The Shapley Value Decomposition of Optimal Portfolios

Haim Shalit*

January 30, 2017

Ben-Gurion University of the Negev, Beer-Sheva, Israel. shalit@bgu.ac.il

Abstract

Investors want to be able to evaluate the true and complete risk of the financial assets they hold in a portfolio. Yet, the current analytic methods provide only partial risk measures. In a different approach, by viewing a portfolio of securities as a cooperative game played by the assets that minimize portfolio risk, investors can calculate the exact value each security contributes to the common payoff of the game. This is known as the Shapley value. It is determined by computing the contribution of each asset to the portfolio risk, by looking at all the possible coalitions in which the risky asset would participate. I develop this concept in order to decompose the risk of mean-variance optimal portfolios and mean-Gini portfolios. This decomposition lets us better rank of risky assets by their comprehensive contribution to the risk of optimal portfolios. Such a procedure allows investors to make unbiased and true decisions when they analyze the inherent risk of their holdings. In an application, the Shapley value is calculated for asset allocation and for portfolios of individual securities. The empirical

*I am grateful to Ofer Zevin for directing me to the Shapley value in finance.

results are contrary to some of the findings of conventional wisdom and beta analysis.

1 Introduction

It is well known that investment managers are concerned with the risk impact of adding securities have to portfolios. Since the inception of modern portfolio theory, investors have measured how securities affect each other. The simplest risk as expressed by asset variance is not sufficient to establish sound investment decisions. Financial theoreticians and practitioners now also take into account correlations, covariances, and betas to establish the cross-effects among investments.

My claim is that these risk measures although sufficient to build efficient portfolios are prone to error in measurement of the true impact of a risky asset upon an optimal portfolio. Hence, when presumably rational and efficient efforts misjudge the true risk of assets in optimal portfolios, a totally new approach is required.

My purpose in response is to apply the concept of Shapley value (Shapley, 1953) to financial management theory and practice. Shapley value theory emerged from cooperative game theory in order to measure the exact contribution of agents playing the game. In a cooperative game, players interact to optimize a common objective whose utility is transferable. The Shapley value concept has been applied successfully in economic theory, politics, sports, and income inequality, to cite a few examples. Its use in financial investments has been limited, and the approach is practically nonexistent in optimal portfolio theory.

The idea behind the Shapley value is to look at all the possible coalitions of players in a cooperative game, and calculate the benefits each player contributes to the various coalitions. As each contribution depends upon the order in which players join the coalition, the Shapley value is calculated by averaging the marginal contributions from the arrival of the various players to the specific coalitions.

In a sense, a portfolio of risky assets is a cooperative game played in order to maximize return or minimize risk. This is a natural way to look at portfolios, and the Shapley value is similarly a natural concept to decompose optimal portfolio risk into its various components. The contribution of assets to the portfolio is true since contributions are derived from all the possible optimal portfolios constructed by the various coalitions. I apply this major insight to standard mean-variance (MV) portfolio management and to the newer mean-Gini portfolio optimization. In an analytic sense, MV portfolio optimization, least-squares minimization, and cooperative game theory share common ground, and the tools used in one field can easily be applied in the others.

The paper is laid out as follows: First, I use Roth's (1988) essay to introduce the Shapley value theory. The notion of applying the Shapley value to decompose some attribute by sources of contribution was formulated by Shorrocks (2013) in a working paper circulated since 1999. Shorrocks presented a general framework to decompose poverty and inequality measures by sources of income using the Shapley value. The same approach was further elaborated by Sastre and Trannoy (2002). Applying the decomposition theory to financial risk and portfolios would follow naturally because inequality measures and risk measures are closely related. Mussard and Terraza (2007) (2008) were the first to use the Shapley value to decompose the risk of given portfolios, although their methodology does not to consider optimal portfolios.

Then, I use Merton's (1972) derivation of the mean-variance efficient frontier to calculate the contribution of each security to the various portfolio coalitions and formulate the Shapley value in optimal portfolios. Finally, I calculate Shapley values for mean-Gini (MG) portfolios. I use Shalit and Yitzhaki's (2005) analytic derivation of the MG efficient frontier to obtain Shapley values that follow stochastic dominant optimal portfolios.

2 On Shapley Value

First, I describe the concept of Shapley value decomposition for use in the mean-variance portfolio and the mean-Gini portfolio. This section draws considerably from Shorrocks (2013) who developed an unified framework to decompose an attribute by its factors. In the standard investment model, I present the portfolio of stocks as an n-person cooperative game with transferable utility where the financial assets are players in the game. The aim is to measure the exact contribution of each player to the general outcome. For a portfolio of securities, the optimization outcome is the risk inherent in the portfolio. Hence, the Shapley value allows us to extract the true and exact contribution of each stock to the portfolio's total risk.

Harsanyi (1977) has enunciated the postulates that lead to the Shapley value theorem as the solution to a cooperative game, where a joint payoff is the specific characteristic function. Here, the joint payoff function defined is the risk borne by the players (i.e., the stocks) in the game. Shapley value theory ensures that the risk decomposition attributed to the various shares in the portfolio is *anonymous* (or *symmetric*), so that the marginal contributions are independent of the order in which the shares are added to the portfolio and *exact* in the sense that all the securities bear the entire portfolio risk.

Consider a portfolio of securities that play a cooperative game whose purpose is to minimize the risk of the portfolio. For a set N of n securities, the Shapley value calculates the contribution of each and every security in the portfolio. To capture the symmetric and exact way each security contributes to the portfolio, we compute the risk v for each and every subset $S \subset N$. In total we have 2^n coalitions including the empty set.

