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PORTFOLIO SELECTION PROBLEMS CONSISTENT WITH GIVEN PREFERENCE ORDERINGS

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This paper theoretically and empirically investigates the connection between portfolio theory and ordering theory. In particular, we examine three different portfolio problems and the respective orderings used to rank investors' choices: (1) risk orderings, (2) variability orderings, and (3) tracking-error orderings. For each problem, we discuss the properties of the risk measures, variability measures, and tracking-error measures, as well as their consistency with investor choices. Finally, for each problem, we propose an empirical application of several admissible portfolio optimization problems using the US stock market. The proposed empirical analysis permits us to evaluate the ex-post impact of the optimal choices, thereby deriving completely different investors' preference orderings during the recent financial crisis.

Keywords: Probability metrics; tracking-error measures; stochastic orderings; coherent measures; linearizable optimization problems; behavioral finance ordering.

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1. Introduction

Stochastic dominance rules have been used to justify the reward-risk approaches proposed in the portfolio selection literature [1]. Moreover, several behavioral finance studies have tried to characterize investors' behavior and preferences [2–8]. In these studies, we can define a particular ordering for investors' choices for any characterization of the preferences of investors. Most of these researchers suggest that investors prefer more than less and are neither risk averters nor risk lovers. As an alternative to these behavioral finance studies, we want to evaluate the impact of different ordering preferences on the portfolio selection problem. With these purposes in mind, we review several single-period portfolio problems proposed in the literature,[1] emphasizing those which are computationally simple (portfolio problems that can be reduced at least to convex programming problems) for different categories of risk measures, variability measures, and tracking-error measures.

Portfolio selection problems can be characterized and classified based on the motivations and intentions of investors. We generally refer to reward-risk problems as problems in which investors weigh the advantages and disadvantages of a choice by optimizing a probability functional that considers both reward and risk measures [12]. Similarly, we refer to target-based approaches (or tracking-error type portfolio problems) when investors want to optimize the distance of their choice relative to some financial benchmarks. Moreover, in other cases it could be as important to optimize a measure of randomness for a given portfolio in order to maximize the returns of the portfolio choices. In all these portfolio selection approaches, we rank the different investor preferences. Hence, ordering theory provides some intuitive rules that are consistent with expected utility theory under uncertainty conditions.

The first macro-classification we consider is between risk orderings/measures and variability orderings/measures. According to [13], we should use operational definitions of risk and uncertainty measures which are derived by investors' perception of risk and uncertainty. Moreover, the definition of risk has to take into account two essential components of observed phenomena: exposure and uncertainty. This is in stark contrast to the Knightian definition of risk and of uncertainty where a probabilistic model can be given for risk, while the uncertainty arises from our lack of perfect knowledge and no probabilistic model can be given. In contrast, the uncertainty that is perceived is generally modeled using a variability measure [14]. Thus, in this paper we distinguish among the different usages of risk orderings/measures and variability measures/orderings in portfolio problems.

Typically, we refer to risk orderings as the stochastic orderings of random variables that are implied by the monotonic order.[2] Variability orderings are characterized by different degrees of dispersion. Since the dispersion of variate X is a measure of the variability of X, we expect its presence to be in the same (or proportional)

[1]For example, the mean-Gini approach (see [34]), the reward-CVaR approach (see [10]), and the reward-spectral measure approach (see [11]).
[2]X is preferred to Y with respect to the monotonic order if and only if $X > Y$.

quantity in its opposite sign [9, 15], denoted by $-X$. Thus, we say that X exhibits higher variability than another variate Y when X dominates Y for a given risk ordering and also $-X$ dominates $-Y$ with respect to the same or another risk ordering.

Roughly speaking, we define a risk (variability) measure by a probability functional that is consistent with a risk (variability) ordering. In other words, if a random variable X is preferred to Y with respect to a given risk (variability) ordering, the risk (variability) measure of X should be lower than the measure of Y. Vice versa, a reward measure is a functional v isotonic with a risk ordering, i.e. if $X > Y$, implies that $v(X) \geq v(Y)$. Generally, a portfolio problem is consistent with a given ordering when all the admissible optimal portfolio choices cannot be dominated with respect to that particular ordering. Therefore, portfolio problems can be consistent with either risk ordering or with variability ordering or with both. Since we do not know *a priori* all possible risk/variability orderings, several authors have proffered alternative definitions of risk/variability measures based on properties that identify risk and variability [16]. For most measures, it is possible to show their consistency with a particular risk/variability ordering.

As shown by proponents of behavioral finance [2, 3, 5–7, 17], not all investors are nonsatiable and risk-averse. Thus, it is important to classify the optimal choices for any admissible ordering of preferences. Once we know the orderings to be used in a portfolio problem, we can identify a probability functional that characterizes an ordering among all admissible choices and that is consistent with investors' preferences.

The purpose of this paper is twofold. First, we show the connection between ordering theory and the theory of probability functionals for various portfolio selection problems. Second, we discuss the computational complexity of selection problems which are consistent with investor's preferences and compare their performance using the major US stock markets.

