In this formulation, the vectorial nature of the EM waves is neglected because we assume the beam to be primarily linearly polarized. For a proper description of the vector wave equations, see [42].

When the laser emits a pulse of finite time duration rather than a monochromatic, continuous-wave (CW) beam, the electromagnetic field is no longer time-harmonic, and the propagation involves temporal dynamics. In practice, for laser pulses of nanosecond duration or longer, the temporal dynamics can be neglected, hence their propagation can still be described by the stationary model for CW beams (2.1). However, in the case of ultrashort pulses (a few picoseconds or shorter), temporal dynamics become important. The leading order linear effect is the group velocity dispersion (GVD) which describes the variation of the phase velocity with the frequency. By the Kramers-Krönig relations, GVD is accompanied by absorption in some part of the spectrum. If the linear absorption occurs at frequencies very different from the optical frequency ω_0, the pulse propagation is governed by

$$2ik_0 A_z(z,\vec{x},\tau) + \nabla^2 A - \beta_2 A_{\tau\tau} + \frac{k_0^2}{n_0^2}\Delta n\left(|A|^2\right)A = 0, \tag{2.3}$$

where $\beta_2 = \frac{\partial^2 k(\omega)}{\partial\omega^2}\Big|_{\omega_0}$ is the GVD coefficient and $\tau = t - z/v_g$ is the time coordinate[1] in a frame of reference moving at the group velocity $v_g = \left(\frac{\partial k}{\partial\omega}\Big|_{\omega_0}\right)^{-1}$. In the case of anomalous dispersion $\beta_2 < 0$, the time coordinate has the same mathematical form as the spatial transverse coordinate. Accordingly, τ can be viewed as an additional transverse coordinate

[1] It is sometimes referred to as the retarded-time coordinate.

so that Eq. (2.3) can be rewritten as Eq. (2.1) where now $\vec{x}$ and ∇^2 are the transverse coordinate and Laplacian operator in d spatial and/or temporal transverse dimensions. In the case of normal dispersion, no solitons can be formed in a self-focusing medium. They can form in a self-defocusing medium, however, this case is out of the scope of my Thesis.

As can be seen from Eq. (2.1), the paraxial approximation creates a distinction between the direction of propagation z and the transverse coordinates. Hence, it is useful to adopt the nomenclature $D+1$ NLSE which means that the beam can diffract and/or disperse in D (spatial and or temporal) transverse dimensions as it propagates in the z direction.

2.2 Types of nonlinearities

In this Section, the most common nonlinearities are described according to their spatial and temporal dependencies.

2.2.1 Instantaneous, local nonlinearities

The most common nonlinearity in optical materials is the cubic nonlinearity, usually named after the Scottish physicist John Kerr (1824-1907). Many physical mechanisms can give rise to a cubic nonlinearity, e.g., non-resonant electrons excitation [22, Chapter 4.3], reorientational [22, Chapter 4.4] or electrostriction nonlinearities [22, Chapter 9.2]. Kerr nonlinearity is found in all materials in nature, and it is the leading order nonlinear effect in fluids and in solids with inversion symmetry. It is characterized by

$$\Delta n_{cubic} = 4n_0 n_2(x,y,z)|A|^2(x,y,z,t), \tag{2.4}$$

where n_2 is the nonlinear refractive index or Kerr coefficient. If the nonlinearity is derived from a microscopic model of a single atom (such as a quantum mechanical computation of the susceptibility of a single atom [22, Chapter 4.3]), then n_2 is related to the third-order susceptibility $\chi^{(3)}$ through [22]

$$n_2 = \frac{3}{4n_0}\chi^{(3)}.$$

The Kerr coefficient can vary in space (see Sections 4 or 7.1) but for media with instantaneous response, it does not depend on time.

One of the important and less obvious mechanisms which can lead to an effective cubic nonlinearity is the *quadratic* nonlinearity. Such nonlinearities give rise to the process of Second Harmonic Generation (SHG), which is modeled by a set of two coupled differential equations for the dynamics of the fundamental frequency (FF) and second-harmonic (SH) fields [22, 85]. The efficiency of the interaction depends on the phase mismatch parameter $\Delta k \equiv 2k_1 - k_2$ where $k_j = \omega_{0j} n_{0j}(\omega_{0j})/c$, ω_{0j} and n_{0j} are the wavevectors, central frequencies and refractive indices of the FF and SH fields, respectively. The interaction is efficient close to the point of phase matching ($\Delta k = 0$). In this case, genuine "quadratic" parametric interaction occurs. Far from phase matching ($\Delta k \approx \pi$), the parametric interaction leads to a rapid exchange of energy between the two frequency components. In this so-called *cascading limit*, the coupled equations for the amplitudes of the FF and SH fields can be approximated as an effective cubic NLSE for each frequency component, i.e., Eq. (2.1) with

a Kerr nonlinearity (2.4), where the cubic coefficient is inversely proportional to the phase mismatch Δk. Since processes involving quadratic nonlinearities require significantly lower light intensities compared with processes involving cubic nonlinearities, quadratic materials are sometimes a preferable substitute for experimental realization of processes typical to cubic nonlinearities [135].

The Kerr nonlinearity is sometimes accompanied by additional high-order nonlinearities. These can be divided into two types. The first type is of high-order defocusing nonlinearities, e.g.,

$$\Delta n_{cubic-quintic} \sim n_2(x,y,z)|A|^2(x,y,z,t) - n_4(x,y,z)|A|^4(x,y,z,t) + \cdots ,$$

where $n_4 > 0$ is the coefficient of the quintic nonlinearity. Such cubic-quintic nonlinearity can be found in chalcogenide glasses [120, 20] or in organic materials [138, 129] and results in a saturation of the nonlinear index change, i.e., the maximal index change is bounded from above. Purely quintic nonlinearity

$$\Delta n_{quintic} \sim -n_4(x,y,z)|A|^4(x,y,z,t) \tag{2.5}$$

arises in the context of a Tonks-Girardeau gas, which is a system of one-dimensional bosons with "impenetrable core" repulsive interactions [73, 25]. It can also appear in quadratic materials if the competing internal and cascaded third order nonlinearities are perfectly balanced [90]. Purely quintic nonlinearity is also commonly used as a pedagogical model for the study of the critical and supercritical NLSE in a one-dimensional setting, see e.g., [44, 9, 78].

The second type is of nonlinear (high-order) absorption. Indeed, at frequencies far from any resonances the Kerr coefficient is very weak; it becomes stronger only near a Four-Wave resonance, i.e., if some Four-Wave mixing combination coincides with an electronic level. In this case, the Kerr coefficient is also accompanied by two-photon absorption, i.e., Δn becomes complex[2].

In some cases, the number of photons absorbed in the nonlinear process is higher. This number is determined by the ratio between E_{gap}, the energy difference between the valence and conduction bands (or more generally, the energy difference between the ground-state and the excited electronic level) and $\hbar\omega$, the photon energy. For semiconductors, for which the energy gap is not large, two-photon absorption is the leading order nonlinear absorption process, see [22, Chapter 4.6], and it can have a non-negligible effect on the nonlinearity, see e.g., [89]. On the other hand, in dielectric materials, for which the energy gap is relatively large, a large number of photons is needed for multi-photon absorption, e.g., six-photon-absorption in fused silica [13] or eight-photon-absorption for atmospheric gases [119]. However, such multi-photon absorption processes are significantly weaker.

2.2.2 Non-instantaneous nonlinearities

All nonlinearities so far were described by an instantaneous (temporal) nonlinear response to the electric field[3]. However, many materials have a *non*-instantaneous nonlinear response.

[2]By the Kramers-Krönig relations, in this case, short pulses can experience also nonlinear dispersion, see Section 2.2.2.

[3]All nonlinearities are actually non-instantaneous, because even in the fastest nonlinear medium, the shortest response time is the life-time (or the dephasing time) of the electronic levels. Hence, by instantaneous

In such media, ultrashort pulses experience a somewhat different nonlinearity compared to that experienced by CW beams due to the associated nonlinear dispersion and nonlinear absorption.

For cubic (2.4) and isotropic materials, the change in the index of refraction due to the third-order polarization can be derived from a microscopic quantum-mechanical model of a single-atom, see e.g. [57, 22]. It is described by

$$\begin{aligned}\Delta n_{non-inst}(t) &\sim \int_0^\infty \int_0^\infty \int_0^\infty d\tau_1 d\tau_2 d\tau_3 R^{(3)}(\tau_1,\tau_2,\tau_3) E(t-\tau_1)E(t-\tau_2)E(t-\tau_3) \qquad (2.6)\\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty d\omega_1 d\omega_2 d\omega_3 \chi^{(3)}(\omega_1,\omega_2,\omega_3)\hat{E}(\omega_1)\hat{E}(\omega_2)\hat{E}(\omega_3)e^{-i(\omega_1+\omega_2+\omega_3)t},\end{aligned}$$

where the third-order susceptibility $\chi^{(3)}$ is the Fourier transform of the third-order response function $R^{(3)}$. Note that in order to maintain casuality, i.e., to ensure that fields of future times (i.e., for $\tau < 0$) do not affect the polarization at time t, the response for earlier times must vanish, i.e., $R^{(3)}(\tau_j < 0) = 0$. Hence, the lower limits of the integrals in Eq. (2.6) are set to 0[4].

From the Kramers-Krönig relations, it follows that for a delayed response ($R^{(3)} \neq \delta(\tau_1)\delta(\tau_2)\delta(\tau_3)$), the susceptibility $\chi^{(3)}$ must be complex and must depend on the frequencies ω_j. In other words, the nonlinear delayed response corresponds to a nonlinear dispersion and a conjugate nonlinear absorption. The easiest way to see that is in the CW limit where it follows from Eq. (2.6) that the contribution of the fundamental frequency to the change in the index of refraction scales as

$$\Delta n \sim \chi^{(3)}(\omega_0,\omega_0,-\omega_0)E^3,$$

i.e., for a complex $\chi^{(3)}$, Δn includes absorption[5], see also Section 2.2.1. Indeed, it can be shown that in this case, the number of photons $\mathcal{N} = \int |A|^2 d\vec{x}dt$ and the total energy $\mathcal{E} = \int E^2 d\vec{x}dt$ are not conserved. This absorption is significant only near a Four-Wave resonance. This occurs, e.g., for UV radiation in dielectrics or optical radiation in semiconductors. Otherwise, the absorption and the delayed part of the electronic response are negligible so that the index change is essentially instantaneous and real.

Despite being most general, the nonlinearity described by Eq. (2.6) is cumbersome and thus, seldom used. Fortunately, usually Eq. (2.6) can be simplified using the Born-Oppenheimer (BO) approximation [57]. In this approximation, the computation of the quantum average of any observable, in particular, the third-order polarization, is made under the assumption that the nuclei-lattice/molecules are frozen. Then, taking the degrees of freedom of the nuclei-lattice/molecule into account gives rise to electronic states dressed with vibrational levels known as *phonons* (in solid-state) or with molecular rotational levels

response we mean that the life-time of the electronic level is significantly shorter compared with the other time scales of the problem, i.e., the pulse duration and the lifetime of the vibrational (phonon) levels.

[4]This requirement on the response function corresponds to the requirement that the corresponding susceptibility would not have poles in the upper-half of the complex plane. This requirement is satisfied by standard expressions for the corresponding susceptibility such as Two-Level-System based susceptibilities with a phenomenological dissipation mechanism in the form of a finite line-width transition, see e.g., [22].

[5]The absorption originates from the finite spectral width of the pulse and also from the finite spectral width of the electronic level.

(in fluids). These levels have slightly higher energy than the corresponding purely electronic state but they are *always* well separated from the other electronic levels.

Typically, optical frequencies lie well below resonance with the excited electronic levels but still well above the vibrational/rotational levels[6]. As noted above, in such cases, the absorption due to electronic transitions is negligible so that the response of the electrons to the electric field is essentially instantaneous. Nevertheless, a third-order process can be resonant with one of the dressed levels. In such a process, a photon is scattered off the nuclei-lattice/molecule, excites a vibrational/rotational mode and becomes red-shifted. The process is usually referred to as a Stimulated-Raman-Scattering (SRS) process and is the leading order effect of the nonlinear dispersion.

In Appendix A it is shown that under the BO approximation, the change in the nonlinear index of refraction due to SRS can be approximated by

$$\Delta n_{SRS} \sim 3(1-f_R)|A|^2 + 2f_R \int_0^\infty h_R(\tau)|A(t-\tau)|^2 d\tau, \tag{2.7}$$

where h_R is the response function computed from the ground-state electronic level and the vibrational/rotational levels only. It should be emphasized that as described in detail in Appendix A, previous derivations of the change in the nonlinear index of refraction due to SRS either had algebraic mistakes or gave an incomplete justification of the derivation. Hence, Appendix A provides the *only* fully-justified derivation of Eq. (2.7). It can be shown that in this case there is no change in the photon number $\mathcal{N}$, in agreement with the physical interpretation of the SRS process. However, the total electromagnetic energy $\mathcal{E}$ decreases due to the energy transfer to the vibrational/rotational degrees of freedom.

We note that SRS occurs only if the corresponding selection rules allow it. For example, SRS occurs in silica and in molecular gases but not in AlGaAs or water. Naturally, it is also absent in noble gases.

For sufficiently long pulses, for which the pulse duration exceeds the lifetime or inverse frequency of the vibrational/rotational level, the integral law (2.7) can be reduced to a differential form, see e.g., [5]. However, it should be noted that in many cases, SRS can cause the pulse to develop modulations which are much shorter than its initial duration (e.g., due to soliton fission or due to radiation related to high-order dispersion [8]). In these cases, the differential model might not be valid anymore, see e.g. [23].