Computations proceed by looking at the marginal contribution of each security to the risk of a portfolio it is a member of. For a given coalition (portfolio) a security k in S contributes marginally to the portfolio by $v(S)-v(S\setminus\{k\})$, where $v(S)$ is the risk of portfolio S, and $v(S\setminus\{k\})$ is the risk of the portfolio composed of S minus the security k. Portfolios are arranged in some given order, and all the orderings are equally probable. Hence, $S\setminus\{k\}$ is the portfolio of securities that precedes k, and its contribution to coalition S is computed when all the orderings of S are accounted for. Thus, given all the equally probable orderings, one can calculate their expected marginal contribution.

For that purpose, one needs the probability that, for a given ordering, the portfolio $S \subset N$, $k \in S$ is seen as the union of security k and the securities that precede it. Two probabilities are used here: First, the probability that k is in s (s being the number of stocks in S) which equals $1/n$, and second, that $S\setminus\{k\}$ arises when $s-1$ securities are randomly chosen from $N\setminus\{k\}$, that is $(n-s)!(s-1)!/(n-1)!$.

The Shapley value for security k is obtained by averaging the marginal contributions to the risk of all portfolios for a set of N securities and the risk function v, which in mathematical terms is written:

$$Sh_k(N,v) \ = \sum_{S\subset N, k\in S} \frac{(n-s)!(n-1)!}{n!}[v(S)-v(S\setminus\{k\})] \tag{1}$$

or

$$Sh_k(N,v) \ = \sum_{S\subset N, k\in S} \frac{s!(n-s-1)!}{n!}[v(S\cup k)-v(S)]) \ . \tag{2}$$

Naturally, the sum of all the Shapley values of the assets equals the total risk of the portfolio that comes from all the securities:

$$v(S)=\sum_{k=0}^{n} Sh_k(N,v) \ . \tag{3}$$

These equations are the basic formulas for the Shapley value computation. In what follows I show how to to define a cooperative game in portfolio optimization and how to set up the common payoff of that game whenever it is played in mean-variance or mean-Gini.

3 On Risk Decomposition of Optimal Mean-Variance Portfolios

Second, I develop the Shapley value for the stocks that constitute an optimal mean-variance portfolio. Given that Shapley value theory considers a single attribute to be allocated among all game participants, I use the Markowitz (1952) MV model by looking mainly at optimal portfolios. Furthermore, Shapley value theory was developed primarily to allocate benefits, i.e., returns, and less to distribute costs, i.e., risk. Hence, by looking at efficient portfolios and minimizing their variance, one ensures that expected returns are always at their best *à la* Markowitz.

To proceed, I calculate first the Shapley value of securities that constitute the global minimum-variance portfolio (MVP). This is a easier task as it requires only minimizing portfolio risk regardless of the required expected return. Thereafter, I address the entire set of frontier portfolios delineated in the MV space. Frontier portfolios are generated by minimizing the portfolio variance for a given expected return. MVP is the frontier portfolio that has the least variance. Once the optimal portfolios are calculated, Shapley values are produced for all the assets along the efficient frontier.

To construct a portfolio frontier in the MV space, I consider N risky assets with returns r that are assumed to be linearly independent. This ensures that the variance-covariance matrix of asset returns $\varSigma$ is non-singular. We denote by μ the vector of the asset's expected returns, and by w the vector of portfolio weights, such that $\sum_{i=1}^{N} w_i = 1$. We assume $w \lesseqgtr 0$, hereby allowing for short sales. A frontier portfolio is obtained by minimizing the variance portfolio σ_p^2 subject to a required mean μ_p. We minimize $\frac{1}{2}w'\Sigma w$ subject to $\mu_p = w'\mu$ and the portfolio constraint $1 = w'l$, where l is an N-vector of ones. As Huang and Litzenberger (1988) show, the solution is obtained by minimizing the Lagrangian that includes the two constraints and deriving the first-order conditions (FOC) for a minimum, and the second-order conditions are satisfied by the non-singularity of Σ.

For the sake of presentation, let us define the quadratic forms: $A = l'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\mu$, $C = l'\Sigma^{-1}l$, and $D = BC - A^2$. All these scalars are positive since the matrix Σ is positive-definite. From the FOC for a minimum variance the optimal portfolio weights for a given mean μ_p are derived as:

$$w_p^* = \frac{1}{D}[B \cdot \Sigma^{-1}l \; - \; A \cdot \Sigma^{-1}\mu] + \frac{1}{D}[C \cdot \Sigma^{-1}\mu \; - \; A \cdot \Sigma^{-1}l]\mu_p. \tag{4}$$

The frontier portfolios delineate an hyperbola in the standard deviation-mean space. Thus, the frontier portfolio variance for a given μ_p is formulated by:

$$\sigma_p^2 = w_p'\Sigma w_p = \frac{C}{D}(\mu_p \; - \; \frac{A}{C})^2 + \frac{1}{C}. \tag{5}$$

Equation (5) is the basic formula for representing the frontier of optimal MV portfolios used to calculate the Shapley value of the stocks. I examine two specific cases: (1) the MVP, and (2) the portfolios for a given mean. The reason for this distinction is that for MVP, the expected value equals A/C, and therefore the variance of MVP equals $1/C$. This simplifies the computation of the Shapley value for the securities in the MVP as outlined below:

1. Establish all the 2^N subsets of the securities in set N.

2. Compute the variance-covariance matrix Σ and $C = l'\Sigma l$ for all the subsets in set N.

3. The variance of the MVP for each subset is $\sigma^2_{MVP} = 1/C$.

4. Following Equation (2), the Shapley value for each stock i in MVP is obtained as:

$$Sh_i(MVP) = \sum_{s=1}^{N-1} \sum_{S \subset N \setminus i} \frac{(N-s-1)!s!}{N!} \left(\frac{1}{C_{S \cup i}} - \frac{1}{C_S} \right) \tag{6}$$

5. The sum of the Shapley values for all the stocks in the MVP is:

$$\sum_{i=1}^{N} Sh_i(MVP) = \frac{1}{C_N}. \tag{7}$$

Although this formulation seems simple enough when applied to the MVP, it is algorithmically demanding as the number of subsets increases exponentially with the number of financial assets. Because we are using portfolios that fulfill the optimality conditions, Shapley values will measure the exact contributions of the stocks to the risk inherent in the MVP.