Thus, we try to answer the following questions:

- What are the "right" preference orderings to be used?
- How can we design some practical large-scale portfolio optimization problems for various preference orderings?
- How in practice can we value the impact of these portfolio optimization models?

Regarding the first question, we want to characterize the investor's behavior. In this context, several utility theory and behavioral finance studies have either analyzed the individual risk profile [18] or proposed alternative solutions to different observable patterns of investors' behavior [19, 20]. In this paper, we do not analyze the decision maker's risk profile, but instead discuss three specific portfolio selection problems characterized by different investors' preferences. Moreover, since there is a heated debate on the computational complexity of the portfolio problem [10, 21, 22], we address the second and third questions by proposing linearizable portfolio problems consistent with new potential orderings. Specifically, we propose solutions to the following three problems:

(1) Consider an investor who wants to outperform a benchmark by taking into account the distributional characteristics of portfolio returns. Here, we use a risk ordering among portfolios by taking into account its performance relative to the benchmark. An investor should choose the appropriate ordering for this active tracking-error strategy.
(2) Consider an investor who wants to maximize profits by trading a portfolio of put and call options on some asset class indexes. Option pricing theory tells us that if an investor maximizes the volatility of the underlying log-return indexes, the investor's opportunities to exercise the options are implicitly optimized. In this case, an investor wants to solve an optimal portfolio problem that uses a variability ordering on the underlying log-return indexes taking into account proper distributional characteristics of the portfolio of derivatives.
(3) Suppose an investor seeks to track a benchmark as closely as possible by following an indexing strategy. In this case, the distance with respect to the benchmark is ordered. An investor should minimize a probability distance to mimic the behavior of the benchmark with a proper portfolio of stocks.

The three portfolio problems use different typologies of ordering among random variables: risk orderings, variability orderings and tracking-error orderings. These orderings are consistent with different investor's preferences. For each of these problems, we identify several practical optimization problems that differ either for the investor's preferences or for the investor's rank of risk (variability) aversion, or for the computational complexity of the optimization problem. In particular, for the first and the third problems, we propose new linearizable portfolio optimization solutions. For the second problem, it is not possible to find a computationally simple optimization solution that maximizes the variability of portfolio log-returns because most of the variability measures are convex functionals [23] and their maximization in the compact set of admissible choices could give local solutions which are not necessarily global.

The same problem exists when we minimize a behavioral finance functional such as those proposed by [5] which is neither convex nor concave. Typically, to get linearizable optimization problems, we can use the Lorenz curve and an integral rule derived from fractional integral theory [24, 25]. The Lorenz curve, which was first applied to measure income inequality, is generally used in its absolute form to capture the essential descriptive features of risk and stochastic dominance [10, 21, 26]. Since stochastic dominance orders, behavioral finance orderings, and dual stochastic dominance rules based on Lorenz orders are linked [5, 11, 27, 47], we are able to introduce linearizable portfolio selection models that are consistent with some of these orders. Moreover, in this context, we show that many historical and recent studies on risk measures in portfolio theory are particular cases that can be simplified from the point of view of computational complexity. We review the Gini and spectral coherent measures [11] and we show how we can linearize the classic

Markowitz mean-variance problem (with no short sales constraints), that is usually solved as quadratic problem.

Finally, we evaluate the performance of the proposed portfolio problems, based on the performance of US stocks during the recent crisis years 2007–2010. Since the number of observations needed to perform a robust analysis is too small with respect to the number of stocks [28, 29], we reduce the dimensionality of the problem by preselecting the first 150 stocks from the universe of stocks we study with the greatest return-risk ratio[3] and then reduce the dimension of the problem by applying a principal component analysis (PCA) to the Pearson correlation matrix of the preselected stock returns. By doing so, we offer an ex-post comparison between the different portfolio selection problems. For investors who want to outperform a benchmark (or seek to track a benchmark), we solve different portfolio problems of typology 1 (or 3). In this case, we compare the ex-post wealth obtained by investing in the optimal solutions of the different portfolio problems of typology 1 or 3. In order to value the ex-post impact of portfolio models of typology 2, we price European options by means of the Black–Scholes model using the historical observations of stock returns. As an experiment, we create out-of-the-money puts and calls and preselect those options with the greatest reward-risk ratio. This first experiment is not based on real data of options and serves only to evaluate the impact of typology 2 portfolio models for investors who trade portfolios of the preselected European derivatives in a "Black and Scholes world". As a further confirmation, we compare the option portfolio model assuming that whoever invests in a portfolio of call options expects the prices of the portfolio of underlying stocks to increase. Thus, for all call options chosen, we compare the ex-post wealth obtained by investing in those underlying stocks.

The paper is organized as follows. In Sec. 2, we derive practical reward-risk models based on measures consistent with the most common risk orders. In Sec. 3, we describe the practical portfolio selection problems associated with variability orderings and tracking-error measures and orderings. In Sec. 4, we first introduce a preliminary analysis of our dataset and illustrate how we empirically compare the different models that have been proposed. Second, we empirically compare the portfolio problems consistent with different risk orderings. Finally, we compare portfolio selection problems which are consistent with variability orderings and with tracking-error orderings. The last section summarizes our contribution.