An additional contribution to the nonlinear index change which is unique to intense ultrashort pulses is the effect of Self-Steepening. Under this effect, the nonlinear index change can be approximated as

$$\Delta n_{SS} \sim \left(1+\delta\frac{\partial}{\partial t}\right)^2 \Delta n_{SRS} \cong \left(1+2\delta\frac{\partial}{\partial t}\right)\Delta n_{SRS}. \tag{2.8}$$

This term originates from the second derivative in time of the third-order polarization in

[6]Specifically, the detuning ($\Delta \equiv \omega_{elec} - \omega_0$) of the laser (central) frequency ω_0 from the characteristic frequency for electronic transitions, ω_{elec} is usually much larger than the pulse spectral width $\Delta\omega$, i.e., $\Delta \gg \Delta\omega$ or $T \gg 2\pi/\Delta$ where $T \sim 2\pi/\Delta\omega$ is the pulse duration. It can be shown that for most dielectric materials, this approximation is valid even for pulses as short as a few fs.

the wave equation, see e.g., [22, Chapter 13][7]. This effect results in a shock formation in the trailing edge of the pulse and an asymmetry in the Self-Phase-Modulation spectral broadening [5]. Still, Self-Steepening does not prevent the formation of *symmetric* solitons with a tilted, asymmetric phase front [36].

Finally, it should be noted that the nonlinear response of materials such as photorefractives has no instantaneous component at all, and the typical response time can be as long as several milliseconds[8]. In this case, the nonlinear response of the medium can be slower than the random phase fluctuations of the incident optical fields. Then, the medium responds only to the time-averaged intensity of the pulse (where the average is to be taken over times much longer than the response time of the nonlinearity), and the optical fields should be treated as incoherent light, see e.g., [71, Chapter 13].

2.2.3 Non-local nonlinearities

An additional class of nonlinearities which has attracted a lot of interest in recent years is a nonlinearity which is non-local in space [121]. Such nonlinearities appear, e.g., in photorefractive materials [109], liquid crystals [70] and materials that exhibit laser-induced thermal nonlinearities [53][9]. The nonlinearity in such materials is usually derived from a macroscopic phenomenological model of many molecules or an atom-lattice model, see e.g., [22]. In a similar manner to short pulses in non-instantaneous media, narrow beams experience a somewhat different nonlinearity compared to that experienced by wide beams (plane waves).

In such materials, rather than depending on the local intensity, the change of the refractive index depends on the beam power [121][10]

$$\Delta n_{non-local} = \Delta n_{local} - \alpha|\vec{x}|^2 P, \qquad P \sim \int_{-\infty}^{\infty} E^2(\vec{x})d\vec{x} \cong 2\int_{-\infty}^{\infty} |A(\vec{x})|^2 d\vec{x}, \tag{2.9}$$

where Δn_{local} and α are constants, or more generally, on [137, 51, 14]

$$\Delta n_{non-local} \sim \int_{-\infty}^{\infty} R(\vec{\xi})|A(\vec{x}-\vec{\xi})|^2 d\vec{\xi}, \tag{2.10}$$

where the paraxial approximation (i.e., $E^2 \cong 2|A|^2$) was used[11]. In Eq. (2.10), the function R represents the spatial response of the medium and is usually modeled by a localized function (e.g., a Gaussian) whose integral is normalized to 1. The strength of the non-locality is

[7] In several works, the origin of the Self-Steepening is attributed to the intensity-dependence of the group velocity. Indeed, one can see from Eq. (2.8) that in the purely instantaneous limit ($f_R \to 0$), the Self-Steepening term is proportional to $|A|^2 A_t$. However, the Self-Steepening term includes an additional contribution which is proportional to $A^2 A_t^*$. Therefore, the above claim is only partially correct.

[8] The nonlinearity in such materials is derived from a phenomenological model of the electron dynamics. Accordingly, the Kramers-Krönig relations do not necessarily apply to these materials.

[9] In such materials, like in materials with non-instantaneous nonlinear response, the Kramers-Krönig relations do not necessarily apply.

[10] The beam power should not be confused with the photon number $\mathcal{N} = \int |A|^2 d\vec{x}dt$ which involves integration over $|A|^2$ rather than on E^2 and additional integration over time.

[11] In fact, the correct response integral should look like the non-instantaneous response (2.6), i.e., with each field appearing with its own variable (as in [109]) and also include the second harmonic term $A^2 e^{2ik_0 z}$.

determined by the spatial extent of the function R (e.g., the Gaussian width) and is usually called the Debye length.

When the beam width exceeds the Debye length, the effect of the non-locality is weak and the electric field in Eq. (2.10) can be expanded in a Taylor series for small values of ξ, see e.g., [109]. The resulting nonlinearity has a local and usually a saturable nature. For example, for photorefractive materials [110], the nonlinear index change is given by

$$\Delta n_{photorefractive} \sim E_{bias} \frac{1}{1+|A|^2},$$

and for photovoltaic materials [132], the nonlinear index change is given by

$$\Delta n_{photovoltaic} \sim E_{bias} \frac{|A|^2}{1+|A|^2}.$$

The externally applied (bias) DC field E_{bias} can be polarized in the same or in a different plane as the optical field. It determines the sign and magnitude of the nonlinear index change and serves as an easily tunable parameter. For a detailed discussion of the complex physics in these types of materials, see e.g., [22, Chapters 11.5,11.6] and [33].

3 Bright solitons in homogeneous media

In order to discuss solitons in a general setting, we derive the dimensionless NLSE by introducing the standard non-dimensional variables

$$\tilde{z} = \frac{z}{2k_0 r_0^2}, \quad \tilde{x} = \frac{x}{r_0}, \quad \tilde{A} = \frac{n_0}{k_0 r_0} A, \tag{3.1}$$

where r_0 is the beam width. Substituting the rescaling (3.1) in Eq. (2.1) gives

$$iA_z(z, \vec{x}) + \nabla^2 A + \Delta n(|A|^2)A = 0, \tag{3.2}$$

where for simplicity of notation, the tildes were dropped. In order to find the equation for solitons, we seek nonlinear bound-states of Eq. (3.2) of the form

$$A(z, \vec{x}) = e^{i\nu z} u(\vec{x}; \nu),$$

where u is a real function. Therefore, the equation for the soliton profile is

$$\nabla^2 u(\vec{x}; \nu) + \Delta n(|u|^2)u - \nu u = 0. \tag{3.3}$$

The parameter ν is sometimes called the propagation constant or the soliton eigenvalue (a term which originates from the context of the linear Schrödinger equation). It describes the phase accumulated due to the nonlinear effect, or equivalently, the change of the linear propagation constant ($k_0 = \vec{k} \cdot \hat{z}$) due to the nonlinear effect.

Since the soliton amplitude is a radially-symmetric localized function[12], the boundary conditions which accompany the soliton equation (3.3) are

$$u'(|\vec{x}| = 0) = 0, \qquad u(\infty) = 0.$$

3.1 Types of solitons

The simplest solution of Eq. (3.3) is the single-hump soliton. Generically, the single-hump soliton is the eigenfunction corresponding to the smallest eigenvalue of the nonlinear Schrdinger operator

$$L_- = -\nabla^2 - \Delta n(|u|^2) + \nu.$$

Accordingly, it is sometimes referred to as the *ground-state* or zero-order soliton. The single-hump soliton is also typically the solution of Eq. (3.3) with the least power where in this context (i.e., under the paraxial approximation, see Section 2.2.3), the soliton power (2.9) reduces to

$$P(\nu) := \int |u|^2 d\vec{x}. \tag{3.4}$$

For $(1+1)$D NLSE with a power nonlinearity

$$\Delta n_{power-nl} = u^{p-1}, \tag{3.5}$$

[12]The soliton *phase* can be non-radially-symmetric, e.g., for vortex solitons or for Self-steepening solitons [36].

there is an exact solution for the soliton equation (3.3) given by

$$u(x;\nu) = \left(\frac{p+1}{2}\nu\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}\sqrt{\nu}x\right). \tag{3.6}$$

These soliton are positive and even[13]. In other nonlinearities and/or $D > 1$ settings, there is generally no analytical expression for the single-hump solitons[14]. Then, the profiles of the ground-state solitons can be computed only numerically (see also discussion in Section 5.1) and are found to be radially-symmetric and positive. In such cases, the soliton equation (3.3) can have additional "high-order" soliton solutions. For example, the $(2+1)$D cubic NLSE has an infinite number of "high-order" soliton solutions which are radially-symmetric, *non-*positive and whose number of zeros equals their serial number. However, these particular "high-order" solitons seem to play no significant role in the dynamics of the nonlinear waves (i.e., they are dynamically unstable and do not serve as attractors), hence, they did not attract a lot of interest.

Nevertheless, several other types of "high-order" solitons did attract a significant amount of interest. One type is the vortex soliton which is a solution of the $(2+1)$D NLSE (2.1) for the ansatz

$$A(z,r,\theta) = u(r)e^{im\theta}e^{i\nu z},$$

where r and θ are the polar coordinates and the topological charge m is a measure of the phase winding. This helical-phase corresponds to angular momentum and it imposes a strictly zero amplitude at the phase singularity at the origin [123].

Even more complex type of solitons are the composite/vector solitons, first suggested by Manakov [87] which consist of two (or more) components that mutually self-trap. Most mutual trapping mechanisms are incoherent, i.e., the coupling between the components depends only on the local intensities and is, therefore, phase insensitive. For example, the nonlinear coupling of different modes in a multi-mode fiber or distinct carrier frequencies is incoherent. On the other hand, the mutual trapping mechanism can be also coherent (phase sensitive) as, for example, for different polarization modes in weakly birefringent semiconductor planar waveguides [71, Chapter 9.1.4][15]. Exact solutions of vector solitons can be found for some models, see [71, Chapter 9.1.3] and references therein.

3.2 Optical solitons - prediction and experiment

This Section provides a brief review of the experimental work on optical solitons in the nonlinear media discussed in Section 2.2. When considering such experimental results, one should bear in mind that in optics, standard experimental techniques allow for the measurement of the beam/pulse properties only after it had passed through the sample. Hence, measurement of the beam profile after any propagation distance and hence, true evidence for

[13] We note that the ground-state must not be always positive, see e.g., the ground-state of the biharmonic NLSE [43].

[14] An exception is the $(1+1)$D NLSE with cubic-quintic nonlinearity for which there is also an exact solution, see e.g. [52].

[15] In that sense, quadratic solitons can also be thought of as coherent vector solitons since in the cascading limit, the equations for FF and SH fields become coherently coupled NLS equations.

solitonic propagation can be obtained only with more sophisticated experimental techniques, such as near-field imaging [79] where the evanescent tails of the beam in the air above the sample were measured at any point along the propagation, or experiments made in fluids inside a length-controlled syringe (see e.g., experiments by Gaeta group, Cornell University). Even then, it should be noted that the beam spatial profile can sometimes remain unchanged while the overall power decreases due to dissipation [18] or the spectral profile undergoes Kerr-induced broadening (Self-Phase-Modulation) [16]. In these cases, the light is *self-trapped*, but diffraction and nonlinearity are not perfectly balanced and no genuine solitons are formed.

3.2.1 Spatial optical solitons

The possibility of soliton formation was first suggested in 1962 by Askar'yan, who predicted that "the strong ... effects of the ray on the medium can be used to set up waveguide propagation conditions and to eliminate divergence of the beam (self-focusing)" [12]. Shortly after, Chiao, Garmire and Townes computed the profile that exhibits a perfect balance between Kerr nonlinearity and diffraction, known as the $(2+1)$D (Townes) soliton [30]. However, a year later, Kelley showed that for sufficiently strong powers, the nonlinearity can dominate the diffraction and hence, the $(2+1)$D Kerr solitons can collapse [68]. Indeed, it became quickly clear that observation of *stable* spatial solitons in Kerr media is only possible in planar geometry $D = 1$. In addition, it was predicted that stable solitons in bulk ($D \geq 2$) can form in media with a saturable nonlinearity.

The earliest experimental evidence of spatial self-trapping was provided by Bjorkholm and Ashkin in 1974 in bulk vapor of sodium atoms [18]. This self-trapping was accompanied by significant absorption, hence, it was not true solitonic propagation.

Surprisingly, it took 11 years until the first observation of true solitons in CS2 gas [15] which has a reorientational nonlinearity [22, Chapter 4.4]. In further studies, Kerr solitons have been observed in planar waveguides of glass [7], semiconductors [6, 69] and polymers [16]. Finally, genuine bulk spatial solitons were observed in polymers with a cubic-quintic nonlinearity [129].

Studies of spatial solitons have made rapid progress since the mid-1990's, when two new soliton-supporting nonlinear optical media became available to experiments [125]. First, generation of solitons in quadratic media which was identified theoretically in the mid-1970's [62], was demonstrated experimentally in a series of works of the group of G. Stegeman. In particular, an effective Kerr coefficient in a quadratic medium was first produced and measured experimentally by De Salvo *et al.* [104]. Then, $(2+1)$D and $(1+1)$D spatial quadratic solitons were generated experimentally by Torruellas *et al.* [130] and Schiek *et al.* [105], respectively, see also [124, 26, 128] for reviews.

Second, Segev *et al.* [109] predicted that the saturable nonlinearities of photorefractive materials can support stable soliton propagation. Soon afterwards, a variety of photorefractive solitons, of both $(1+1)$D and $(2+1)$D types and for both focusing and defocusing nonlinearities, have been explored, see [108] for a review.

Vortex soliton dynamics have been theoretically investigated in numerous contexts [75, 38] and the angular momentum and spatial dynamics have been studied experimentally in defocusing Kerr [127, 122], photorefractive [40] and quadratic nonlinear media.

3.2.2 Temporal optical solitons

Temporal solitons in single-mode optical fibres were predicted in 1973 by Hasegawa and Tappert [56], and first observed experimentally in 1980 by Mollenauer *et al.* [88].