Now, we can develop the Shapley value for the stocks of all optimal portfolios on the frontier. Since the MV efficient frontier is a function of the required mean return μ_p the variance of a frontier portfolio is provided by Equation (5), which can be written equivalently as:

$$\sigma_p^2 = \frac{1}{D}(C\mu_p^2 - 2A\mu_p + B). \tag{8}$$

The Shapley value is now computed as follows:

1. Establish all the 2^N subsets of the securities in set N.

2. Compute the variance-covariance matrix Σ, $A = l'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\mu$, $C = l'\Sigma^{-1}l$, and $D = BC - A^2$ for all the subsets.

3. Establish an arbitrary set of required mean returns $\mu_p > A_N/C_N$ where A_N and C_N are the quadratic forms for the entire set N. Compute the frontier

portfolio variance for each subset $S \subseteq N$ and for all mean returns μ_p using Equation (8).

4. Following Equation (2) the Shapley value for each stock i in an optimal frontier portfolio subject to a given μ_p is obtained as:

$$Sh_i(\mu_p) = \sum_{s=1}^{N-1} \sum_{S \subset N \backslash i} \frac{(N-s-1)!s!}{N!} [\sigma_p^2(\mu_p, S \cup i) - \sigma_p^2(\mu_p, S)] \,. \tag{9}$$

5. For a given expected return μ_p, the sum of the Shapley values adds to the optimal portfolio variance at μ_p:

$$\sum_{i=1}^{N} Sh_i(\mu_p) = \sigma_p^2(\mu_p) \,. \tag{10}$$

To validate the Shapley value decomposition of efficient portfolios variance, I compare it to the conventional methods used today, namely "natural" decomposition of portfolio variance by its securities. This "natural" decomposition would seem to be tautological unless we address the basic notion of risk in combining securities and the need to build portfolios in order to reduce that risk.[1] The fundamental idea behind portfolio analysis according to Samuelson (1967) is that diversification pays as it reduces risk. I use the standard analysis to decompose σ_p^2 of Equation (5) into components that attribute the variation to the assets in the portfolio as follows:

$$\sigma_p^2 = \sum_{i=i}^{N} \delta_i = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \mathrm{cov}(r_i, r_j) \,. \tag{11}$$

Hence, the variation attributed to asset i is :

$$\delta_i = w_i \sum_{j=1}^{N} w_j \mathrm{cov}(r_i, r_j) = w_i \mathrm{cov}(r_i, r_p) \,. \tag{12}$$

[1]Shorrocks (1982) coined the "natural decomposition" terminology when he developed the decomposition of income inequality by its factors.

The share of the variance attributed to asset i becomes:

$$\varphi_i = \frac{\delta_i}{\sigma_p^2} = w_i \frac{\text{cov}(r_i, r_p)}{\sigma_p^2} = w_i \beta_i \tag{13}$$

which sums to unity. This variance decomposition is a function of the stock weights in the portfolio and their betas. This is the basic result when decomposing the risk of optimal portfolios. If, on the other hand, Equation (13) were concerned with the variance of the MVP, then $\varphi_i = w_i$ since $\text{cov}(r_i, r_p) = \sigma_p^2$ for any portfolio or asset. This is valid only as a special case for the MVP.

Applying the Shapley Value to Asset Allocation

To demonstrate the advantages of the Shapley value in portfolio analysis, I construct the efficient frontier for six classes of US assets using Ibbotson SBBI's aggregate data on stocks, bonds, and bills. The data consist of 1060 monthly nominal returns from January 1926 through April 2014 for six indices of US assets: large-company stocks (LCS), small-company stocks (SCS), long-term corporate bonds (LCB), long-term government bonds (LGB), intermediate-term government bonds (IGB), and U.S. Treasury bills (TB). The summary statistics are presented in Table 1, together with two normality tests, the standard Jarque-Bera statistic and the newer Kolmogorov-Smirnov statistic for the ordinary least-squares (OLS) test by Shalit (2012).

Table 1: Ibbotson's Monthly Returns 1926-2014

Statistic	LCS	SCS	LCB	LGB	IGB	TB
Mean	0.95%	1.29%	0.51%	0.48%	0.44%	0.29%
Std Dev	5.47%	8.29%	2.16%	2.41%	1.26%	0.25%
Gini (GMD)	2.78%	4.07%	1.08%	1.25%	0.64%	0.14%
JB-stat	3979.6	7201.1	2283.8	1128.7	3655.1	262.5
KS-OLS	0.129	0.803	0.557	0.207	0.312	0.091

The means and the variance-covariance matrix are computed using these data, and the MV efficient frontier is calculated from Equation (5) and depicted in Figure 1. The minimum variance portfolio (MVP) allocation weights are provided in Table 2 for $\mu_{MVP} = 0.2884\%$ and $\sigma_{MVP} = 0.2521\%$. For that allocation, most of the weights go to T-bills and only a very small part of the portfolio to other bonds.