2. Practical Reward-Risk Portfolios that are Consistent with Risk Orderings

In this section, we discuss the first typology of one period portfolio models explained in Sec. 1. In particular, we examine mean-risk portfolio selection problems that are

[3]Several return risk ratio can be used in this framework (see, among others, [17]). In this empirical analysis we use the Rachev ratio (see [30]).

consistent with the most commonly used risk orderings in the finance literature, such as, stochastic dominance orderings, inverse stochastic dominance orderings, and behavioral finance orderings. Clearly, a large part of the following discussion could be also extended to a dynamic framework using proper dynamic reward and risk measures [10]. We assume T independent and identically distributed (i.i.d.) observations of returns $r_{(k)} = [r_{1,k}, \ldots, r_{n,k}]'$ and of the benchmark $r_{Y,k}$, $k = 1, \ldots, T$.

2.1. *Practical portfolio selection problems consistent with stochastic dominance orderings*

Recall that X dominates Y with respect to the α stochastic dominance order $X \underset{\alpha}{\geq} Y$ (with $\alpha \geq 1$) if and only if $E(u(X)) \geq E(u(Y))$ for all u belonging to a given class U_α of utility functions [1, 24, 25, 27]. Moreover, the derivatives of u satisfy the inequalities $(-1)^{k+1}u^{(k)} \geq 0$ where $k = 1, \ldots, m-1$ for the integer m that satisfies $m - 1 \leq \alpha < m$. The ordering $X \geq_\alpha Y$ is also equivalent to saying that for every real t:

$F_X^{(\alpha)}(t) := E((t-X)_+^{\alpha-1})/\Gamma(\alpha) \leq F_Y^{(\alpha)}(t)$ when $\alpha > 1$, and $F_X(t) = \Pr(X \leq t) \leq F_Y(t) = \Pr(Y \leq t)$ when $\alpha = 1$, where $\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1}\mathrm{e}^{-z}dz$. In particular, first-degree stochastic dominance (FSD) (that corresponds to $\alpha = 1$) is an ordering of preferences for nonsatiable agents while second-degree stochastic dominance (SSD) ($\alpha = 2$) is an ordering of preferences for nonsatiable risk-averse investors. Moreover, we refer to α bounded stochastic dominance order between X and Y (namely, $X \overset{b}{\underset{\alpha}{\geq}} Y$) when $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$ for any t belonging to the support of X and Y.

Consider T i.i.d. observations of the returns $r = [r_1, \ldots, r_n]'$ and of the benchmark return r_Y. Then, a consistent estimator of $\Gamma(\alpha)F_{x'r-r_Y}^{(\alpha)}(t)$ is given by $\hat{G}_{x'r-r_Y}^{(\alpha)}(t) = \frac{1}{T}\sum_{k=1}^{T}(t - x'r_{(k)} + r_{Y,k})^{\alpha-1}I_{[t>x'r_{(k)}-r_{Y,k}]}$ where $I_{[t>x'r_{(k)}-r_{Y,k}]} = \begin{cases} 1 \text{ if } t > x'r_{(k)} - r_{Y,k} \\ 0 \text{ otherwise} \end{cases}$. When no short sales are allowed, the support of all admissible portfolios is given by the interval $(\min_{x\in S}\min_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k}),$ $\max_{x\in S}\max_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k}))$ where $S = \{x \in R^n | \sum_{i=1}^n x_i = 1; x_i \geq 0\}$. For all $t \in (\min_{x\in S}\min_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k}), \max_{x\in S}\min_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k}))$, minimizing $\hat{G}_{x'r-r_Y}^{(\alpha)}(t)$ yields the value 0 and the optimal weights $\hat{x} = \arg(\max_{x\in S}\min_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k}))$. In order to find portfolios that are not first-order stochastically dominated, we minimize $\hat{G}_{x'r-r_Y}^{(1)}(t) = \frac{1}{T}\sum_{k=1}^{T} I_{[t>x'r_{(k)}-r_{Y,k}]}$ for any $t \geq c = \max_{x\in S}\min_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k})$. In order to find in a mean-risk space the optimal portfolios that are nondominated with respect to α stochastic dominance order for $\alpha > 1$ for a given mean equal to or greater than m and a parameter $t \geq c = \max_{x\in S}\min_{1\leq k\leq T}(x'r_{(k)} - r_{Y,k})$, one must solve the following optimization problem with linear constraints:

$$\min_{x,v_k} \frac{1}{T}\sum_{k=1}^{T} v_k^{\alpha-1} \quad \text{subject to}$$

$$\frac{1}{T}\sum_{t=1}^{T} x' r_{(t)} \geq m; \quad \sum_{j=1}^{n} x_j = 1; \;\; x_j \geq 0; \;\; j = 1, \ldots, n$$
$$v_k \geq 0; \quad v_k \geq t + r_{Y,k} - x' r_{(k)}, \;\; k = 1, \ldots, T. \tag{2.1}$$