However, it was soon observed that when the pulse becomes short enough, high-order dispersion and high-order nonlinear effect such as SRS and SS become important and destroy the solitonic nature of the pulse[16]. However, even then, solitons play an important role in the analysis of the dynamics, especially due to their role in Super-Continuum-Generation in microstructured optical fibers, see [39] for a review.

3.2.3 Optical spatio-temporal solitons

One of the major goals in the study of optical solitons is the possibility to generate pulses that are localized in the transverse dimensions of space, as well as in time. Such solitons of the $(3+1)$D NLSE are called spatiotemporal solitons or "light bullets", a term coined by Silberberg [111] which stresses their particle-like nature.

In contrast to the extensive studies of spatial and temporal $(1+1)$D and spatial $(2+1)$D solitons, experimental progress toward the production of $(3+1)$D solitons has been slow. One of the reason for that is that the conditions for formation of stable spatiotemporal solitons were not identified theoretically yet. Indeed, evidence for spatiotemporal solitons in one spatial dimension has been given by Wise group in quadratic media [81, 80] and Barad group in cubic media [79, 29]. However, even in these cases the beam splits in space and in time after propagating a few characteristic lengths[17]. To date, true $(3+1)$D spatiotemporal solitons have not been observed. For a discussion of the current status of the problem, see [85].

[16] It should be noted, however, that still a self-similar beams can be formed, as e.g., an asymmetric profile in the differential model of SRS [5], see [49] or a symmetric profile in the presence of fourth-order dispersion [65]. In both cases, these profiles form despite losses to the associated energy dissipation due to coupling to optical phonons or linear waves.

[17] For example, in the latter case, SRS causes a frequency red-shift, and consequently, the effective index of refraction is changed. This, in turn, causes the nonlinear coefficient to change and the spatial profile of the pulse is thus modified.

4 Optical solitons in inhomogeneous media

Since the 1980's, there has been a growing interest in soliton propagation in inhomogeneous media, i.e., in media where the refractive index varies in space. Indeed, by that time, advancements in microfabrication technology made it possible to impress changes with high spatial frequency on the *linear* refractive index of various dielectrics and semiconductors.

The most popular structures are those where the linear refractive index is periodically modulated. They are known as Photonic Crystals or waveguide arrays. The original motivations behind the design of such structures were to trap light, to guide it in one direction, to route and switch light signals between various channels, or to increase reflections into a gain medium, and thus decrease the power needed for lasing [35]. Apart from these reasons, it became quickly clear that the equation that describes the evolution of the light in these materials is similar to the equation that describes the dynamics of electrons in a solid, i.e., the fabricated/induced modulation of the refractive index is equivalent to the periodic potential of the atom-lattice[18]. This analogy has raised several interesting theoretical questions and enabled the investigation in optics of phenomena which are not accessible in solid-state (such as the Andersson localization [106, 76]). Moreover, it allowed for a straightforward application of the knowledge acquired in the study of condensed-matter physics to the field of optics, e.g., the computation of the bandgap structure or defect modes, concepts such as discrete breathers and intrinsic localized modes, and theoretical methods such as the Coupled-Mode-Theory and the Tight-Binding (discrete) approximation.

Although originally the research interests described above involved *linear* waves, it was equally interesting to study *nonlinear* effects in such materials. Moreover, nonlinear effects in inhomogeneous media open the way to new phenomena such as discrete solitons [32] and lattice solitons [46]. Another important motivation for studying nonlinear wave propagation in inhomogeneous media was the possibility that the inhomogeneity would stabilize the nonlinear waves and arrest collapse in bulk Kerr media.

The modeling of light propagation in inhomogeneous media depends on the strength of the modulation: In the case of strong modulations, the light propagation is modeled by the full vectorial Maxwell equations, which in the CW limit can be reduced to two evolution equations for transverse electric and magnetic field components which are coupled through the longitudinal component (see e.g. [61, Chapter 8]), or directly to each other (see e.g. [82]). Alternatively, the full vectorial Maxwell equations can be reduced to the vector Nonlinear Helmholtz Eq. model [42][19].

In the case of weak modulations, then in a similar manner to homogeneous media, the Slowly-Varying-Envelope/paraxial Approximation can be used to reduce the above models to the nonlinear Schrödinger equation model. The propagation is then modeled by Eq. (3.2) where the change in the refractive index due to the linear potential/lattice/inhomogeneity is given by

$$\Delta n_{linear\ potential} = -V_l(z, \vec{x}) - \Delta n\left(|A|^2\right). \tag{4.1}$$

[18] Accordingly, the terms of potential and lattice are freely interchanged in this context.

[19] In the latter case, it was shown by several authors that if the modulation is linear and periodic, although the Slowly-Varying-Envelope/paraxial approximation does not necessary hold, still the NLSE governs the dynamics of the envelope of the solution, see e.g. [37].

Here V_l is a linear potential term that is proportional to the variations of the linear refractive index[20] and Δn is any of the nonlinearities discussed in Section 3.1.

Eq. (4.1) assumes that the potential can vary either in the direction of propagation z or in the direction perpendicular to the propagation $\vec{x}$. However, naturally, in the former case, standard solitons cannot form. Indeed, in the context of solitons and the NLSE, most studies were dedicated to materials where the change in the index of refraction occurs only in the direction *perpendicular* to the propagation. In this case, the propagation is modeled by

$$iA_z(z,\vec{x}) + \nabla^2 A + \Delta n\left(|A|^2\right)A - V_l(\vec{x})A = 0. \tag{4.2}$$

Most of the research efforts on this model focused on the analysis of the corresponding linear problem (i.e., Eq. (4.2) with $\Delta n = 0$) and its relation to the dynamics of the solitons studied in the various available nonlinear homogeneous media (those discussed in Section 3.1). For our purposes, it should be noted that studies of solitons in Kerr media with a linear potential in the $\vec{x}$ direction showed that under some conditions, the linear potential can stabilize the solitons in bulk ($D \geq 2$) media, see e.g., [85] for a review.

In addition to all the "standard" homogeneous media solitons, materials with a periodic linear refractive index support the formation of two additional types of solitons. Solitons of the first type are called gap solitons and they appear in the forbidden gaps of the corresponding linear problem. They have attracted an immense amount of interest, especially due to their variety and their relation to defect modes of a perfectly periodic structure. These solitons are, however, out of the scope of my research. A second type of solitons, which are unique to inhomogeneous periodic media, are the multi-hump solitons, which were encountered in [9] in media with a linear lattice and also in my research [115] for media with a nonlinear lattice. However, they have received very little attention so far.

Later studies of linear potentials considered several generalizations of periodic potentials, such as periodic potentials with defects, potentials with a quasycrystal structure and random potentials, see e.g. [118] for a discussion.

Since 2005, there has been a growing interest in the possibility of modulating the *nonlinear* refractive index, i.e.,

$$\Delta n_{nonlinear\ potential} = \left(1 - V_{nl}(z,\vec{x})\right)\Delta n\left(|A|^2\right), \tag{4.3}$$

where V_{nl} is proportional to the variations of the nonlinear coefficient[21]. As for linear potentials, relatively few studies were done on potentials that vary along z compared with potentials that vary along $\vec{x}$[22]. In this case, the propagation is modeled by

$$iA_z(z,\vec{x}) + \nabla^2 A + \left(1 - V_l(\vec{x})\right)\Delta n\left(|A|^2\right)A = 0. \tag{4.4}$$

[20] For example, $V_l = \chi^{(1)}(z,\vec{x}) - \left\langle \chi^{(1)}(z,\vec{x})\right\rangle$ in MKS/SI units.

[21] For example, for Kerr media, $V_{nl} = 4n_0 n_2(z,\vec{x})$ in MKS/SI units.

[22] A periodic modulation of the *nonlinear* refractive index in the z direction is referred to as *nonlinearity management*, in analogy to *dispersion management*, where the dispersion term is modulated in space. Light dynamics in such materials was studied theoretically by several authors in various limits [17, 93, 103, 139] and more recently also experimentally in [27]. A more recent work [34] addressed the coupling between small amplitude waves to the resonant modes of the management period. However, the analysis did not always provide a satisfying agreement with numerical or experimental results.

The most systematic study of solitons in media with nonlinear potentials in the transverse direction $\vec{x}$ was done in my research [44, 116], see Section 7.1.

The first studies on nonlinear potentials have led to the natural extension of these works to materials where both the linear and nonlinear coefficients are modulated in space. So far, relatively few studies considered these materials (see e.g., [100, 1]), however, as shown in my research, these materials can be studied with exactly the same framework as was used for materials with only one type of inhomogeneity, see a detailed discussion in Section 3.

5 Soliton theory

The theoretical research on solitons was directed mostly towards studying soliton existence, soliton profiles, stability and dynamics. My research touched upon the last three subjects, initially for the new class of materials having a transverse periodic modulation of the Kerr coefficient (4.3) and later on, for more general linear and nonlinear inhomogeneities. This Section provides a brief review of the main results on soliton profiles and stability, with emphasize on the contributions made in my research.

5.1 Soliton profiles

In the presence of inhomogeneities, light propagation in the material is modeled by

$$iA_z(z,\vec{x}) + \nabla^2 A + (1 - V_{nl}(\vec{x}))\,\Delta n\left(|A|^2\right) A - V_l(\vec{x})A = 0. \tag{5.1}$$

and the soliton profile is the solution of

$$\nabla^2 u(\vec{x};\nu) + (1 - V_{nl}(\vec{x}))\,\Delta n(|u|^2)u - \nu u - V_l(\vec{x}) = 0. \tag{5.2}$$

Unfortunately, in this case, the soliton profile cannot be computed analytically even in those few cases where there are explicit analytic solutions in homogeneous media. Hence, the soliton profiles can be computed either numerically or using perturbation methods. In fact, very few results of perturbative computations of soliton profiles are available. Among them, outstands the "effective mass approximation" [24, 28] for the computation of the profile of solitons near the band-edge of a periodic linear potential. More recently, Fibich and Wang [45], and later in the papers of my Thesis, it was shown how to use the soliton width as a small parameter in a perturbation analysis for the soliton profiles in both nonlinear and linear potentials in the limits of narrow and wide beams, see a detailed discussion in [44].

Significantly more results are available from numerical computation of soliton profiles. In $D = 1$, Eq. (5.2) is an ordinary differential equation that can be solved with the shooting method. However, the shooting method becomes very difficult to use for $D > 1$ NLSE or for NLSE with a potential. Hence, in our research, we used a different numerical method which recently has become popular for the computation of soliton profiles. It is based on the iterative method of Petviashvili [97]. In Appendix B, an original derivation of this method is given, as well as an elaborate discussion of its convergence properties. Most of the material given in Appendix B *was not* published in the papers of my Thesis.

Other methods which are frequently used for computation of the soliton profile are the Newton iterations method [67, 21], imaginary-time method [31] and recently, an improved version of the Petviashvili method [77].

5.2 Soliton stability - theoretical basis

The usual notion of soliton stability in NLSE theory is that of Orbital Stability[23]:

[23]The more strict *asymptotic* stability was studied in several papers, see e.g., [48] for a discussion.

Definition 5.1 *Let $u(\vec{x})$ be a solution of Eq. (5.2) with propagation constant ν. Then, $u(\vec{x})e^{i\nu z}$ is an orbitally stable solution of the NLSE (5.1) if for all ε, exists $\delta(\varepsilon) > 0$ such that for any initial condition $A_0(z = 0, \vec{x})$ such that $\inf_{\gamma\in\mathbb{R}} \|A_0 - ue^{i\gamma}\|_{H_1} < \delta$, the corresponding solution $A(z, \vec{x})$ of Eq. (5.1) satisfies*

$$\sup_{z\geq 0} \inf_{\gamma\in\mathbb{R}} \|A(\vec{x}, z) - u(\vec{x})e^{i\gamma}\|_{H_1} < \varepsilon \ ,$$

where the H_1 norm is defined as $\|f\|^2_{H_1} := \int(|f|^2 + |\nabla f|^2)d\vec{x}$.

The first analytic result on soliton stability was obtained by Vakhitov and Kolokolov [131]. Based on a *linear* stability analysis, they showed that a necessary condition for soliton stability is that $dP/d\nu > 0$, i.e., the soliton power increases with increasing propagation constant ν. Subsequently, this result was derived from a rigorous *nonlinear* stability analysis [133].

Grillakis, Shatah and Strauss (GSS) further extended and generalized the theory of soliton stability for a general Hamiltonian system [54, 55]. In the case of positive solitons ($u > 0$), the GSS stability theory can be stated as follows[24]. Let

$$\begin{aligned} d(\nu) &= \mathcal{H} - \nu\mathcal{P} \\ &= \int \left[|\nabla u|^2 + (V_l(\vec{x}) + \nu)\, u^2 - (1 - V_{nl}(\vec{x}))\, G(u)\right] d\vec{x}, \end{aligned}$$

where $G = \int_0^u F(u^2)du'$ and $F = \Delta n$, let $p(d'') = 1$ if $d''(\nu) > 0$ and $p(d'') = 0$ if $d''(\nu) < 0$, and let $n_-(L_+)$ be the number of negative eigenvalues of the linearized operator

$$L_+^{(V)} = -\nabla^2 + \nu - (1 - V_{nl}(\vec{x}))\left(F(u^2) - 2u^2F'\right) + V_l. \tag{5.3}$$

Then, $A = ue^{i\nu z}$ is orbitally stable if $n_-(L_+) = p(d'')$, and orbitally unstable if $n_-(L_+) - p(d'')$ is odd [54, 55].