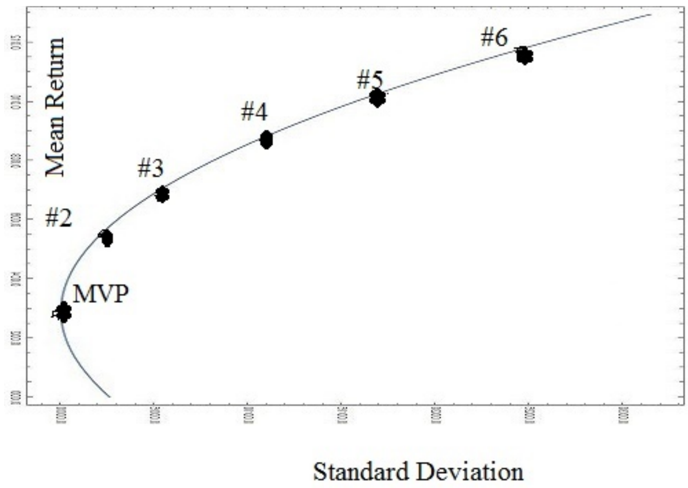
The Shapley values for the MVP assets are computed using Equation (6). These values are reported in terms of standard deviations in Table 2 together with the share of the Shapley value of each class toward the total standard deviation of the MVP. Large stocks have a Shapley value of 0.80% and contribute 317% of the MVP risk; small stocks have a Shapley value of 1.37% and contribute 543% to the risk of the MVP. T-bills have a negative Shapley value of 1.42%, meaning that they reduce the total risk exposure by 563% in terms of standard deviation.

These staggering results could not be predicted by looking only at the assets' standard deviation and their composition in the MVP. As Equation (13) shows, the share of variance attributed to assets in the MVP is given by their holdings in the MVP. This is exhibited in the last two rows of Table 2. T-bills, being the main MVP component, also bear 99% of the risk, which provides a completely different picture when considering the results for the Shapley value in the MVP.

Table 2: MVP: Weights, Shapley Values, and Variance Decomposition

	LCS	SCS	LCB	LGB	IGB	TB	$\mu = 0.29\%$
MVP weights	0.02%	0.16%	1.30%	-0.35%	-1.06%	99.93%	100%
SV	0.80%	1.37%	-0.03%	0.02%	-0.50%	-1.42%	0.25%
SV Share	317%	543%	-13%	10%	-195%	-563%	100%
Stdev Dcmp	0.004%	0.01%	0.03%	-0.01%	-0.03%	0.25%	0.25%
Stdev Share	0.02%	0.16%	1.30%	-0.35%	-1.06%	99.93%	100%

Figure 1: Mean-Variance Efficient Frontier for Asset Classes



Let us next explore the Shapley values of portfolio assets on the efficient frontier. For five arbitrary given means, I minimize the portfolio variance and compute the assets' optimal weights as reported in Table 3. As the required mean return increases, the results show a short position in government bonds and in T-bills and an increasing position in large stocks, small stocks, and corporate bonds. It appears that the main positive weight is allocated to corporate bonds.

Table 3: Optimal Weights of Assets for Each of the MV Frontier Portfolios

Prtfl	Mean	Std Dev	Weight	Weight	Weight	Weight	Weight	Weight
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.29%	0.25%	0.02%	0.16%	1.30%	-0.35%	-1.06%	99.93%
2	0.49%	1.15%	5.42%	5.68%	17.54%	-29.13%	83.76%	16.74%
3	0.69%	2.26%	10.81%	11.20%	33.79%	-57.92%	168.57%	-66.45%
4	0.90%	3.38%	16.20%	16.71%	50.03%	-86.70%	253.39%	-149.64%
5	1.09%	4.50%	21.59%	22.23%	66.28%	-115.48%	338.20%	-232.82%
6	1.29%	5.62%	26.99%	27.75%	82.52%	-144.26%	423.02%	-316.01%

The Shapley values for the assets on the optimal frontier are computed following Equation (9). They are reported in Table 4, together with their shares from total risk as shown in Table 5. In terms of standard deviation, the Shapley values of large and small stocks decline as one moves along the efficient frontier from lower risk to higher risk portfolios.

Table 4: Shapley Values of Assets on the Efficient Frontier

Prtfl	Mean	Std Dev	SV	SV	SV	SV	SV	SV
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.29%	0.25%	0.90%	1.37%	0.30%	0.45%	-0.48%	-2.28%
2	0.49%	1.15%	0.80%	1.36%	-0.03%	0.11%	-0.51%	-0.58%
3	0.69%	2.26%	0.55%	1.05%	0.34%	0.69%	0.05%	-0.43%
4	0.90%	3.38	0.44%	0.85%	0.89%	1.43%	0.48%	-0.72%
5	1.09%	4.50%	0.41%	0.74%	1.45%	2.18%	0.85%	-1.11%
6	1.29%	5.62%	0.42%	0.65%	1.99%	2.92%	1.20%	-1.55%

Table 5: Shares of Shapley Values of Assets on the Efficient Frontier

Prtfl	Mean	Std Dev	SV%	SV%	SV%	SV%	SV%	SV%
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.29%	0.25%	352%	546%	117%	178%	-191%	-903%
2	0.49%	1.15%	69.85%	118.00%	-2.78%	9.54%	-44.29%	-50.00%
3	0.69%	2.26%	24.32%	46.57%	15.07%	30.62%	2.25%	-18.83%
4	0.90%	3.38%	13.08%	25.27%	26.37%	42.38%	14.14%	-21.23%
5	1.094%	4.50%	9.12%	16.23%	32.13%	48.39%	18.86%	-24.73%
6	1.29%	5.62%	7.41%	11.49%	35.47%	51.89%	21.26%	-27.53%