In particular, in order to obtain optimal choices for nonsatiable risk-averse investors, we solve the previous linear programming (LP) problem for $\alpha = 2$. The optimal choices consistent with an α-stochastic dominance order (with $\alpha > 2$) can be obtained as a solution of a linear problem. This is always possible since we can apply the following fractional integral property[4] for any $\alpha > v \geq 1$:

$$F_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha - v)} \int_{-\infty}^{t} (t-u)^{\alpha - v - 1} F_X^{(v)}(u) du. \tag{2.2}$$

Therefore, if $\alpha > 2$ and $v = 2$ in Eq. (2.1) then

$$\tilde{G}^{(\alpha)}_{x'r - r_Y}(t) \overset{M,T\to\infty}{\longrightarrow} \Gamma(\alpha - 2) F^{(\alpha)}_{x'r - r_Y}(t) = \int_{-\infty}^{t} (t-u)^{\alpha-3} E((u - x'r + r_Y)_+) du$$

for any $t \geq c = \max_{x \in S} \min_{1 \leq k \leq T}(x' r_{(k)} - r_{Y,k})$, where

$$\tilde{G}^{(\alpha)}_{x'r - r_Y}(t) = \frac{t-c}{M} \sum_{i=1}^{M-1} \left((M - i) \left(\frac{t-c}{M} \right) \right)^{\alpha - 3}$$
$$\times \frac{1}{T} \sum_{k=1}^{T} \left(c + i \left(\frac{t-c}{M} \right) - x' r_{(k)} + r_{Y,k} \right)^{\alpha - 1} I_{[c + i(\frac{t-c}{M}) > x' r_{(k)} - r_{Y,k}]}.$$

Thus, $\tilde{G}^{(\alpha)}_{x'r - r_Y}(t)$ is a consistent estimator of $\Gamma(\alpha - 2) F^{(\alpha)}_{x'r - r_Y}(t)$ when M is large enough.[5] So, in a mean-risk space, we get nondominated portfolios with respect to α ($\alpha > 2$) stochastic dominance bounded order by solving the following LP problem for $t \geq c = \max_{x \in S} \min_{1 \leq k \leq T}(x' r_{(k)} - r_{Y,k})$, and a mean equal to or greater

[4] See [24, 25]. Moreover, as suggested by the authors in [27, 47], formula (2.2) can be used for different characteristic functionals $\rho_X^{(\alpha)}$ instead of $F_X^{(\alpha)}$.

[5] We should choose M equal to or larger than $(t - c)/d$ where

$$d = \min_x \inf\{z = ((x'r - r_Y)_{k:T} - (x'r - r_Y)_{k-1:T}) \mid z > 0; k = 2, \ldots, T\}$$

and $(x'r - r_Y)_{k:T}$ is the kth observation of T ordered observations of $(x'r - r_Y)$.

than m:

$$\min_{x,v_k} \frac{t-c}{M} \sum_{i=1}^{M-1} \left((M-i) \left(\frac{t-c}{M} \right) \right)^{\alpha-3} \frac{1}{T} \sum_{i=1}^{T} v_{k,i} \quad \text{subject to}$$

$$\frac{1}{T} \sum_{t=1}^{T} x' r_{(t)} \geq m; \quad \sum_{j=1}^{n} x_j = 1; \ \ x_j \geq 0; \ \ j = 1, \ldots, n; \ \ v_{k,i} \geq 0; \tag{2.3}$$

$$v_{k,i} \geq c + i \left(\frac{t-c}{M} \right) - x' r_{(k)} + r_{Y,k}, \quad k = 1, \ldots, T; \ \ i = 1, \ldots, M.$$

2.2. *Practical portfolio problems that are consistent with inverse stochastic dominance orderings*

As an alternative to classic stochastic orders, we can use the dual (also called inverse) representations of stochastic dominance rules [27]. Namely, we say X dominates Y with respect to α inverse stochastic dominance order $X \underset{-\alpha}{\geq} Y$ (with $\alpha \geq 1$) if and only if for every $p \in [0, 1]$,

$$F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)} \int_0^p (p-u)^{\alpha-1} dF_X^{-1}(u) \geq F_Y^{(-\alpha)}(p) \ \text{ when } \alpha > 1, \quad \text{and}$$