Most works done by the mathematics/analysts community were based on the GSS Theorem. However, in my research, we had relied on the following Stability Theorem:

Theorem 5.1 *Let $u(\vec{x})$ be a positive solution of Eq. (3.3) with propagation constant ν. Then, $A = u(\vec{x})e^{i\nu z}$ is an orbitally-stable solution of the NLSE (5.1) if and only if both of the following conditions hold:*

1. *The* **slope** *(Vakhitov-Kolokolov)* **condition**

$$\frac{dP}{d\nu} > 0, \tag{5.4}$$

2. *The* **spectral condition**

$$n_-(L_+) = 1, \tag{5.5}$$

i.e., L_+ has exactly one negative eigenvalue of multiplicity one.

[24] The GSS theory is valid also for high-order solitons, see [44, Remark 11].

Theorem 5.1 was proved initially for linear potentials which are bounded and decay to zero at infinity [101, 134] and in the narrow soliton (semi-classical) limit in the subcritical case [92]. In my research, we have extended the proof to nonlinear potentials [44]. The proof for the case of linear potentials bounded form below (i.e., not necessarily periodic or decaying to zero) will appear in the near future [60]. However, in our research we assumed that Theorem 5.1 holds for any type of lattice/potential.

Theorem 5.1 is an extension of the classical stability theory of GSS [54, 55]. Indeed, since $d'(\nu) = P(\nu)$, the sign of d'' is the same as the sign of the power slope. Hence, in the GSS theory, stability and instability depend on a *combination* of the slope condition (5.4) and the spectral condition as follows: If both the slope condition and the spectral condition are satisfied, the soliton is stable, whereas if either the slope condition is satisfied and $n_-(L_+)$ is even, or if the slope condition is violated and $n_-(L_+)$ is odd, the soliton is unstable. There are two cases not covered by the GSS theory: When the slope condition is satisfied and $n_-(L_+)$ is odd, and when the slope condition is violated and $n_-(L_+)$ is even[25]. Theorem 5.1 shows that in these two cases, the solitons are unstable. Hence, Theorem 5.1 implies that *there is a "decoupling" of the slope and spectral conditions, in the sense that both are needed for stability, and violation of either of them would lead to instability.*

[25]Note, however, that despite the coupling of the slope and spectral conditions, the slope condition is still a necessary condition in the GSS formalism.

6 Soliton stability - main results

6.1 Brief overview

The study of solitons and their stability has been a very active field of research for many years and hundreds of papers were published on the problem. This large body of research can be roughly divided into two approaches: The first approach, adopted by the "applied physics" community, consists of checking the slope condition (5.4) only and solving the NLSE numerically order to observe the actual dynamics. The spectral condition (5.5), which appears in both the GSS and our stability Theorems, was "ignored", most likely because it was relatively unknown among the "applied physics" community. As we show in our research (see e.g., [118]), the spectral condition is always satisfied for positive solitons in homogeneous media, a fact that gives an *a posteriori* justification for this approach.

However, in the presence of any type of inhomogeneity, "ignoring" the spectral condition is justified only for solitons centered at lattice minima, since only then the spectral condition is satisfied, see e.g. [92, 95, 44, 116, 78, 113, 118, 100]. In all other cases, checking only the slope condition might lead to incorrect conclusions regarding stability. In that sense, this approach provides only partial stability results for solitons in inhomogeneous media.

The second approach, adopted by the community of mathematicians/analysts, was based on the GSS theory. Due to the complexity of the problem and the need to have complete rigorous proof for every detail, the progress made by following this approach was very slow. In particular, most results were obtained for power-law nonlinearities (3.5), and very little results are available for the more complicated nonlinearities such as the saturable or non-local nonlinearities or for high-order solitons. In particular, results on solitons in inhomogeneous media are scarce. Moreover, this approach does not provide any information on the actual instability dynamics.

In addition to the above, in both approaches there is a major deficiency: most, if not all, studies were aimed to say whether the soliton is stable or not. However, while the dynamics of the orbitally-stable solitons is relatively straightforward - the solution remains close to the unperturbed soliton, there are several possible ways for a soliton to become unstable: it can undergo collapse, complete diffraction, drift, breakup into separate structures, etc.. Moreover, even an unstable soliton can be practically stable if the instability develops very slowly. Conversely, a weakly-stable soliton can become unstable by relatively small perturbations. Hence, knowing if the soliton is stable or not gives only a partial description of the actual dynamics.

6.2 Our approach

According to the description above, there seemed to be a need for a *simple and complete* theory with which the stability and instability dynamics can be predicted *qualitatively* and *quantitatively*. With such a theory in hand, general trends in this class of problems can be identified, and there will be no need to study each new physical configuration of nonlinearity, lattice or dimension "from the beginning". Our analytical results were corroborated by numerical simulations. This combination of analysis and numerics is a crucial component in the evolution of our theory.

Such a theory was the output of my PhD research. In the rest of this Section, our qualitative and quantitative approaches are described briefly and in Section 7, its evolution through our papers.

6.2.1 Qualitative approach

Theorem 5.1 provides the theoretical basis for the classification of the instability dynamics, since it shows that the two *coupled* conditions for stability in the GSS theory are, in fact, completely *independent*. Indeed, it turns out that the instability dynamics depends on which of the two conditions for stability is violated. In particular, we show in our series of papers that a violation of the slope condition leads to an amplitude instability whereby infinitesimal changes of the soliton can result in large changes of the beam amplitude, and can even lead to collapse or total diffraction. On the other hand, when the soliton is unstable due to a violation of the spectral condition, it undergoes a *drift instability* whereby infinitesimal shifts of the initial soliton location lead to a lateral movement of the soliton away from its initial location.

The drift dynamics has an intuitive physical explanation. According to Fermat's Principle, light bends towards regions of higher refractive-index. Positive values of the potentials correspond to negative values of the refractive index, hence, Fermat's principle implies that beams bend towards regions of lower potential. Moreover, since generically, the spectral condition is satisfied for solitons centered at a lattice minimum but violated for solitons centered at a lattice maximum, one sees that the drift instability of solitons centered at lattice maxima and the drift stability of solitons centered at lattice minima is a manifestation of Fermat's principle. An additional mathematical motivation for the drift instability can be found in [118].

Currently, we do not have similar arguments that can motivate and justify the link between the slope condition and the amplitude instability. However, we have provided several numerical evidences for this link in each of our papers.

The upshot of the qualitative approach is that it can be used to predict the soliton dynamics even if the location and width of the solution change significantly during the propagation. For example, suppose that for a certain soliton the slope condition is satisfied when it is centered at a lattice maximum but violated if it is centered at a lattice minimum and suppose the soliton is initially centered at the lattice maximum. Then, the soliton initially drifts toward the potential minimum but when it becomes drift-stable, it develops an amplitude instability and can even collapse [44, 116, 118]. This example shows that even if the amplitude and drift instabilities develop or "disappear" dynamically during the propagation, our approach is still useful and gives a simple explanation for soliton dynamics which, *a priory*, seems complicated and counter-intuitive.

6.2.2 Quantitative approach

While the qualitative approach classifies the type of instability, the quantitative approach addresses questions such as how fast does the instability develop or how large should a perturbation be in order to overcome the stabilization effects. Surprisingly, despite the multitude of papers on soliton stability and dynamics, it seems that such important questions

were not discussed before.

The quantitative theory for the soliton location is described in detail in [114, 118]. It is based on the following linear oscillator equation

$$\frac{d^2}{dz^2}\langle x_j\rangle = \Omega_j^2\left(\langle x_j\rangle - x_{0,j}\right), \quad \Omega_j^2 = -C_j\lambda_{0,j}^{(V)}, \quad C_j > 0, \tag{6.1}$$

where $\langle x_j\rangle$ is the Center-of-Mass of the soliton, $x_{0,j}$ is the location of the lattice extremum in the jth direction and C_j is known positive constant.

At present, the quantitative relation between the magnitude of the slope and the strength of the amplitude stability is not known, i.e., we do not have a relation such as (6.1). However, we have provided many numerical evidences for this link, see detailed description in [118].

The quantitative approach is especially important in identifying cases of weak stability/instability. For example, consider a soliton for which the two conditions for stability are met, but for which $\lambda_{0,j}^{(V)}$ or the slope are very small in magnitude. Such a soliton is orbitally stable, yet it can become unstable under perturbations which are quite small compared with typical perturbations that exist in experimental setups. Hence, such a soliton is "mathematically stable" but "physically unstable", see e.g., [44]. Conversely, consider an unstable soliton for which either $\lambda_{0,j}^{(V)}$ or the slope are positive but small. In this case, the instability develops so slowly so that it can be sometimes neglected over the propagation distances of the experiment. Such a soliton is therefore "mathematically unstable" but "physically stable" [113].

7 Description of the contents and main results of my papers

In this Section, a concise description of the motivation and main results of each of our papers is given. The results described here were presented in six papers, four out of which have been published and are those on which my Thesis is based. An additional summary paper, which has been recently submitted for publication, is included as well because it gives the most comprehensive description of our approach. The description of each paper is followed by the paper itself.

7.1 Nonlinear lattices

In our first two papers, we gave the first systematic study of the stability and instability of solitons in media with *nonlinear* potentials. This study was motivated by recent advances in fabrication techniques that make it possible to manufacture materials with a spatially-varying nonlinear index change.

In particular, purely nonlinear lattices (i.e., without any modulations to n_0, Eq. (4.4)) can be created in materials with a quadratic nonlinearity by varying the phase mismatch parameter Δk, a technique known as Quasi-Phase-Matching [22]. In this case, the quadratic nonlinear coefficient, which in homogeneous media depends only on the FF and SH frequencies and the corresponding refractive indices, can be controlled in any point in space, see, e.g., [11]. Then, in the cascading limit, the effective cubic nonlinearity (see Section 2.2) becomes spatially modulated itself[26]. Another context where nonlinear lattices are created is for ultrashort pulse propagation in planar waveguides. In this case, the variation of the effective index along the propagation due to the SRS-induced red-shift gives rise to a spatially-dependent cubic coefficient [29].

Another technique for the creation of purely nonlinear lattices arises in the context of BEC where the scattering length is modulated in space using Feshbach resonance with non-uniform magnetic fields [96].

Nonlinear lattices can also be accompanied by linear lattices. For example, the spatial distribution of n_0 and n_2 can be controlled in highly anisotropic semiconductor heterostructures [59]. Another technique is by focusing an intense pulse to a dielectric waveguide. This pulse causes optical breakdown in the focal volume which impresses permanent changes to the refractive indices. Using these two techniques it was shown that the induced changes to n_2 can be greater than the induced changes to n_0. In such cases, the study of purely nonlinear lattices can be viewed as a first stage towards a unified theory for the combined effects of linear and nonlinear lattices. Indeed, in years following the publication of our work, several studies of a combined linear and nonlinear potential were published, see e.g., [100, 1].

By the time of the initiation of this study, it was known that linear potentials can stabilize the solitons of the $(2+1)$D cubic NLSE (i.e., Eq. (4.2) and a cubic nonlinearity (2.4)). In contrast to the large body of research on linear lattices (see, e.g., [46, 2, 74, 85] as well as Section 4), hardly any research has been devoted to the effect of nonlinear lattices. Hence, our

[26] In fact, this technique was brought to our attention only after the publication of the two papers [44, 116] on nonlinear lattices.

specific goal was to find the conditions for stabilizing solitons in materials with a transverse nonlinear lattice, i.e., Eq. (4.4).

7.1.1 Fibich, Sivan and Weinstein, "Bound states of nonlinear Schrödinger equations with a periodic nonlinear microstructure", Physica D 217, 31 (2006) [44]

In this paper, we chose the $(1+1)$D NLSE as a toy model since it is the simplest and easiest case to study analytically and numerically. Moreover, using the appropriate power, we could study the subcritical, but also the critical and supercritical cases in this simpler $D = 1$ setting.

We have used a combination of rigorous analysis, asymptotic analysis, and numerical simulations to study the profile and stability and instability dynamics of the solitons. We introduced and emphasized the importance of the dimensionless parameter N, which measures the ratio of the input beam width to the lattice period / characteristic length. This provides a small parameter in the narrow and wide beam limit, and opens the way to the computation of the soliton profiles and the analysis of the different regimes. Our study appears to be the first wherein the three regimes: wide ($N \gg 1$), narrow ($N \ll 1$) and intermediate ($N = \mathcal{O}(1)$) beams were systematically considered. We showed that the same lattice may stabilize beams of a certain width while destabilizing beams of a different width.

Our detailed analysis showed that unlike a linear lattice, a nonlinear lattice can only stabilize narrow beams centered at minima of potentials that satisfy several strict conditions. Even then, the stabilization is very weak since it scales as $N^4 \ll 1$. Hence, this stabilization is not likely to be observed in experiment.

Despite this "disappointing result", in this study we set the foundations for our qualitative and quantitative approaches. These foundations relied mostly on numerical simulations and physical intuition. Our next papers consist of 1) a series of generalizations of the ideas that appeared in this paper to different physical configurations of dimension and lattice type and 2) improving the theoretical basis for our approach.

7.1.2 Sivan, Fibich and Weinstein, "Waves in Nonlinear Lattices: Ultrashort Optical Pulses and Bose-Einstein Condensates", Physical Review Letters 97, 193902 (2006) [116]

In this paper, we extended the results of [44] to a $(2+1)$D NLSE setting. We chose to focus on lattices which appear only in one transverse dimension out of the two[27]. Such materials are relevant for ultrashort pulses propagation in planar waveguide arrays where the refractive index is modulated in the spatial but not in the temporal transverse coordinate; such materials were relatively less explored.

Beyond the analysis of this specific configuration, our results show that the effect of the nonlinear lattice on solitons in such materials is qualitatively similar regardless of the exact type of the lattice, be it e.g., a radially-symmetric (isotropic) lattice, a lattice with square (or any other) topology or even a one-dimensional lattice in the one-dimensional critical case studied in [44]; the differences between all these cases are only quantitative.

[27] We used the terminology "anisotropic lattices".