This unexpected result is also reflected in the shares of Shapley values shown in Table 5. In general, bonds, whether corporate, government, or T-bills, become more important with higher-variance portfolios.[2] These portfolios being on the efficient frontier also yield higher expected return. When we compare these results with the "natural" decomposition of risk as seen on Tables 6 and 7, we get an entirely opposite picture. The standard deviation attributed to stocks and corporate

[2]For conservative portfolios, popular advice recommends to allocate more wealth to bonds and cash, and less to stocks. For aggressive portfolios popular advice recommends more stocks and less bonds and cash. Using Ibbotson SBBI's data for an earlier period, these recommendations were found to be not inefficient by Shalit and Yitzhaki (2003).

bonds increases along the frontier although their shares stay the same possibly due to the way the betas in Equation (13) are computed for each portfolio.

It would be difficult to assert that the natural decomposition measures the true contribution of each asset to portfolio risk since it ignores the basic fact that individual assets can alter risk in a series of alternative portfolios. Shapley values reflect the true contribution of assets to the risk of the portfolio because all these alternatives are considered. The comparison of Tables 5 and 7 clearly shows the superior advantage of Shapley values in evaluating risky assets and pricing them accordingly.

Table 6: Standard Deviation Decomposition of Assets on the MV Frontier

Prtfl	Mean	Std Dev	Std Dev	Std Dev	Std Dev	Std Dev	Std Dev	Std Dev
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.29%	0.25%	0.004%	0.01%	0.03%	-0.01%	-0.026 %	0.25%
2	0.49%	1.15%	0.48%	0.60%	0.51%	-0.61%	0.92%	0.09%
3	0.69%	2.26%	0.95%	1.19%	0.98%	-1.19%	1.81%	-0.15%
4	0.90%	3.38%	1.42%	1.78%	1.46%	-1.78%	2.70%	-0.17%
5	1.09%	4.50%	1.90%	2.37%	1.93%	-2.37%	3.60%	-0.09%
6	1.29%	5.62%	2.37%	2.96%	2.41%	-2.95%	4.49%	0.19%

Table 7: Shares of Std Dev Decomposition of Assets on the MV Frontier

Prtfl	Mean	Std Dev	σ %	σ %	σ %	σ %	σ %	σ %
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.29%	0.25%	0.02%	0.16%	1.30%	-0.35%	-1.06%	99.93%
2	0.49%	1.15%	17.22%	27.24%	19.26%	-27.70%	63.35%	0.61%
3	0.69%	2.26%	17.69%	27.72%	18.82%	-27.84%	64.04%	-0.43%
4	0.90%	3.38%	17.76%	27.73%	18.57%	-27.73%	63.91%	-0.24%
5	1.09%	4.50%	17.77%	27.71%	18.43%	-27.65%	63.78%	-0.04%
6	1.29%	5.62%	17.78%	27.68%	18.33%	-27.59%	63.68%	0.11%

Thus, if one believes that asset classes will continue to behave as in the past 90 years, it is my contention that the most valuable assets in building optimal portfolios are the classes of government and corporate bonds and not as one would expect the large- and small-stock classes. It's not like US professional sports; the Most Valuable Player is determined not by voting but by an actual true contribution to the portfolio game. Because optimal portfolio composition depends on the level of asset required mean return, Shapley valuation of financial assets changes accordingly. For lower portfolio mean returns and therefore lower-variance portfolios, it is the class of small stocks that is the most valuable. For the higher-mean returns and higher-variance portfolios, it is the class of government bonds.

Why should we care about this valuation? Because as we compute the benefits of a specific asset to the optimal portfolio, we evaluate its true theoretical price and compare that to the market price, revealing some possible arbitrage opportunities. The true risk valuation ranking of financial assets would follow accordingly.

It is well established in financial economics literature that to be valid with expected utility the MV model is limited to assets that follow a normal probability distribution. As we see in Table 1, none of the Ibbotson's asset classes are like this. Indeed, the Jarque-Bera test and the OLS-Shalit test both reject normality at the highest significance level available. The alternative is to use a two-parameter investment model that follows stochastic dominance such as mean-Gini.

4 On Risk Decomposition of Optimal Mean-Gini Portfolios

Finally, I develop the Shapley value for assets that make up an optimal mean-Gini (MG) portfolio. In constructing MG portfolios, investors use Gini's mean difference (GMD) as a measure of risk. GMD is defined as half the expected absolute difference between the returns of two randomly drawn amounts invested in the portfolio. The Gini may be more conveniently defined as the covariance between returns and cumulative probability distribution function (CDF):

$$\mathit{\Gamma} = 2\text{cov}[r\ ,\ F_r(r)], \tag{14}$$

where $\mathit{\Gamma}$ is the Gini, $F_r(r)$ is the CDF, and r the returns. In practice to estimate Equation (14), we rank the returns in ascending order and calculate the sample covariance between the returns and their relative position. The use of the Gini in financial economics is rooted in its advantage over the variance as a measure of risk. With the mean, the Gini provides necessary and sufficient conditions for second-degree stochastic dominance (SSD) as proved by Yitzhaki (1982). Indeed, for two portfolios with means μ_1, μ_2 and Ginis $\mathit{\Gamma}_1, \mathit{\Gamma}_2$, $\mu_1 \geqq \mu_2$ and $\mu_1 - \mathit{\Gamma}_1 \geqq \mu_2 - \mathit{\Gamma}_2$ are necessary conditions for SSD and for portfolios whose CDF intersect at most once, these conditions are also sufficient. MG theory in finance was established by Shalit and Yitzhaki (1984), and the MG efficient frontier was delineated by Shalit and Yitzhaki (2005) and further illustrated by Cheung et al. (2007).