$$F_X^{-1}(p) \geq F_Y^{-1}(p) \quad \text{when } \alpha = 1,$$

where $F_X^{-1}(0) = \lim_{p \to 0} F_X^{-1}(p)$ and $F_X^{-1}(p) = \inf\{x : \Pr(X \leq x) = F_X(x) \geq p\}$ $\forall p \in (0,1]$ is the left inverse of the cumulative distribution function F_X. In this case, $-F_X^{(-\alpha)}(p)$ is the risk measure associated with this risk ordering. In the risk management literature, the opposite of the p-quantile $F_X^{-1}(p)$ of X is referred to as value-at-risk (VaR) of X, i.e. $\text{VaR}_p(X) = -F_X^{-1}(p)$. VaR refers to the maximum loss among the best $1-p$ percentage cases that could occur for a given horizon. A consistent statistic of the p-quantile of X is given by the order statistic $X_{[pT]:T}$ that indicates the $[pT]$th observation of T ordered observations of X. In order to find the optimal portfolio that is nondominated at the first inverse order, we minimize $-X_{[pT]:T}$ under the usual constraints. Moreover, using the fractional integral property (2.2) for any $\alpha > 1$, we get $\Gamma(\alpha-1) F_X^{(-\alpha)}(p) = \int_0^p (p-u)^{\alpha-2} F_X^{-1}(u) du$. Assuming equally probable T scenarios, a consistent estimator of $\Gamma(\alpha-1) F_{x'r-r_Y}^{(-\alpha)}(p)$ when $p = s/T$ for a given integer $s \leq T$ is given by $\frac{1}{T} \sum_{i=1}^{s-1} (\frac{s-i}{T})^{\alpha-2} (x'r - r_Y)_{i:T}$. Thus in a mean-risk space, we get portfolios that are optimal with respect to the αth inverse stochastic dominance order for any $\alpha > 1$ by solving the following optimization problem:

$$\min_x - \sum_{i=1}^{s-1} \left(\frac{s-i}{T} \right)^{\alpha-2} (x'r - r_Y)_{i:T} \quad \text{subject to}$$

$$\frac{1}{T} \sum_{t=1}^{T} x' r_{(t)} \geq m; \quad \sum_{j=1}^{n} x_j = 1; \ \ x_j \geq 0; \ \ j = 1, \ldots, n \tag{2.4}$$

for a given $s \leq T$ and a mean equal to or greater than m. Recall that first- and second-stochastic dominance rules are equivalent to their inverse orderings.[6] Furthermore, $F_X^{(-2)}(p) = L_X(p) = \int_0^p F_X^{-1}(t)dt$ is the absolute Lorenz curve (or absolute concentration curve) of stock X with respect to its distribution function F_X. The absolute concentration curve $L_X(p)$ valued at p shows the mean return accumulated up to the lowest p percentage of the distribution. Both measures and $L_X(p)$ have important financial and economic interpretations and are widely used in the recent risk literature. In particular, the negative absolute Lorenz curve divided by probability p is a coherent risk measure in the sense of [16] that is called conditional value-at-risk (CVaR), or expected shortfall (or average value-at-risk) [10, 22], and is expressed as

$$\mathrm{CVaR}_p(X) = -L_X(p) = \inf_u \left\{ u + \frac{1}{p} E((-X-u)_+) \right\}, \tag{2.5}$$

where the optimal value u is $\mathrm{VaR}_p(X) = -F_X^{-1}(p)$. As a consequence of Eq. (2.5), Pflug in [22] demonstrated that we can minimize CVaR for a fixed mean by solving a LP problem and that coherent risk measures using specific functions for the Lorenz curve can be easily obtained. In particular, we observe that some classic Gini-type measures are coherent measures. By definition, for every $v > 1$ and for every $\beta \in (0,1)$ we have that

$$\begin{aligned} GT_{(\beta,v)}(X) &:= -\Gamma(v+1) F_X^{(-(v+1))}(\beta)/\beta^v \\ &= -(v-1)v \int_0^\beta (\beta-u)^{v-2} L_X(u)du/\beta^v \end{aligned}$$

is consistent with $\underset{-(v+1)}{\geq}$. Then using the coherency of CVaR, we can easily prove that $GT_{(\beta,v)}(X)$ is a coherent functional associated with the $(v+1)$ inverse stochastic dominance order and the following remark holds.

Remark 2.1. For every $v \geq 1$ and for every $\beta \in (0,1)$, the measure $GT_{(\beta,v)}(X) = -\Gamma(v+1)F_X^{(-(v+1))}(\beta)/\beta^v$ is a linearizable coherent risk measure that is consistent with the $\underset{-(v+1)}{\geq}$ order.

The measure $GT_{(\beta,v)}(X)$ generalizes the CVaR that we get when $v = 1$. Since the measures $GT_{(\beta,v)}(X)$ are strictly linked to the extended Gini mean difference (as we will show in the next section), we refer to them as Gini-type coherent measures. Moreover, from Remark 2.1, when we minimize the measure $GT_{(\beta,v)}(x'r - r_Y)$ and $\beta = s/T$ for a given integer $s \leq T$, then by using Eq. (2.5) we can linearize its consistent estimator $\frac{-v(v-1)}{\beta^v T} \sum_{i=1}^{s-1} (\frac{s-i}{T})^{v-2} \hat{L}_{x'r-r_Y}(\frac{i}{T})$ where $\hat{L}_{x'r-r_Y}(\frac{i}{T}) = \frac{1}{T}\sum_{k=1}^{i} (x'r - r_Y)_{k:T}$. In particular, we can determine choices that