7.2 Linear lattices

After the detailed study of nonlinear lattices, we wanted to apply the qualitative and quantitative approaches also to the much better explored materials with linear potentials. The possibility to do so was based on three arguments. First, our approach was based on the stability Theorem 5.1 which is the same for linear and nonlinear lattices. Second, the mathematical and physical arguments that were used to justify our results for nonlinear lattices were valid also for linear lattices (e.g., the structure of the associated eigenvalue problem or the Ehrenfest law for the drift dynamics, see [118]). Finally, it was shown in [99] that a nonlinear lattice in a purely focusing media can be mapped into a linear lattice by a simple transformation[28].

In contrast to most studies on solitons in linear lattices, we did not focus on the analysis of the linear problem and the relation between the dynamics of linear and nonlinear waves in such systems. Instead, following the studies of solitons in nonlinear lattices [44, 116], we focused on the effect of the lattice on the soliton profile and dynamics compared with the case of a homogeneous medium (i.e., in the absence of a potential).

In the first paper [78, Le-Coz, Fukuizumi, Fibich, Ksherim and Sivan, *Physica D* (2008)], we studied the stability and instability of lattice solitons in a one-dimensional medium with power nonlinearity (3.5) in the presence of a point defect, described by a delta-function potential. In this case, the NLSE is given by

$$iu_z(z,x) + u_{xx} + \gamma\delta(x)u + |u|^{p-1}u = 0, \tag{7.1}$$

where γ is a real constant. This work consists of two parts: The first was an analytical study of the stability properties of solitons in such media based on GSS theory. It was conducted by our collaborators, Stefan Le-Coz from Université de Franche-Comté, France, and Reika Fukuizumi from Hokkaido University, Japan. The second part was a series of numerical examples which I was involved in their choice and interpretation. The numerical work itself was done by a fellow student, Baruch Ksherim. Accordingly, since my contribution to this work was relatively small, it is not included as one of the papers of my Thesis.

In the second paper [113, **Sivan, Fibich, Efremidis and Barad, "Analytic theory of narrow lattice solitons", *Nonlinearity* 21, 509 (2008)**], we studied the stability and instability of lattice solitons which are *narrow* with respect to the lattice characteristic lengthscale (e.g., period).

The reason for the choice of these two cases is that in both of them, it is possible to compute the soliton profile and consequently, the power, slope and perturbed zero-eigenvalues, either analytically [78] or asymptotically [113]. Hence, in the absence of a complete proof for Theorem 5.1 for the general linear potentials, in these two papers we provide its proof for two specific cases.

[28] It should be noted that this transformation adds an additional term to the model. However, the effect of this term on the soliton profiles and stability is not well understood yet. Moreover, this transformation also raises the question of the existence of gap solitons in a purely nonlinear lattice. A first step towards the answer to this question might be found in [1].

7.3 Sivan, Fibich and Ilan, "Drift instability and tunneling of lattice solitons", *Phys. Rev. E* 77, 045601(R) (2008) [114]

In parallel to the works that extended the results of [44], in [114] we made further progress in establishing our results rigorously. In particular, we proved that for solitons of the D-dimensional NLSE with a linear and/or nonlinear potential (5.1), a violation of the spectral condition leads to a drift instability. This provides a proof for the claims raised in [44, 116, 113, 78] based only on numerical simulations and on physical intuition and expand their validity to a more general setting.

While having a fundamental theoretical importance, the proof of the relation between the spectral condition and the drift instability has practical implications as well. Our main result, the derivation of an equation that describes the dynamics of the soliton Center-of-Mass (6.1), enabled us to compute analytically the drift rate of solitons initially centered near a lattice maximum, and the restoring force that the lattice exerts upon solitons initially centered near a lattice minimum. Thus, we obtained a *quantitative* prediction of the lateral dynamics. Based on this relation, we explained the failure to observe the drift instability in the experiment described in [89].

We note that different reduced equations for the lateral dynamics were previously derived under the assumption that the beam remains close to the initial soliton profile (see e.g. [66]), by allowing the soliton parameters to vary along the propagation (see e.g., [71] and references therein) or use the Inverse Scattering Transform to account also for radiation effects (see e.g. [64, 50]). These approaches, as well as ours, are valid only as long as the beam profile remains close to a soliton profile. In these cases, the Center-of-Mass describes the dynamics well. However, unlike these previous approaches, Eqs. (6.1) are based on the link between the linear stability theory and the lateral dynamics. This link shows that in contrast to the ansatz used in previous works, the beam profile evolves as a soliton perturbed by the eigenfunction $f_{0,j}^{(V)}$. The validity of this perturbation analysis was manifested by the excellent fit between the reduced Eqs. (6.1) and the numerical simulations for a variety of lattice types. To the best of our knowledge, such an agreement was not achieved with the previous approaches.

In this paper we also gave two related results: the first concerns the occurrence of a drift instability in Petviashvili's iterative method for the computation of soliton profiles [97], see also Appendix B. The second result concerns the implications of our theoretical prediction on the lateral dynamics to soliton tunneling. Tunneling occurs when a soliton initially centered at a lattice minimum is perturbed strongly enough so that it moves sideways and goes across the nearest lattice maximum (potential barrier). Clearly, this stage lies beyond the regime of validity of the predictions made on the basis of the stability Theorem 5.1. Still, we show that our approach can give rough estimates for the critical (i.e., minimal) initial transverse velocity needed for tunneling in cases where the standard approach based on the Peierls-Nabarro Potential (PNP) approach [72] fails.

It should be noted that the results of this paper can be used to obtain several additional important applicative results. For example, for periodic potentials, one can concatenate trajectories computed from Eq. (6.1) at each potential maximum and minimum and thus, to obtain a prediction of the full trajectory of the beam. Such a result can provide a simple, even if crude model for switching and routing of signals in such systems. Such a model is highly desirable for applications but currently, is not available.

To summarize, this paper provided, apparently for the first time, a unified theory for the mobility of lattice solitons by showing the intrinsic connection between soliton mobility and soliton stability, two key properties that so far were studied separately.

7.4 Sivan, Ilan and Fibich, "Qualitative and quantitative analysis of stability and instability dynamics of positive lattice solitons", submitted to *Phys. Rev. E* [118]

This fourth paper provides a summary paper to our unified theory for soliton stability and instability dynamics. In this paper, we give the most coherent and complete description of our qualitative and quantitative approaches. We then demonstrate our approach on simple examples of periodic lattices and then, on more complicated examples of periodic lattices with defects, lattices with quasi-crystal structures etc. with emphasize on the quantitative theory for the lateral dynamics [114]. These results show that our approach applies to any type of linear lattices and for solitons of any width. This paper has been submitted for publication only recently. Nevertheless, it is included in my Thesis because of its seminal character.

8 Summary

In my Thesis, I have presented a series of six papers which had led to the formulation of a unified approach for the stability and instability dynamics of positive bright solitons. This approach consists of a *qualitative* characterization of the type of instability, and a *quantitative* estimation of the instability rate and the strength of stability. This approach was applied to a wide variety of physical configurations and summarized in [118] by several general rules. The variety of these configurations implies that our qualitative and quantitative approaches apply to positive solitons in any dimension, any type of nonlinearity (e.g., saturable or non-local) as well as for other lattice configurations, e.g., "surface" or "corner" solitons [84]. Thus, our theory reveals the strong similarity between these configurations which, *a priory*, look very different from each other and were thus studied separately until now.

Moreover, we believe that a formulation of a qualitative and quantitative theories for "high-order" solitons such as gap solitons or vortex solitons may be possible, but requires further study. Indeed, the instability dynamics of such solitons is expected to be richer and to include additional types of instabilities.

Finally, we note that our quantitative approach is still incomplete, as we currently do not have an equation that describes the evolution of the soliton amplitude/width. With such a model in hand, we believe that our approach can be applied also for cases where the potentials vary along the propagation direction ($V(z)$).

A Nonlinear delayed response due to Stimulated-Raman-Scattering

Stimulated-Raman-Scattering is described mathematically under the Born-Oppenheimer (BO) approximation. Hellwarth *et al.* [57] use quantum mechanical perturbation analysis for an electric-dipole potential to show that under the BO approximation, the third-order response function scales as

$$R^{(3)}(\tau_1, \tau_2, \tau_3) \sim \delta(\tau_1) R_{BO}(\tau_2 - \tau_1) \delta(\tau_3 - \tau_2), \tag{A.1}$$

where $R_{BO}(\tau)$ is a real function that satisfies

$$\int_0^\infty R_{BO}(\tau) d\tau = 1. \tag{A.2}$$

The physical meaning of this approximation is that between τ_2 and τ_1 the system is excited to a vibrational/rotational level whose life time is comparable to the pulse duration; between τ_3 and τ_2 and τ_1 and t, the system is in a virtual excitation, i.e., it is excited to an electronic level.

Substituting Eq. (A.1) in Eq. (2.6) shows that the third-order polarization is given by

$$P^{(3)}(z, x, y, t) = E(z, r, t) \int_0^\infty R_{BO}(\tau) E^2(z, x, y, t - \tau) d\tau. \tag{A.3}$$

In the SRS process described by Eqs. (A.1) and (A.3), a photon is scattered off the nuclei-lattice/molecule, excites a vibrational/rotational mode and becomes red-shifted (Stokes-scattered[29]), in agreement with the interpretation above. In this process, the generated phonons are incoherent in the sense that their wavevectors are randomly distributed.

We are interested only in the contribution of the third-order polarization to the fundamental harmonic. Hence, in a similar manner to Eq. (2.2), we separate $P^{(3)}$ into the fundamental and third harmonics

$$P^{(3)}(z, r, t) \;=\; p^{(3)}_{\omega_0}(z, r, t) e^{-i\omega_0 t} + p^{(3)}_{3\omega_0}(z, r, t) e^{-3i\omega_0 t} + c.c..$$

Using Eq. (2.2) and neglecting third-harmonic terms, we get that

$$p^{(3)}_{\omega_0} \;=\; 2A \int_0^\infty R_{BO}(\tau) |A(t - \tau)|^2 d\tau + A^* \int_0^\infty R_{BO}(\tau) e^{2i\omega_0 \tau} A^2(t - \tau) d\tau. \tag{A.4}$$

For pulses shorter than a few picoseconds but not shorter than a few femtoseconds, the function R_{BO} can be separated into instantaneous contributions from transitions involving only electronic levels, and delayed contributions from the transitions involving also vibrational/rotational levels (see discussion in Section 2.2.2), i.e.,

$$R_{BO}(\tau) = (1 - f_R)\delta(\tau) + f_R h_R(\tau), \tag{A.5}$$

[29] The conjugate process of Anti-Stokes scattering can be accounted for only in a model that describes the phonon dynamics.

where f_R is the relative magnitude of the delayed response. Since $h_R(\tau)$ describes only the delayed response, then, $h_R(0) = 0$ and by Eq. (A.2), $\int_0^\infty h_R(\tau)d\tau = 1$. In this case, $p^{(3)}_{\omega_0}$ reduces to

$$p^{(3)}_{\omega_0} = 3(1-f_R)|A|^2A(t) + 2f_RA\int_0^\infty h_R(\tau)|A(t-\tau)|^2d\tau + f_RA^*\int_0^\infty h_R(\tau)e^{2i\omega_0\tau}A^2(t-\tau)d\tau. \tag{A.6}$$

In most materials, the vibrational/rotational frequency is well defined[30]. In such cases, the delayed response function $h_R(\tau)$ can be approximated using a standard Two-Level-System susceptibility based on the electronic ground-state and the vibrational/rotational level. In other cases, e.g., for fused silica, the spectrum of the generated phonons can be quite wide and requires a more sophisticated treatment, see e.g. [58][31].

In the limit of long pulses, it can be seen that Eq. (A.6) reduces to the standard Kerr nonlinearity (2.4). Hence, the two contributions to the the delayed response as given by Eq. (A.6) become important only for sufficiently short pulses. The first contribution (the second term Eq. (A.6)) becomes important when the spectral content of the pulse $\Delta\omega \sim 2\pi/T$ becomes comparable with the vibrational/rotational frequency[32]. Indeed, this happens when the Fourier transforms $\chi_R(\omega) = \mathcal{F}[h_R(\tau)]$, which is centered at $\omega = \Omega_{vib/rot}$, and $\mathcal{F}[|A|^2]$ overlap.

The second contribution (the third term in Eq. (A.6)) does not appear in any of the standard derivations of the nonlinear index change due to SRS [19, 86, 47]. Peculiarly, the absence of the this term was not intentional but rather, a result of three completely different algebraic mistakes in each of these derivations[33]. This term *did* appear in the derivation of Karasawa *et al.* in [63]. However, it was then neglected by claiming that it is small with no

[30]For example, in solid-state, the photon wavelength is much longer than the lattice constant. Hence, since the intersection of the photon and optical phonon dispersion curves dictates the phonon energy and momentum (wavenumber), the phonon wavenumber is near the center of the Brillouin zone.

[31]However, despite that, it should be noted that a Two-Level-System approximation of the response in silica is used after all. In particular, in [126], $h_R(\tau)$ was approximated as a Two-Level-System susceptibility

$$h_R(\tau) = \frac{\tau_1^2+\tau_2^2}{\tau_1\tau_2^2}e^{-\frac{\tau}{\tau_2}}\sin(\tau/\tau_1), \tag{A.7}$$

where $\tau_1 = 12.2$fs is the inverse phonon frequency and $\tau_2 = 32$fs is the phonon lifetime. Although this expression for h_R is in common use (see e.g. [5]), we note that this Two-Level-System approximation suggests that the phonon wavelength is $\lambda_{vib/rot} = \frac{2\pi c}{n\Omega_{vib/rot}} = \frac{c}{n}\tau_1 \approx 2.7\mu$m. However, the characteristic wavelengths of lattice vibrations in silica are of the order of 10μm or longer, i.e., almost an order of magnitude longer, see e.g.,the Sellmeier formula in [5]. Moreover, this wavelength $\lambda_{vib/rot} \approx 2.7\mu$m coincides with the absorption line due to residual water molecules which, at the time, were an unavoidable impurity in the process of optical fibers manufacturing [112]. As far as we understand, the similarity between the wavelength implied by the value assigned to the phonon inverse frequency τ_1 in [126] and the water absorption line was incidental. Indeed, a corrected model for the silica response function which takes into consideration the correct silica absorption lines was given by [58].