The MG portfolio frontier is obtained by minimizing the Gini of the portfolio $\mathit{\Gamma}_p$ subject to a required portfolio mean return and a portfolio constraint. Let $r_p$be the portfolio return defined as $r_p = \sum_{i=1}^{N} w_i r_i = w'r$, where r_i are asset returns. The portfolio Gini:

$$\mathit{\Gamma}_p = 2\text{cov}[r_p\ ,\ F_r(r_p)] = 2\sum_{i=1}^{N}\ w_i \text{cov}[r_i,\ F_p(r_p)] \tag{15}$$

is minimized subject to $\mu_p = w'\mu$, $1 = w'l$, and $w \lesseqgtr 0$. The solution to this problem is more complex to solve than the MV optimization outlined in Section 3 because one needs to calculate the covariance between each asset's return and the portfolio CDF, which is not a trivial task. Furthermore, contrary to the variance of a sum of two variables that produces one covariance and one Pearson coefficient of correlation, the Gini of two variates generates two coefficients of correlation, namely, the Gini correlations:

$$\begin{cases} \rho_{ij} = \dfrac{\text{cov}[i\,,\,F_j(j)]}{\text{cov}[i\,,\,F_i(i)]} \\ \\ \rho_{ji} = \dfrac{\text{cov}[j\,,\,F_i(i)]}{\text{cov}[j\,,\,F_j(j)]} \end{cases} \tag{16}$$

Schechtman and Yitzhaki (1999) in a study of the properties of Gini correlations and determined that if the variates are exchangeable up to a linear transformation, the Gini correlations are equal to each other.[3] This simplifies the portfolio optimization problem as the MG optimal frontier can derived analytically as in Shalit and Yitzhaki (2005). Under exchangeability, we define the Gini squared of the portfolio as:

$$\mathit{\Gamma}_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \rho_{ij} \mathit{\Gamma}_i \mathit{\Gamma}_j, \tag{17}$$

where $\rho_{ii} = 1$. Denote as $\boldsymbol{R}$ the matrix of Gini correlations, as $\boldsymbol{\Gamma}$ the diagonal matrix of asset Ginis, and as $\boldsymbol{V}$ the matrix $\boldsymbol{V} = \boldsymbol{\Gamma R \Gamma}$. Then, the Gini-squared of the portfolio in Equation (17) can be written as $\mathit{\Gamma}_p^2 = w'\boldsymbol{V}w$. The optimization problem becomes:

$$\begin{aligned} Min \quad & w'\boldsymbol{V}w \\ s.t. \quad \mu_p = & \quad w'\mu \\ 1 \quad = & \quad w'l \end{aligned} \tag{18}$$

The first-order conditions (FOC) for a minimum Gini portfolio are similar to those presented in Section 3 for the MV optimization model. Hence, an analytical

[3] A set of random variables is said to be exchangeable if, for every permutation of the variates, the joint distributions are identical.

solution to problem (18) can be produced if one uses the quadratic forms $A = l'\boldsymbol{V}^{-1}\mu$, $B = \mu'\boldsymbol{V}^{-1}\mu$, $C = l'\boldsymbol{V}^{-1}l$, and $D = BC - A^2$. All these scalars are positive because matrix $\boldsymbol{V}$ is positive-definite. From the FOC for a minimum portfolio Gini square, the optimal weights for a given μ_p are obtained as:

$$w_p^{\Gamma *} = \mathbf{x} + \mu_p \mathbf{y} \tag{19}$$

where $x = [B(\boldsymbol{V}^{-1}l) - A(\boldsymbol{V}^{-1}\mu)]/D$ and $y = [C(\boldsymbol{V}^{-1}\mu) - A(\boldsymbol{V}^{-1}l)]/D$. Equation (19) is used to generate the MG portfolio frontier delineated as a hyperbola in the mean-Gini squared space:

$$\Gamma_p^2 = w_p' \boldsymbol{V} w_p = \frac{1}{D}(C\mu_p^2 \; - \; 2A\mu_p + B) \tag{20}$$

The Gini-square equation (20) helps us determine the Shapley value of assets on the optimal MG frontier. Assuming exchangeability simplifies the procedure. An analytical solution is provided as follows:

1. Establish all the 2^N subsets of the securities in set N.

2. For all these subsets, compute the matrix $\boldsymbol{R}$ of Gini correlations, the diagonal matrix of asset Ginis $\boldsymbol{\Gamma}$, and the matrix $\boldsymbol{V} = \boldsymbol{\Gamma R \Gamma}$. Calculate the appropriate quadratic forms $A = l'\boldsymbol{V}^{-1}\mu$, $B = \mu'\boldsymbol{V}^{-1}\mu$, $C = l'\boldsymbol{V}^{-1}l$, and $D = BC - A^2$.

3. For an arbitrary set of required mean returns μ_p, compute the frontier portfolio Gini-squared for each subset $S \subseteq N$ and for all mean returns μ_p as $\Gamma_p^2 = \frac{1}{D}(C\mu_p^2 - 2A\mu_p + B)$.