[6]That is, $X\,FSD\,Y \Leftrightarrow F_X^{-1}(p) \geq F_Y^{-1}(p)$, $p \in [0,1]$ and $X\,SSD\,Y \Leftrightarrow F_X^{(-2)}(p) \geq F_Y^{(-2)}(p)$, $p \in [0,1]$.

are optimal with respect to α inverse stochastic dominance orders for any $\alpha > 2$ by solving the following LP problem:

$$\min_{x,b_1,\dots,b_{s-1},v_{t,i}} \sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{\alpha-3}\left(\frac{i}{T}b_i + \frac{1}{T}\sum_{t=1}^{T} v_{t,i}\right) \quad \text{subject to}$$

$$\frac{1}{T}\sum_{t=1}^{T} x' r_{(t)} \geq m; \quad \sum_{j=1}^{n} x_j = 1; \;\; x_j \geq 0; \;\; j = 1,\dots,n; \;\; v_{t,i} \geq 0; \tag{2.6}$$

$$v_{t,i} \geq -x' r_{(t)} + r_{Y,t} - b_i; \quad t = 1,\dots,T; \;\; i = 1,\dots,s-1,$$

for a given mean equal to or greater than m.

Other typical coherent measures are the spectral measures proposed by the author in [11]. Any spectral measure

$$M_\phi(X) = -\int_0^1 \phi(u) F_X^{-1}(u)du \tag{2.7}$$

is a coherent risk measure identified by its risk spectrum ϕ that is an *a.e.* non-negative decreasing and integrable function such that $\int_0^1 \phi(u)du = 1$. Since to define a stochastic ordering it is sufficient to prove the invariance in law property and its consistency with a preference order [27], then for any spectrum ϕ with $\phi > 0$, $\phi' < 0$, in $(0, 1)$, we can introduce a new ordering that we call a *ϕ-spectral risk ordering*. In particular, we say that X dominates Y with respect to the ϕ-spectral risk ordering (namely, $X\phi$ *spectral* Y) if and only if

$$ST_{(\phi,X)}(t) := \frac{-1}{\int_0^t \phi(u)du}\int_0^t \phi(u)F_X^{-1}(u)du \leq ST_{(\phi,Y)}(t) \quad \forall\, t \in (0,1]. \tag{2.8}$$

This is a risk ordering because it is consistent with monotone ordering and $ST_{(\phi,Y)} = ST_{(\phi,X)}$ if and only if $F_X = F_Y$. Moreover, using Eq. (2.5) we can always find the optimal choices consistent with a *ϕ-spectral ordering* by solving a LP problem.

2.3. *Practical portfolio selection problems that are consistent with behavioral finance orderings*

The first behavioral orderings introduced in the finance literature are those orderings that were deduced from Markowitz' studies on investors' utility (referred to as *Markowitz orderings*) and the orderings based on prospect theory (referred to as *prospect orderings*) [3–5]. Choices consistent with these orderings are optimal for nonsatiable investors who are neither risk-averse nor risk-loving. According to the definition of prospect ordering given by [5], X dominates Y in the sense of prospect theory ($X\, PSD\, Y$) if and only if

$$\forall\,(a,y) \in [0,1]\times(-\infty,0], \quad g_X(a,y) := a g_X(y) + (1-a)\tilde{g}_X(y) \leq g_Y(a,y),$$

where $g_X(y) := \int_0^{-y} F_X(u)du = E((-y-X)I_{[X\in(0,-y]]}) - yF_X(0)$ and $\tilde{g}_X(y) := \int_y^0 F_X(u)du = -E((X-y)I_{[X\in(y,0]]}) - yF_X(0)$.

A consistent estimator of $g_X(a,y)$ is given by

$$\hat{g}_{x'r-r_Y}(a,y)$$
$$= \frac{1}{T}\left(a\sum_{k=1}^{T}(-y-x'r_{(k)}+r_{Y,k})I_{[x'r_{(k)}-r_{Y,k}\in(0,-y]]}\right.$$
$$\left.-y\sum_{k=1}^{T}I_{[x'r_{(k)}-r_{Y,k}<0]}-(1-a)\sum_{k=1}^{T}(x'r_{(k)}-r_{Y,k}-y)I_{[x'r_{(k)}-r_{Y,k}\in(y,0]]}\right),$$

and we get nondominated portfolios with respect to the prospect theory order minimizing $\hat{g}_{x'r-r_Y}(a,y)$ for different values of $(a,y)\in[0,1]\times[c,0]$, where $c=-\max(|\min_{x\in S}\min_{1\le k\le T}(x'r_{(k)}-r_{Y,k})|,|\max_{x\in S}\max_{1\le k\le T}(x'r_{(k)}-r_{Y,k})|)$ and $S=\{x\in R^n\mid\sum_{i=1}^{n}x_i=1;x_i\ge 0\}$.