[32]For example, for fused silica for which the characteristic phonon wavelengths are $\sim 10\mu$m, this occurs for pulses shorter than a couple of 100fs.

[33]Generally speaking, the reason for these mistakes is that the derivations were made in the frequency domain, where the equation of propagation is rather complicated. Here, we derive $p^{(3)}_{\omega_0}$ in the time domain, an approach which makes the computations much simpler.

further justification. In what follows, I provide a systematic justification of the neglect of this term which as far as we know, was not given elsewhere in the literature.

In the same manner as for the second term in Eq. (A.6), one can see that this term becomes important when the Fourier transforms $\chi_R(\omega - 2\omega_0) = \mathcal{F}\left[h_R(\tau)e^{2i\omega_0\tau}\right]$, which is centered at $\omega = \Omega_{vib/rot} - 2\omega_0$, overlaps with $\mathcal{F}\left[|A|^2\right]$. This happens for $\Omega_{vib/rot} \sim 2\omega_0 \pm \Delta\omega$ or assuming that the pulse is quasi-monochromatic (i.e., $2\omega_0 \gg \Delta\omega$), this occurs for $\Omega_{vib/rot} \sim 2\omega_0$, i.e., at a two-photon resonance with the vibrational/rotational level. This shows that this term represents the dispersion due to transitions involving phonon levels and the reduction in the photon number $\mathcal{N} = \int |A|^2 d\vec{x}dt$ resulting from phonon generation. Indeed, it can be shown that the photon number $\mathcal{N}$ is not conserved due to the presence of this term.

The most energetic Raman transition possible $\lambda_{vib/rot} = \frac{2\pi c}{n\Omega_{vib/rot}} \cong 2.4\mu$m is found in H_2 molecules for which the atom-mass is minimal and hence, the rotational frequency is highest [41]. This wavelength implies that the two-photon resonance with the vibrational/rotational level occurs at wavelengths well in the mid-IR regime, a regime which is not accessible by most standard light sources. Moreover, this spectral regime is not reached even in the extreme case of Super-Continuum-Generation where the pulse spectrum broadens significantly and the pulse profile undergoes temporal changes on the scale of a few fs.

Accordingly, despite the mistakes in the derivations of [19, 86, 47], the third term in Eq. (A.6) *can* be neglected so that Eq. (A.4) reduces to Eq. (2.7)[34]. In that sense, this neglect is equivalent to adopting the Rotating-Wave approximation, see e.g. [22, 10].

[34]Most likely, this is the reason for the survival of the algebraic mistakes mentioned above.

B Petviashvili's / renormalization method for computation of soliton profiles

This numerical method was first introduced by Petviashvili [97] and more recently by Ablowitz and co-workers in a series of papers, see e.g. [3, 91, 4]. Here, we derive the method using a different approach which, we believe, makes it somewhat more intuitive. In addition, we provide simple explanations for the convergence properties of the scheme.

B.1 Derivation of the iteration scheme

For simplicity, we derive the iteration scheme for $D = 1$ solitons with a power nonlinearity (3.5). Let u be the nontrivial solution of

$$-\partial_x^2 u(x) - |u|^{p-1}u + \nu u = 0, \tag{B.1}$$

where $p > 1$ and let $\mathcal{F}(u) = \int_{-\infty}^{\infty} u(x)e^{-ikx}dx$ be the Fourier transform of u. Taking the Fourier transform of Eq. (B.1) and rearranging yields

$$\mathcal{F}(u) = \frac{1}{k^2+\nu}\mathcal{F}(|u|^{p-1}u). \tag{B.2}$$

This equation can be solved with the fixed point iterations

$$\mathcal{F}(u_{m+1}) = \frac{1}{k^2+\nu}\,\mathcal{F}(|u_m|^{p-1}u_m), \qquad m = 0, 1, \ldots, \tag{B.3}$$

so that $u_{m+1} = \mathcal{F}^{-1}\big(\frac{1}{k^2+\nu}\,\mathcal{F}(|u_m|^{p-1}u_m)\big)$. Naturally, the solution of the iteration scheme (B.3) u satisfies

$$\mathcal{F}(u) = \frac{1}{k^2+\nu}\mathcal{F}(|u|^{p-1}u). \tag{B.4}$$

Unfortunately, numerical simulations show that the iteration scheme (B.3) usually diverges to the fixed points $u_\infty \equiv 0$ or $u_\infty \equiv \infty$, rather than converge to u. This divergence can be understood in the following way. Suppose, for example, that at some stage in the iterations we obtain $u_m = Cu$ where u is the solution of Eq. (B.1) and C is a complex constant. In this case, by Eq. (B.4)

$$\mathcal{F}(u_{m+1}) = \frac{1}{k^2+\nu}\,\mathcal{F}(C^{p-1}|u|^{p-1}\,Cu),$$

i.e., $u_{m+1} = C^p u$. The next iteration will yield $u_{m+2} = (C^p)^p\,u = C^{p^2}u$ so that by induction,

$$u_{m+n} = (C)^{p^n}\,u. \tag{B.5}$$

Therefore, the iterations will diverge to $u_\infty \equiv 0$ if $|C| < 1$ and to $u_\infty \equiv \infty$ if $|C| > 1$.

The argument above shows that in order to make sure that the iterations converge to u, we need to prevent the norm of u_m from going to zero or to infinity. To do that, it was originally suggested by Petviashvili [97] and later by Ablowitz *et al.* [4] to renormalize the

solution in each iteration so that it satisfies an integral identity. In particular, multiply Eq. (B.2) by $\big[\mathcal{F}(u)\big]^*$ and integrate over k, resulting in the integral identity

$$\int |\mathcal{F}(u)|^2 dk = \int \frac{1}{k^2+\nu}\mathcal{F}(|u|^{p-1}u)\big[\mathcal{F}(u)\big]^* dk. \tag{B.6}$$

Let us define

$$\mathrm{SL} \equiv \int |\mathcal{F}(u)|^2 dk, \qquad \mathrm{SR} \equiv \int \frac{1}{k^2+\nu}\mathcal{F}(|u|^{p-1}u)\big[\mathcal{F}(u)\big]^* dk. \tag{B.7}$$

In general, u_m does not satisfy condition (B.6). Therefore, it was suggested to define $u_{m+\frac{1}{2}} = C_m u_m$ where the real constant C_m is chosen so that $u_{m+\frac{1}{2}}$ will satisfy identity (B.6). Specifically, in a similar manner to Eq. (B.6) we define

$$\mathrm{SL}_m \equiv \int |\mathcal{F}(u_m)|^2 dk, \qquad \mathrm{SR}_m \equiv \int \frac{1}{k^2+\nu}\mathcal{F}(|u_m|^{p-1}u_m)\big[\mathcal{F}(u_m)\big]^* dk, \tag{B.8}$$

so that

$$\mathrm{SL}_{m+\frac{1}{2}} = C_m^2\ \mathrm{SL}_m = C_m^{p+1}\ \mathrm{SR}_m = \mathrm{SR}_{m+\frac{1}{2}}.$$

This equation has three real solutions: $C_m = 0$ (corresponding to $u_\infty = 0$), $C_m = \infty$ (corresponding to $u_\infty = \infty$) and the nontrivial solution

$$C_m = \left(\frac{\mathrm{SL}_m}{\mathrm{SR}_m}\right)^{\frac{1}{p-1}}, \tag{B.9}$$

corresponding to $u_\infty = u$. Therefore, we can avoid the divergence to $u_\infty = \infty$ or $u_\infty = 0$ by applying the iterations (B.3) to $u_{m+\frac{1}{2}}$ rather than to u_m, i.e.,

$$\mathcal{F}(u_{m+1}) = \left(\frac{\mathrm{SL}_m}{\mathrm{SR}_m}\right)^{\beta^*} \frac{1}{k^2+\nu}\mathcal{F}(|u_m|^{p-1}u_m), \qquad \beta^* = \frac{p}{p-1}. \tag{B.10}$$

In other words, by choosing C_m as in (B.9) (i.e., so that the integral identity in the frequency domain (B.6) is satisfied), we restrict the iterations to the family of solutions $\{u|\ u \text{ satisfies } (B.6),\ u \not\equiv 0,\ u \not\equiv \infty\}$ and obtain the desired solution.

Although, in all the studies that used this scheme the above identity was used, we have shown that one can use any other identity in order to suppress the divergence from the desired solution u. In that sense, the name given to the method by Ablowitz and Muslimanni [4] can be changed from "spectral renormalization method" to simply "renormalization method".

It is also worth mentioning that numerical simulations show that the iterations (B.10) converge to $u_\infty = u$ also for some values of β different from β^*. In order to see why, let us assume, as before, that at some stage of the iterations we obtain $u_m = Cu$. Then, by Eq. (B.8)

$$\mathrm{SL}_m = C^2\mathrm{SL}(u), \qquad \mathrm{SR}_m = C^{p+1}\mathrm{SR}(u),$$

and the next iteration will be

$$\begin{aligned}\mathcal{F}(u_{m+1}) &= \left(\frac{\mathrm{SL}_m}{\mathrm{SR}_m}\right)^{\beta} \frac{1}{k^2+\nu}\, \mathcal{F}(|u_m|^{p-1}\, u_m) = \left(\frac{C^2}{C^{p+1}}\right)^{\beta} \frac{1}{k^2+\nu}\, \mathcal{F}(C^p|u|^p) \\ &= C^{\beta-\beta p+p}\mathcal{F}(u), \end{aligned} \tag{B.11}$$

where we used relation (B.4). For $\gamma = \beta - \beta p + p$, we get that $u_{m+1} = C^{\gamma} u$ and by induction

$$u_{m+n} = (C)^{\gamma^n}\, u.$$

Since the iterations converge (i.e., $\lim_{m\to\infty} u_m = u$) only for $C^{\gamma^n} \to 1$, a necessary condition for convergence is $|\gamma| < 1$. This corresponds to $1 < \beta < \frac{p+1}{p-1}$. The fastest rate of convergence is obtained for $\beta = \beta^*$ since in this case $\gamma = 0$ and $u_{m+1} = u$.

B.2 Numerical code

A sample Matlab numerical code for the Petviashvili method is as follows:

```
function [PSI,x]=SR_SL(Nx,XMAX,P,nu,thresh,iter)

dx      = 2*XMAX/Nx;                    %
x       = [-XMAX:dx:+XMAX-dx];          %
dk      = pi/XMAX;                      %
kx      = fftshift([-Nx/2:Nx/2-1]*dk); %
denom   = 1./(nu + kx.^2);              %
beta    = P/(P-1);                      % normalization power
PSI     = exp(-x.^2);                   % initial guess

n=1; ratio=1;

while ( abs(ratio)>thresh & n<iter )
    old_psi = PSI;
    psi_hat = fft(old_psi);
    NL_hat  = fft((abs(old_psi)).^(P-1).*(old_psi));

    RHS     = denom.*(NL_hat);
    SL      = Integrate(conj(psi_hat).*psi_hat,x);
    SR      = Integrate(conj(psi_hat).*RHS,x);
    psi_hat = (SL/SR)^beta*RHS;
    PSI     = ifft(psi_hat);

    ratio   = abs(abs(SL)/abs(SR)-1);
    n=n+1;
end
```

In this code, *Integrate* is some routine that computes the integral over the input vector, e.g., Simpson integration. Typically, this scheme converges after a few iterations.

B.3 Generalizations

One of the big advantages of using the (spectral) renormalization method is its easy application to NLS-type equations with higher dimension or with potentials.

For example, the renormalization method for the 2D NLSE

$$-\partial_x^2 u(x,y) - \partial_y^2 u - |u|^{p-1}u + \nu u = 0, \tag{B.12}$$

can be obtained by replacing k with $\mathbf{k} = (k_x, k_y)$ in (B.10). In this case, the iteration scheme becomes

$$\mathcal{F}(u_{m+1}) = \left(\frac{\mathrm{SL}_m}{\mathrm{SR}_m}\right)^{\beta} \frac{1}{k_x^2 + k_y^2 + \nu} \mathcal{F}(|u_m|^{p-1}u_m), \tag{B.13}$$

where

$$\mathrm{SL}_m \equiv \int |\mathcal{F}(u_m)|^2 dk_x dk_y, \qquad \mathrm{SR}_m \equiv \int \frac{1}{k_x^2 + k_y^2 + \nu} \mathcal{F}(|u_m|^{p-1}u_m)\big[\mathcal{F}(u_m)\big]^* dk_x dk_y, \tag{B.14}$$

and $\mathcal{F}$ is the two-dimensional Fourier transform.

In my research we used the renormalization method in order to compute solitons in media with a nonlinear potential [44, 116] and with a linear potential. In the latter case, the scheme should be slightly modified, see e.g., [4].