4. Following Equation (2) the Shapley value for each stock i in an optimal

frontier portfolio, given μ_p , is obtained as:

$$Sh_i(\mu_p) = \sum_{s=1}^{N-1} \sum_{S \subset N \backslash i} \frac{(N-s-1)!s!}{N!} [\mathit{\Gamma}_p^2(\mu_p, S \cup i) - \mathit{\Gamma}_p^2(\mu_p, S)] \tag{21}$$

5. For a given return μ_p the sum of Shapley values adds to the optimal portfolio Gini-squared at μ_p:

$$\sum_{i=1}^{N} Sh_i(\mu_p) = \mathit{\Gamma}_p^2(\mu_p) \, . \tag{22}$$

To validate the importance of Shapley values in MG optimal portfolios I compare them to the current Gini decomposition called the "natural decomposition" by Shorrocks (1982). For MG portfolios, this decomposition was established by Shalit and Yitzhaki (1984) as Equation (15) here above. This implies that the risk attributed to asset i is:

$$\delta_i^{\mathit{\Gamma}} \; = \; 2w_i \operatorname{cov}[r_i \, , \; F_p(r_p)] \tag{23}$$

and the relative risk attributed to asset i is:

$$\varphi_i^{\mathit{\Gamma}} \; = \; \frac{2w_i \operatorname{cov}[r_i \, , \; F_p(r_p)]}{\mathit{\Gamma}_p} \; = \; w_i \, \beta_i^{\mathit{\Gamma}} \; , \tag{24}$$

where $\beta_i^{\mathit{\Gamma}}$ is the MG beta of asset with respect to portfolio p. Hence, the " MG natural decomposition of risk" is basically identical to what was seen with MV portfolios but including MG betas.

Computing the Shapley Value for MG portfolios

We again use Ibbotson's data on six US assets classes as presented in Table 1 to construct mean-Gini efficient portfolios under the assumption of exchangeability. This postulation simplifies the procedure, as an analytic solution can be provided, and the optimization results facilitate calculation of the Shapley values. The Ginis of each asset are exhibited in Table 1. [4]

The weights expressed in Equation (19) delineate the MG efficient frontier in the space mean-Gini square. The first step is to compute the Shapley values of the global minimum Gini portfolio (MGP). The MGP allocation weights are obtained for the mean $\mu_{MGP} = 0.2884\%$ and the Gini $\mathit{\Gamma}_{MGP} = 0.137\%$ shown in Table 8.

As in the case of the MVP most of the weight goes to the T-bills, and only a very small part of the portfolio goes to other bonds and stocks. The Shapley values for the assets at MGP are computed using Equation (21) and are reported with the shares of the Shapley values from the Gini at MGP. Hence, large stocks contribute 300% of the portfolio Gini at MGP, small stocks 491% of the risk, and T-bills reduce the total risk exposure by 515%. These results are similar to the ones obtained for the MVP, therefore supporting the superiority of the Shapley value decomposition of risk over the use of the "natural" decomposition of optimal portfolio variance.

Table 8: The Minimum Gini Portfolio: Weights and Shapley Values

MGP$\mu = 0.29\%$	LCS	SCS	LCB	LGB	IGB	TB	$\mathit{\Gamma}$
MGP weights	0.03%	0.17%	1.42%	-0.26%	-0.79%	99.43%	100%
Shapley Value	0.413%	0.674%	-0.018%	0.022%	-0.247%	-0.706%	0.137%
Shapley Share	301%	491%	-13%	16%	-180%	-515%	100%

The weights of the optimal MG portfolios are reported in Table 9. The positions and behavior of asset classes along the MG efficient frontier are similar to the positions recorded for the MV optimal frontier in Table 3. As the required expected return rises the portfolio Gini is increased and the optimal shares of large stocks, small stocks, corporate bonds, and intermediate-term bonds increase. The government bonds and T-bills exhibit short positions that become more negative as one moves along the efficient frontier.

[4] If the asset returns exhibited a normal distribution, their Gini values would have been near the standard deviation divided by the square root of π, which is not the case with our data.

Table 9: Optimal Weights of Assets for Each MG Frontier Portfolio

Prtl	Mean	GMD	Weight	Weight	Weight	Weight	Weight	Weight
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.289%	0.0002%	0.03%	0.17%	1.42%	-0.26%	0.79%	99.43%
2	0.491%	0.003%	2.80%	7.51%	16.99%	-30.24%	85.93%	17.00%
3	0.692%	0.013%	5.57%	14.85%	32.56%	-60.21%	172.65%	-65.42%
4	0.894%	0.029%	8.34%	22.19%	48.14%	-90.19%	259.37%	-147.85%
5	1.095%	0.051%	11.11%	29.53%	63.71%	-120.17%	346.09%	-230.28%
6	1.297%	0.079%	13.88%	36.87%	79.29%	-150.15%	432.83%	-312.71%

The Shapley values of assets on the MG efficient frontier are computed following Equation (21) and are reported in Table 10. Their shares of total GMD are reported in Table 11. The results show a similar picture as with the MV Shapley values, as stocks become less valuable as required returns increase.

Table 10: Shapley Values of Assets of Portfolios on the MG Efficient Frontier

Prtfl	Mean	GMD	SV	SV	SV	SV	SV	SV
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.289%	0.0002%	0.229%	0.486%	-0.109%	-0.072%	-0.211%	-0.323%
2	0.491%	0.003%	0.229%	0.487%	-0.118%	-0.082%	-0.207%	-0.307%
3	0.692%	0.013%	0.218%	0.476%	-0.110%	-0.064%	-0.196%	-0.310%
4	0.894%	0.029%	0.198%	0.451%	-0.089%	-0.021%	-0.179%	-0.331%
5	1.095%	0.051%	0.167%	0.414%	-0.053%	0.050%	-0.156%	-0.372%
6	1.297%	0.079%	0.126%	0.365%	-0.002%	0.147%	-0.126%	-0.432%

Table 11: Shares of Shapley Values of Assets on the MG Frontier								
Prtfl	Mean	GMD	SV%	SV%	SV%	SV%	SV%	SV%
#			LCS	SCS	LCB	LGB	IGB	TB
1	0.289%	0.0002%	121451%	258248%	-57770%	-38233%	-111856%	-171741%
2	0.491%	0.003%	6829%	14562%	-3498%	-2438%	-6177%	-91776%
3	0.692%	0.013%	1701%	3709%	-862%	-5036%	-1530%	-2416%
4	0.894%	0.029%	690%	1577%	-311%	-72%	-626%	-1158%
5	1.095%	0.051%	329%	816%	-104%	99%	-307%	-733%
6	1.297%	0.079%	159%	461%	-2.32%	186%	-159%	-545%

As with the MV portfolios there is a need to explore the "natural" decomposition of MG optimal portfolio in order to compare the Shapley value results with other methods in MG space. We reserve this for future work.