According to the definition of Markowitz orderings given by [5], we can say that *X dominates Y in the sense of Markowitz order* $(X\ M\ SD\ Y)$ if and only if for every $y\in(-\infty,0]$

$$m_X(y):=\int_{-\infty}^{y}F_X(u)du=F_X^{(2)}(y)=E((y-X)_+)\le m_Y(y)\quad\text{and}$$

$$\int_{-y}^{+\infty}F_Y(u)-F_X(u)du=E((X+y)_+)-E((Y+y)_+)\ge 0$$

if and only if

$$m_X(a,y):=aE((y-X)_+)-(1-a)E((X+y)_+)\le m_Y(a,y),$$
$$\forall\,(a,y)\in[0,1]\times(-\infty,0].$$

In this case, we obtain optimal portfolios in the sense of Markowitz ordering by solving the following portfolio problem in a mean-risk space:

$$\min_{x,v_k}\frac{a}{T}\sum_{k=1}^{T}v_k-\frac{1-a}{T}\sum_{k=1}^{T}(x'r_{(k)}-r_{Y,k}+y)I_{[x'r_{(k)}-r_{Y,k}>-y]}\quad\text{subject to}$$
$$\frac{1}{T}\sum_{t=1}^{T}x'r_{(t)}\ge m;\quad\sum_{j=1}^{n}x_j=1;\;\;x_j\ge 0;\;\;j=1,\ldots,n;\tag{2.9}$$
$$v_k\ge 0;\quad v_k\ge y+r_{Y,k}-x'r_{(k)};\;\;k=1,\ldots,T,$$

for different values of $(a,y)\in[0,1]\times[c,0]$ with $c=-\max(|\min_{x\in S}\min_{1\le k\le T}(x'r_{(k)}-r_{Y,k})|,|\max_{x\in S}\max_{1\le k\le T}(x'r_{(k)}-r_{Y,k})|)$ and a given mean equal to or greater than m. We now introduce many new kinds of behavioral finance orderings associated with the *aggressive-coherent functionals*

$$S_X(a,t_1,t_2)=a\rho_{1,X}(t_2)-(1-a)\rho_{2,-X}(t_1),\tag{2.10}$$

where $a \in [0,1]$, $t_1 \in B_1$, $t_2 \in B_2$ and $\rho_{1,X}$, $\rho_{2,X}$ are two simple coherent functionals [16]. Even if we minimize one of these aggressive-coherent functionals, we obtain optimal choices for nonsatiable investors who are neither risk-averse nor risk-loving. For example, for any $(a,t,\beta) \in [0,1] \times [0,1] \times [0,1]$ we can consider the Gini type coherent functional $g_X(a,t,\beta) = aGT_{(t,v_1)}(X) - (1-a)GT_{(\beta,v_2)}(-X)$ where $v_1, v_2 > 1$. Thus we define new behavioral finance orderings saying that X dominates Y in the sense of the Gini behavioral finance ordering if and only if $g_X(a,t,\beta) \leq g_Y(a,t,\beta)$ for any $(a,t,\beta) \in [0,1] \times [0,1] \times [0,1]$. Similarly we can define spectral type behavioral finance orderings using spectral measures.

Observe that we cannot linearize the portfolio problems consistent with the proposed behavioral finance orderings because the probability functionals $g_X(a,y)$, $m_X(a,y)$, and $S_X(a,t_1,t_2)$ are defined as the difference of two convex nonlinear probability functionals. We deduce that by minimizing these functionals we can get local optima and the computational complexity of these portfolio selection models is a common problem for all previously proposed models consistent with behavioral finance orderings.

3. Portfolio Problems Consistent with Convex-Type and Tracking-Error Orderings

We now present one period portfolio selection problems where investors optimize the variability (as in Example (2) in Sec. 1) or track a benchmark as closely as possible (as in example (3) in Sec. 1). Moreover, we clarify the link between orderings theory and probability distances/metrics theory.[7] Even in this context we could extend our analysis to a dynamic framework using proper dynamic variability measures.[8]

3.1. *Practical portfolio problems consistent with convex-type orderings*

Probably the most well-known variability ordering in financial economics is the concave ordering commonly referred to as the Rothschild–Stiglitz (R–S) ordering [1]. We say that X dominates Y in the sense of Rothschild and Stiglitz ($X\,RSD\,Y$) if and only if all risk-averse investors prefer the less uncertain variate X to Y, i.e. if and only if $E(X) = E(Y)$ and $X \underset{2}{\geq} Y$. All the variability orderings can also be defined in the opposite direction. We refer to these orderings as *convex-type* orderings which are described as follows: $X\,RSD\,Y$ if and only if Y dominates X in the sense of convex-type orderings, if and only if all risk-loving investors prefer Y to X. More generally, X dominates Y in the sense of α_1, α_2-R–S order (with

[7]The theory of probability metrics is presented in [31] and [48]. For applications of probability metrics to finance, see [32].
[8]See, for example, [33] and for further discussions [10].