Another extension of the scheme (B.10) is to the biharmonic NLSE

$$\partial_x^4 u(x) - |u|^{p-1}u + \nu u = 0. \tag{B.15}$$

where the scheme is obtained by replacing k^2 by k^4 in (B.10). In this case, the iteration scheme becomes

$$\mathcal{F}(u_{m+1}) = \left(\frac{\mathrm{SL}_m}{\mathrm{SR}_m}\right)^{\beta^*} \frac{1}{k^4 + \nu} \mathcal{F}(|u_m|^{p-1}u_m), \tag{B.16}$$

where

$$\mathrm{SL}_m \equiv \int |\mathcal{F}(u_m)|^2 dk, \qquad \mathrm{SR}_m \equiv \int \frac{1}{k^4 + \nu} \mathcal{F}(|u_m|^{p-1}u_m)\big[\mathcal{F}(u_m)\big]^* dk. \tag{B.17}$$

An immediate extension is possible also for the anisotropic equation

$$\partial_x^4 u(x,y) - \partial_x^2 u - \partial_y^2 u - |u|^{p-1}u + \nu u = 0. \tag{B.18}$$

B.4 Convergence

Analytical proof for the conditions of the convergence of iteration schemes such as (B.10) were obtained in [94] for a more general problem. In the context of Eq. (B.1), the main result of [94] can be stated as:

Theorem B.1 *Let $u(x)$ be a solution of Eq. (B.1). Then, the iteration scheme (B.10) converges to $u(x)$ in a small neighbourhood of $u(x)$ if two conditions are met:*

1. $1 < \beta < \frac{p+1}{p-1}$.

2. $n_-(L_+) = 1$ *where* $L_+ = -\nabla^2 - pu^{p-1} + \nu$ *and* n_- *counts the number of its negative eigenvalues.*

The fastest rate of convergence of the iterations is achieved for $\beta = \beta^* = p/(p-1)$. *If any of these two conditions are not met, the iteration scheme (B.10) diverges from* $u(x)$.

A simple and intuitive explanation for the first condition was given in Section B.1. The second condition is, in fact, equivalent to the spectral condition (5.5) for stability of solitons of the NLSE. As shown in Section 6.1, the spectral condition is satisfied only for positive solitons in homogeneous media and for positive solitons centered at potential minima in inhomogeneous media. Indeed, according to our experience, the iteration scheme (B.10) easily converges to the positive ground-state soliton in those two cases[35].

However, despite the above, it was shown in several studies that the iteration scheme (B.10) can sometimes converge to solitons centered at a potential maximum, see e.g. [44, 116, 113]. Moreover, in [44] we showed that if the real part of the iterative solution u_m is taken at each iteration, then the iteration scheme (B.10) can sometimes be forced to converge even to solitons centered at a potential maximum.

These apparent inconsistencies were explained in [114]. In particular, it was shown that the reason for the divergence of the iteration scheme from solitons centered at potential maxima is that the iterative solution u_m undergoes a drift instability in a completely analogous manner to that experienced by solitons of the NLSE. Still, we showed in [114] that if the drift rate is sufficiently slow, the iterations can converge to a soliton centered at a potential maximum after all. In that sense, forcing the solution of the iterative scheme to be real is one way to slow down the drift rate. These results show the universality of the spectral condition (5.5) and shows once again the importance of the quantitative study of the stability/convergence of solitons.

[35] It should be noted that with a proper initial guess, it can also converge to high-order positive solitons such as the vortex solitons [117] or multi-hump solitons [115].

References

[1] F.K. Abdullaev, A. Abdumalikov, and R. Galimzyanov, *Gap solitons in Bose-Einstein condensates in linear and nonlinear optical lattices*, Phys. Lett. A **367** (2007), 149.

[2] F.K. Abdullaev, A. Gammal, A.M. Kamchatnov, and L. Tomio, *Dynamics of bright solitons in a Bose-Einstein condensate*, Int. J. of Mod. Phys. B **19** (2005), 3415–3473.

[3] M.J. Ablowitz and Z.H. Musslimani, *Discrete diffraction managed spatial solitons*, Phys. Rev. Lett. **87** (2001), 254102.

[4] _____, *Spectral renormalization method for computing self-localized solutions to nonlinear systems*, Opt. Lett. **30** (2005), 2140–2142.

[5] G.P. Agrawal, *Nonlinear fiber optics*, third ed., Academic Press, 2001.

[6] J.S. Aitchison, K. Al-Hemyari, C.N. Ironside, R.S. Grant, and W. Sibbett, *Observation of spatial solitons in algaas waveguides*, Electron. Lett. **28** (1992), 1879.

[7] J.S. Aitchison, A.M. Weiner, Y. Silberberg, M.K. Oliver, J.L. Jackel, D.E. Leaird, E.M. Vogel, and P.W.E. Smith, *Observation of spatial optical solitons in a nonlinear glass waveguide*, Opt. Lett. **15** (1990), 471.

[8] N. Akhmediev and M. Karlsson, *Cherenkov radiation emitted by solitons in optical fibers*, Phys. Rev. A **51** (1995), 2602.

[9] G.L. Alfimov, V.V. Konotop, and P. Pacciani, *Stationary localized modes for the quintic nonlinear Schrödinger equation with a periodic potential*, Phys. Rev. A **75** (2007), 023624.

[10] L. Allen and J.H. Eberly, *Optical resonance and two-level atoms*, 1994.

[11] A. Arie and A. Bahabad, *Quasi phase matching in two-dimensional nonlinear photonic crystals*, Optical and Quantum Electronics **39** (2007), 361.

[12] G.A. Askar'yan, *Effect of the gradient of a strong electromagnetic beam on electrons and atoms*, Sov. Phys. JETP **15** (1962), 1088.

[13] S. Bar-Ad, Unpublished.

[14] C. Barsi, W. Wan, C. Sun, and J.W. Fleischer, *Dispersive shock waves with nonlocal nonlinearity*, Opt. Lett **32** (2007), 2930.

[15] A. Barthélémy, S. Maneuf, and C. Froehly, *Propagation soliton et auto-confinement de faisceaux laser par non linearité optique de kerr*, Opt. Commun. **55** (1985), 201.

[16] U. Bartuch, U. Peschel, Th. Gabler, R. Waldhaus, and H.-H. Horhold, *Experimental investigations and numerical simulations of spatial solitons in planar polymer waveguides*, Opt. Commun. **134** (1997), 49.

[17] L. Bergé, V.K. Mezentsev, J.J. Rasmussen, P.L. Christiansen, and Y.B. Gaididei, *Self-guiding light in layered nonlinear media*, Opt. Lett. **25** (2000), 1037–1039.

[18] J.E. Bjorkholm and A. Ashkin, *CW self-focusing and self-trapping of light in Sodium vapor*, Phys. Rev. Lett. **32** (1974), 129.

[19] K.J. Blow and D. Wood, *Theoretical description of transient stimulated raman scattering in optical fibers*, IEEE J. Qu. Elec. **25** (1989), 2665–2672.

[20] G. Boudebs, S. Cherukulappurath, H. Leblond, J. Troles, F. Smektala, and F. Sanchez, *Experimental and theoretical study of higher-order nonlinearities in chalcogenide glasses*, Opt. Commun. **219** (2003), 427.

[21] J.P. Boyd, *Why newton's method is hard for travelling waves: Small denominators, kam theory, arnolds linear fourier problem, non-uniqueness, constraints and erratic failure*, Math. Comput. Simul. **74** (2007), 7281.

[22] R.W. Boyd, *Nonlinear optics*, second ed., Academic Press, 2003.

[23] G. Boyer, *High-power femtosecond-pulse reshaping near the zero-dispersion wavelength of an optical fiber*, Opt. Lett. **24** (1999), 945.

[24] V.A. Brazhnyi and V.V. Konotop, *Theory of nonlinear matter waves in optical lattices*, Mod. Phys. Lett. B **18** (2004), 627.

[25] H. Buljan, O. Manela, R. Pezer, A. Vardi, and M. Segev, *Dark stationary matter waves via parity-selective filtering in a tonks-girardeau gas*, Phys. Rev. A **74** (2006), 043610.

[26] A.V. Buryak, P. Di Trapani, D.V. Skryabin, and S. Trillo, *Optical solitons due to quadratic nonlinearities: From basic physics to futuristic applications*, Phys. Rep. **63** (2002), 370.

[27] M. Centurion, M.A. Porter, P.G. Kevrekidis, and D. Psaltis, *Nonlinearity management in optics: experiment, theory and simulations*, Phys. Rev. Lett. **97** (2006), 033903.

[28] S-C. Cheng, *An effective-mass theory for bright gap solitons in optical lattices*, Chinese J. of Phys. **43** (2005), 835.

[29] D. Cheskis, Y. Linzon, I. Ilsar, and S. Bar-Ad, *Raman-induced localization in kerr waveguide arrays*, Opt. Lett. **32** (2007), 2459.

[30] R.Y. Chiao, E. Garmire, and C.H. Townes, *Self-trapping of optical beams*, Phys. Rev. Lett. **13** (1964), 479.

[31] M.L. Chiofalo, S. Succi, and M.P. Tosi, *Ground state of trapped interacting bose-einstein condensates by an explicit imaginary-time algorithm*, Phys. Rev. E **62** (2000), 7438.

[32] D.N. Christodoulides and R.I. Joseph, *Discrete self-focusing in nonlinear array of coupled waveguides*, Opt. Lett. **13** (1988), 794–796.

[33] B. Crosignani, P. DiPorto, M. Segev, G. Salamo, and A. Yariv, *Nonlinear optical beam propagation and solitons in photorefractive media*, La Rivista del Nuovo Cimento **21** (1998), 1.

[34] S. Cuccagna, E. Kirr, and D.M. Pelinovsky, *Parametric resonance of ground states in the nonlinear Schrödinger equation*, J. Diff Eqs. **220** (2007), 85.

[35] R. de La Rue, *Harnessing the power of the photon*, Opt. Photon. News **17** (2006), 30.

[36] J.R. de Oliviera, M.A. de Moura, J.M. Hickmann, and A.S.L. Gomes, *Self-steepening of optical pulses in dispersive media*, J. Opt. Soc. Am. B **9** (1992), 2025.

[37] C. Martijn de Sterke and J.E. Sipe, *Envelope-function approach for the electrodynamics of nonlinear structures*, Phys. Rev. A **38** (1988), 5149.

[38] A.S. Desyatnikov, L. Torner, and Y.S. Kivshar, *Optical vortices and vortex solitons*, Prog. Opt. **47** (2005), 291.

[39] J.M. Dudley, G. Genty, and S. Coen, *supercontinuum generation in photonic crystal fibers*, Rev. Mod. Phys. **78** (2006), 1135.

[40] G. Duree, M. Morin, G. Salamo, M. Segev, B. Crosignani, P. Di Porto, E. Sharp, and A. Yariv, *Dark photorefractive spatial solitons and photorefractive vortex solitons*, Phys. Rev. Lett. **74** (1995), 1978.

[41] J.R. Ferraro and K. Nakamoto, *Introductory raman spectroscopy*, Academic Press, 1994.

[42] G. Fibich and B. Ilan, *Vectorial and random effects in self-focusing and in multiple filamentation*, Physica D **157** (2001), 113–147.

[43] G. Fibich, B. Ilan, and G.C. Papanicolaou, *Self focusing with fourth-order dispersion*, SIAM J. Appl. Math. **62** (2002), 1437–1462.

[44] G. Fibich, Y. Sivan, and M.I. Weinstein, *Bound states of NLS equations with a periodic nonlinear microstructure*, Physica D **217** (2006), 31–57.

[45] G. Fibich and X.P. Wang, *Stability of solitary waves for nonlinear schrödinger equations with inhomogeneous nonlinearities*, Physica D **175** (2003), 96–108.

[46] J.W. Fleischer, G. Bartal, O. Cohen, T. Schwartz, O. Manela, B. Freedman, M. Segev, H. Buljan, and N.K. Efremidis, *Spatial photonics in nonlinear waveguide arrays*, Opt. Exp. **13** (2005), 1780–1796.

[47] P.L. Francois, *Nonlinear propagation of ultrashort pulses in optical fibers: total field formulation in the frequency domain*, J. Opt. Soc. Am. B **8** (1991), 276.

[48] J. Frölich, S. Gustafson, B.L.G. Jonsson, and I.M. Sigal, *solitary waves dynamics in an external potential*, Comm. Math. Phys. **250** (2004), 613–642.

[49] L. Gagnon and P.A. Bélanger, *Soliton self-frequency shift versus Galilean-like symetry*, Opt. Lett. **15** (1990), 466.

[50] J. Garnier and F.K. Abdullaev, *Transmission of matter-wave solitons through nonlinear traps and barriers*, Phys. Rev. A **74** (2006), 013604.

[51] N. Ghofraniha, C. Conti, G. Ruocco, and S. Trillo, *Shocks in nonlocal media*, Phys. Rev. Lett **99** (2007), 043903.

[52] B.V. Gisin, R. Driben, and B.A. Malomed, *Bistable guided solitons in the cubic-quintic medium*, J. Opt. B **6** (2004), S259.

[53] J.P. Gordon, R.C.C. Leite, R.S. Moore, S.P.S Porto, and J.R. Whinnery, *Long transient effects in lasers with lasers with inserted liquid samples*, J. Appl. Phys. **36** (1965), 3.

[54] M.G. Grillakis, J. Shatah, and W.A. Strauss, *Stability theory of solitary waves in the presence of symmetry i*, J. Funct. Anal. **74** (1987), 160–197.

[55] ———, *Stability theory of solitary waves in the presence of symmetry ii*, J. Funct. Anal. **94** (1990), 308–348.

[56] A. Hasegawa and F. Tappert, *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. anomalous dispersion*, Appl. Phys. Lett. **23** (1973), 142.

[57] R.W. Hellwarth, *Third-order optical susceptibilities of liquids and solids*, Prog. Qu. Elec. (1977), 1–68.

[58] D. Hollenbeck and C.D. Cantrell, *Multiple-vibratinal-mode model for fiber-optic Raman gain spectrum and response function*, J. Opt. Soc. Am. B **19** (2002), 2886–2892.

[59] D.C. Hutchings, *Theory of ultrafast nonlinear refraction in semiconductor superlattices*, IEEE J. of Sel. topics in Q. Elec. **10** (2004), 1124–1132.