5 Concluding Remarks

Theoretical methods and analytical tools have migrated from the field of income inequality to financial economics. These include use of the Gini to improve the optimization of financial portfolios and generalization of the Lorenz curve to expand the use of stochastic dominance in finance, leading eventually to conditional-value-at-risk. These are major advances because, unfortunately, financial returns do not follow a normal distribution.

Applying the Shapley value to portfolio analysis is more intricate because it requires the mental calculus to look at portfolio selection as a cooperative game where assets act to reduce risk while increasing expected return. Once this notion is accepted, risk decomposition comes naturally, in the sense that one looks at the contribution of each security in the portfolio to the overall risk reduction.

But Shapley value theory adds much more than the standard beta analysis when it comes to decomposition of portfolio risk because it looks at the contribution of each asset to all the possible coalitions. Hence, when financial analysts propose adding or eliminating specific securities from present positions the true impact is already expressed within the Shapley value. The brilliance of Lloyd Shapley is that he looked at all available configurations and averaged their marginal effects to obtain a multidimensional contribution that is much meaningful than the standard beta.

The analysis and the results presented here are the offshoots of a new theory of capital asset pricing. It is my contention that as research on the subject evolves, results will arise to contradict common financial wisdom. For sake of simplicity, my analysis has used the first-order conditions of optimal portfolio selection allowing for short sales, in order to compute Shapley values without the need for specific optimization techniques. Many avenues of future research should open to generalize mean-variance portfolio optimizations and to allow for generalized probability distributions to find mean-Gini efficient portfolios. This means that portfolio frontiers will be optimized for each and every coalition and Shapley values calculated accordingly.

As standard Shapley values computation requires huge memory allocation, there will be a need to improve calculation in order to accommodate the risk decomposition of large portfolios that follow market indices. This is the trend of future research if Shapley values are to replace standard beta theory and leave the flat universe of our current financial analysis behind.

References

[1] C. Sherman Cheung, Clarence C. Kwan, and Peter C.P. Miu. Mean-Gini portfolio analysis : A pedagogic illustration. *Spreadsheets in Education(eJSiE)*, 2(2):194–207, 2007.

[2] John C. Harsanyi. *Rational Behavior and Bargaining Equilibrium in Games and Social Situations.* Cambridge University Press, Cambridge, England, 1977.

[3] Chi-fu Huang and Robert H. Litzenberger. *Foundations for Financial Economics.* Elsevier Science Publishing Co, New York NY, 1988.

[4] Harry Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

[5] Robert C. Merton. An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis*, 7(4):1851–1872, September 1972.

[6] Stephane Mussard and Virginie Terraza. The Shapley decomposition for portfolio risk. *Applied Economics Letters*, 15:713–715, 2008.

[7] Alvin E. Roth. *The Shapley Value: Essays in Honor of Lloyd S. Shapley*, chapter Introduction, pages 1–30. Cambridge University Press, New York, 1988.

[8] Paul A. Samuelson. General proof that diversification pays. *Journal of Financial and Quantitative Analysis*, 2(1):1–13, March 1967.

[9] Mercedes Sastre and Alain Trannoy. Shapley inequality decomposition by factor components: Some methodological issues. *Journal of Economics*, 9:51–89, 2002. Suppl.

[10] Haim Shalit. Using OLS to test for normality. *Statistics & Probability Letters*, 82(11):2050–2058, November 2012.

[11] Haim Shalit and Shlomo Yitzhaki. Mean-Gini, portfolio theory, and the pricing of risk assets. *Journal of Finance*, 39:1449–68, 1984.

[12] Haim Shalit and Shlomo Yitzhaki. An asset allocation puzzle: Comment. *The American Economic Review*, 93(3):1002–1008, 2003.

[13] Haim Shalit and Shlomo Yitzhaki. The mean-Gini efficient portfolio frontier. *Journal of Financial Research*, 28(1):59–75, Spring 2005.

[14] Lloyd S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol II*, volume 28 of *Annals of Mathematics Studies*, pages 307–317. Princeton University Press, Princeton, New Jersey, 1953.

[15] Edna Shechtman and Shlomo Yitzhaki. On the proper bounds of the Gini correlation. *Economics Letters*, 63(2):133–138, 1999.

[16] Anthony F. Shorrocks. Inequality decomposition by factor components. *Econometrica*, 50(1):193–211, January 1982.

[17] Anthony F. Shorrocks. Decomposition procedures for distributional analysis: A unified framework based on the Shapley value. *Journal of Economic Inequality*, 11:99–126, 2013.

[18] Virginie Terraza and Stephane Mussard. New trading risk indexes: Application of the Shapley value in finance. *Economics Bulletin*, 3(25):1–7, 2007.

[19] Shlomo Yitzhaki. Stochastic dominance, mean variance, and Gini's mean difference. *American Economic Review*, 72(1):178–185, 1982.