[60] B. Ilan and M.I. Weinstein, Preprint.

[61] J.D. Jackson, *Classical electrodynamics*, third ed., Wiley, 1999.

[62] Y.N. Karamzin and A.P. Sukhorukov, Sov. Phys. JETP **41** (1976), 414.

[63] N. Karasawa, S. Nakamura, N. Nakagawa, M. Shibata, R. Morita, H. Shigekawa, and M. Yamashita, *Comparison between theory and experiment of nonlinear propagation for a-few-cycle and ultrabroadband optical pulses in a fused-silica fiber*, IEEE J. Quant. Elec. **37** (2001), 398.

[64] V.I. Karpman, *Soliton evolution in the presence of perturbation*, Phys. Scr. **20** (1979), 462.

[65] V.I. Karpman and A.G. Shagalov, *Stability of solitons described by nonlinear schrödinger-type equations with higher-order dispersion*, Physica D **144** (2000), 194.

[66] Y. V. Kartashov, A.S. Zelenina, L. Torner, and V.A. Vysloukh, *Spatial soliton switching in quasi-continuous optical arrays*, Opt. Lett. **29** (2004), 766.

[67] H.B. Keller, *Numerical solution of bifurcation and nonlinear eigenvalue problems, in P.H. Rabinowitz, applications of bifurcation theory*, Academic Press.

[68] P.L. Kelley, *Self-focusing of optical beams*, Phys. Rev. Lett. **15** (1965), 1005–1008.

[69] G. Khitrova, H.M. Gibbs, Y. Kawamura, H. Iwamura, T. Ikegami, J.E. Sipe, and L. Ming, Phys. Rev. Lett. **70** (1993), 920.

[70] I.C. Khoo, Prog. Opt. **26** (1988), 107.

[71] Y.S. Kivshar and G.P. Agrawal, *Optical Solitons*, Academic Press, 2003.

[72] Y.S. Kivshar and D.K. Campbell, *Peierls-Nabarro potential barrier for highly localized nonlinear modes*, Phys. Rev. E **48** (1993), 3077.

[73] E.B. Kolomeisky, T.J. Newman, J.P. Straley, and X. Qi, *Low-dimensional Bose liquids: beyond the Gross-Pitaevskii approximation*, Phys. Rev. Lett. **85** (2000), 1146.

[74] V.V. Konotop, *in Dissipative Solitons, edited by n. akhmediev*, Springer, 2005.

[75] V.I. Kruglov, Y.A. Logvin, and V.M. Volkov, *The theory of spiral laser beams in nonlinear media*, J. Modern Optics **39** (1992), 2277–2291.

[76] Y. Lahini, A. Avidan, F. Pozzi, M. Sorel, R. Morandotti, D.N. Christodoulides, and Y. Silberberg, *Anderson localization and nonlinearity in one-dimensional disordered photonic lattices*, Phys. Rev. Lett. **100** (2008), 013906.

[77] T.I. Lakoba and J. Yang, *A generalized petviashvili iteration method for scalar and vector hamiltonian equations with arbitrary form of nonlinearity*, Journal of Computational Physics **226** (2007), 1668.

[78] S. Le-Coz, R. Fukuizumi, G. Fibich, B. Ksherim, and Y. Sivan, *Instability of bound states for a nonlinear schrödinger equation with a dirac potential*, Physica D **2008** (237), 1103.

[79] Y. Linzon, I. Ilsar, D. Cheskis, R. Morandotti, J. S. Aitchison, and S. Bar-Ad, *Near-field imaging of nonlinear pulse propagation in planar silica waveguides*, Phys. Rev. E **72** (2005), 066607.

[80] X. Liu, K. Beckwitt, and F.W. Wise, *Two-dimensional optical spatiotemporal solitons in quadratic media*, Phys. Rev. E **62** (2000), 1328.

[81] X. Liu, L.J. Qian, and F.W. Wise, *Generation of optical spatiotemporal solitons*, Phys. Rev. Lett. **82** (1999), 4631.

[82] Y. Liu, G. Bartal, D.A. Genov, and X. Zhang, *Subwavelength discrete solitons in nonlinear metamaterials*, Phys. Rev. Lett. **99** (2007), 153901.

[83] K.E. Lonngren, Plasma Phys. **25** (1983), 943.

[84] K.G. Makris, S. Suntsov, D.N. Christodoulides, G.I. Stegeman, and A. Hache, *Discrete surface solitons*, Opt. Lett. **30** (2005), 2466.

[85] B.A. Malomed, D. Mihalache, F. Wise, and L. Torner, *Spatiotemporal optical solitons*, J. Opt. B **7** (2005), R53.

[86] P.V. Mamyshev and S.V. Chernikov, *Ultrashort-pulse propagation in optical fibers*, Opt. Lett. **15** (1990), 1076–1078.

[87] S.V. Manakov, Sov. Phys. JETP **38** (1974), 248.

[88] L.F. Mollenauer, R.H. Stolen, and J.P. Gordon, Phys. Rev. Lett. **45** (1980), 1095.

[89] R. Morandotti, U. Peschel, J.S. Aitchison, H.S. Eisenberg, and Y. Silberberg, *Dynamics of discrete solitons in optical waveguide arrays*, Phys. Rev. Lett. **83** (1999), 2726–2729.

[90] J. Moses, B.A. Malomed, and F. Wise, *Self-steepening of ultrashort pulses without self-phase-modulation*, Phys. Rev. E **76** (2007), 021802(R).

[91] Z.H. Musslimani and J. Yang, *Self-trapping of light in a two-dimensional photonic lattice*, J. Opt. Soc. Am. B **21** (2004), 973–981.

[92] Y-G. Oh, *Stability of semiclassical bound states of nonlinear Schrödinger equations with potentials*, Comm. Math. Phys.. **121** (1989), 11–33.

[93] D.E. Pelinovsky, P.G. Kevrekidis, and D.J. Frantzeskakis, *Averaging for solitons with nonlinearity management*, Phys. Rev. Lett. **91** (2003), 240201.

[94] D.E. Pelinovsky and Y.A. Stepanyants, *Convergence of Petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations*, SIAM J. Appl. Math. **42** (2004), 1110–1127.

[95] D.E. Pelinovsky, A.A. Sukhorukov, and Y.S. Kivshar, *Bifurcations and stability of gap solitons in periodic potentials*, Phys. Rev. E **70** (2004), 036618.

[96] C.J. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases*, Cambridge University Press, 2001.

[97] V.I. Petviashvili, *Equation of an extraordinary soliton*, Sov. J. Plasma Phys. **2** (1976), 257.

[98] E. Polturak, P.G.N. de Vegvar, E.K. Zeise, and D.M. Lee, *Solitonlike propagation of zero sound in superfluid 3He*, Phys. Rev. Lett. **46** (1981), 1588.

[99] M.A. Porter, P.G. Kevrekidis, B.A. Malomed, and D.J. Frantzeskakis, *Modulated amplitude waves in collisionally inhomogeneous bose-einstein condensates*, Phyica D **229** (2007), 104.

[100] Z. Rapti, P.G. Kevrekidis, V.V. Konotop, and C.K.R.T. Jones, *Solitary waves under the competition of linear and nonlinear periodic potentials*, J. Phys. A. **40** (2007), 14151.

[101] H.A. Rose and M.I. Weinstein, *On the bound states of the nonlinear Schrödinger equation with a linear potential*, Physica D **30** (1988), 207–218.

[102] J.S. Russell, in 14th Meeting of the British Association Reports (1844).

[103] H. Sakagouchi and B.A. Malomed, *Resonant nonlinearity management for nonlinear Schrödinger solitons*, Phys. Rev. E **70** (2004), 066613.

[104] R. De Salvo, D.J. Hagan, M. Sheik-Bahae, G.I. Stegeman, E.W. Van Stryland, and H. Vanherzeele, *Self-focusing and self-defocusing by cascaded second-order effects in ktp*, Opt. Lett. **17** (1992), 28.

[105] R. Schiek, Y. Baek, and G.I. Stegeman, *One-dimensional spatial solitary waves due to cascaded second-order nonlinearities in planar waveguides*, Phys. Rev. E **53** (1996), 1138.

[106] T. Schwartz, G. Bartal, S. Fishman, and M. Segev, *Transport and anderson localization in disordered two-dimensional photonic lattices*, Nature **446** (2007), 52 – 55.

[107] A. Scott, *Emergence and dynamics of coherent structures*, Oxford University Press, 1999.

[108] M. Segev and D.N. Christodoulides, Opt. Photon. News **13** (2002), 70.

[109] M. Segev, B. Crosignani, A. Yariv, and B. Fischer, *Spatial solitons in photorefractive media*, Phys. Rev. Lett. **68** (1992), 928.

[110] M. Segev, G.C. Valley, B. Crosignani, P. Di Porto, and A. Yariv, *Steady-state spatial screening solitons in photorefractive materials with external applied fields*, Phys. Rev. Lett. **73** (1994), 3211.

[111] Y. Silberberg, *Collapse of optical pulses*, Opt. Lett. **15** (1990), 1282.

[112] Y. Sivan, S. Bar-Ad, and A. Katzir, private communication.

[113] Y. Sivan, G. Fibich, N.K. Efremidis, and S. Bar-Ad, *Analytic theory of narrow lattice solitons*, Nonlinearity **21** (2008), 509–536.

[114] Y. Sivan, G. Fibich, and B. Ilan, *Drift instability and tunneling of lattice solitons*, Phys. Rev. E **77** (2008), 045601(R).

[115] Y. Sivan, G. Fibich, and M.I.Weinstein, *Unpublished.*

[116] Y. Sivan, G. Fibich, and M.I. Weinstein, *Waves in nonlinear microstructures - ultrashort optical pulses and bose-einstein condensates*, Phys. Rev. Lett. **97** (2006), 193902.

[117] Y. Sivan and B. Ilan, *Private communications.*

[118] Y. Sivan, B. Ilan, and G. Fibich, *Qualitative and quantitative analysis of stability and instability dynamics of positive lattice solitons*, submitted to Phys. Rev. E.

[119] S. Skupin, L. Bergé, U. Peschel, F. Lederer, G. Méjean, J. Yu, J. Kasparian, E. Salmon, J.P. Wolf, M. Rodrigues, L .Wöste, R. Bourayou, and R. Sauerbrey, *Filamentation of femtosecond light pulses in the air: turbulent cels versus long-range clusters*, Phys. Rev. E **70** (2004), 046602.

[120] F. Smektala, C. Quemard, V. Couderc, and A. Barthélémy, *Non-linear optical properties of chalcogenide glasses measured by Z-scan*, J. Non-Cryst. Solids **274** (2000), 232.

[121] A.W. Snyder and D.J. Mitchell, *Accessible solitons*, Science **276** (1997), 1538.

[122] A.W. Snyder, L. Poladian, and D.J. Mitchell, *Stable black self-guided beams of circular symmetry in a bulk kerr medium*, Opt. Lett. **17** (1992), 789.

[123] M.S. Soskin and M.V. Vasnetsov, *Singular Optics*, Prog. Opt. **42** (2001), 219.

[124] G.I. Stegeman, D.J. Hagan, and L. Torner, Opt. Quantum Electron. **28** (1996), 1691.

[125] G.I. Stegeman and M. Segev, *Optical spatial solitons and their interactions: universality and diversity*, Science **286** (1999), 1518.

[126] R.H. Stolen, J.P. Gordon, W.J. Tomlinson, and H.A. Haus, *Raman response function of silica-core fibers*, J. Opt. Soc. Am. B **6** (1989), 1159.

[127] G.A. Swartzlander and C.T. Law, *Stable black self-guided beams of circular symmetry in a bulk kerr medium*, Phys. Rev. Lett. **69** (1992), 2503.

[128] L. Torner and A. Barthélémy, *Quadratic solitons: recent developments*, IEEE J. Quantum Electron. **39** (2003), 22.

[129] W.E. Torruellas, B. Lawrence, and G.I. Stegeman, Electron. Lett. **32** (1996), 2092.

[130] W.E. Torruellas, E.W. VanStryland Z. Wang, D.J. Hagan, G.I. Stegeman, L. Torner, and C.R. Menyuk, *Observation of two-dimensional spatial solitary waves in a quadratic medium*, Phys. Rev. Lett. **74** (1995), 5036.

[131] M.G. Vakhitov and A.A. Kolokolov, *Stationary solutions of the wave equation in a medium with nonlinearity saturation*, Radiophys. Quant. Elec. **16** (1973), 783–798.

[132] G.C. Valley, M. Segev, B. Crosignani, A. Yariv, M.M. Fejer, and M.C. Bashaw, *Dark and bright photovoltaic spatial solitons*, Phys. Rev. A **50** (1994), R4457.

[133] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–68.

[134] _____, *The nonlinear Schrödinger equations - singularity formation, stability and dispersion*, Contemporary Mathematics **99** (1989), 213–232.

[135] F. Wise, L. Qian, and X. Liu, *Applications of cascaded quadratic nonlinearities to femtosecond pulse generation*, J. of Non. Opt. Phys. and Mat. **11** (2002), 317.

[136] A. Xie, L. van der Meer, W. Hoff, and R.H. Austin, *Long-lived amide I vibrational modes in myoglobin*, Phys. Rev. Lett. **84** (2000), 5435.

[137] Z. Xu, Y.V. Kartashov, and L. Torner, *Soliton mobility in nonlocal optical lattices*, Phys. Rev. Lett **95** (2005), 113901.

[138] C. Zhan, D. Zhang, D. Zhu, D. Wang, Y. Li, D. Li, Z. Lu, L. Zhao, and Y. Nie, J. Opt. Soc. Am. B **19** (2002), 369.

[139] V. Zharnitsky and D.E. Pelinovsky, *Averaging of nonlinearity-managed pulses*, Chaos 15 **15** (2005), 037105.