$\alpha_1, \alpha_2 \geq 2$) if and only if $X \underset{\alpha_1}{\geq} Y$ and $-X \underset{\alpha_2}{\geq} -Y$; that is, if and only if

$$F_X^{(\alpha_1,\alpha_2)}(b,t) := bF_X^{(\alpha_1)}(t) + (1-b)\bar{F}_X^{(\alpha_2)}(t) \leq F_Y^{(\alpha_1,\alpha_2)}(b,t) \quad \text{for every } b \in [0,1],\ \ t \in R, \tag{3.1}$$

where $\bar{F}_X^{(\alpha)}(t) = E((X-t)_+^{\alpha-1})/\Gamma(\alpha) = F_{-X}^{(\alpha)}(-t) = \frac{1}{\Gamma(\alpha-2)}\int_{-\infty}^{-t}(-t-u)^{\alpha-3} E((X+u)_+)du$. Generally, we refer to α_1, α_2-R–S simply as α-R–S order when $\alpha = \alpha_1 = \alpha_2$. Similarly, we state X dominates Y in the sense of α_1, α_2-inverse R–S order (with $\alpha_1, \alpha_2 \geq 2$) if and only if, $X \underset{-\alpha_1}{\geq} Y$ and $-X \underset{-\alpha_2}{\geq} -Y$, that is, if and only if

$$\begin{aligned} F_X^{(-\alpha_1,-\alpha_2)}(b,p) &:= bF_X^{(-\alpha_1)}(p) + (1-b)\bar{F}_X^{(-\alpha_2)}(p) \\ &\geq F_Y^{(-\alpha_1,-\alpha_2)}(b,p) \quad \forall\, b \in [0,1],\ \ \forall\, p \in [0,1], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \bar{F}_X^{(-\alpha)}(p) &= \frac{1}{\Gamma(\alpha)}\int_p^1 (u-p)^{\alpha-1} dF_X^{-1}(u) \\ &= F_{-X}^{(-\alpha)}(1-p) = \frac{1}{\Gamma(\alpha-2)}\int_0^{1-p}(1-p-u)^{\alpha-3}L_{-X}(u)du. \end{aligned}$$

When $\alpha_1 = \alpha_2 = 2$ in the relations (3.1) and (3.2), we obtain the classic R–S order. By changing the parameters in (3.1) and (3.2), we also obtain variability measures that are consistent with the respective R–S type orderings. For example,

$$\begin{aligned} F_X^{(3,3)}(0.5, E(X)) &= \frac{E((E(X)-X)_+^2) + E((X-E(X))_+^2)}{4} \\ &= \frac{1}{2}\left(\int_{-\infty}^{E(X)} E((t-X)_+)dt + \int_{-\infty}^{-E(X)} E((X+t)_+)dt\right) \end{aligned}$$

gives one fourth of the variance of X, which is obviously a variability measure. Since for all these orderings we assume $\alpha_1, \alpha_2 \geq 2$, the associated functionals satisfy the convexity property and they can be easily linearized when we minimize their consistent estimators for $F_X^{(\pm\alpha_1,\pm\alpha_2)}(b,t)$ as suggested in Sec. 2. For example, as a consequence of the results presented in Sec. 2.2, we can minimize the variance for the mean equal to the value m, thereby solving the classic Markowitz' mean-variance problem with the following LP optimization:

$$\min_{x, v_{k,i}, u_{k,i}} \frac{m-c}{M}\sum_{i=1}^{M-1}\frac{1}{T}\sum_{k=1}^{T} v_{k,i} + \frac{-m-c}{M}\sum_{i=1}^{M-1}\frac{1}{T}\sum_{k=1}^{T} u_{k,i} \quad \text{subject to}$$

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} x' r_{(t)} = m; \quad \sum_{j=1}^{n} x_j = 1; \;\; x_j \geq 0; \;\; j = 1, \ldots, n; \\
&\qquad v_{k,i} \geq c + i\frac{1}{M}\left(\frac{1}{T}\sum_{t=1}^{T} x' r_{(t)} - c\right) - x' r_{(k)}; \\
&u_{k,i} \geq c + i\frac{1}{M}\left(-c - \frac{1}{T}\sum_{t=1}^{T} x' r_{(t)}\right) + x' r_{(k)}; \\
&\qquad u_{k,i} \geq 0; \;\; v_{k,i} \geq 0; \;\; k = 1, \ldots, T; \;\; i = 1, \ldots, M,
\end{aligned}
\tag{3.3}
$$

where $c = -\max(|\min_{x \in S} \min_{1 \leq k \leq T}(x' r_{(k)} - r_{Y,k})|, |\max_{x \in S} \max_{1 \leq k \leq T}(x' r_{(k)} - r_{Y,k})|)$, and M is large. It is well known that there exists a closed-form solution to the mean-variance portfolio problem when unlimited short sales are permitted. However, the mean-variance problem does not allow for a closed-form solution with no short sales and, thus, the linear optimization portfolio problem (3.13) suggests an improvement of the computational complexity of the mean-variance portfolio problem.

We now propose a variability measure to solve problem (2.2) from Sec. 1. To do this, we use the variability measure to optimize the choice between n European options on n indexes with log-returns $z = [z_1, \ldots, z_n]'$. We assume that European calls and puts at time t with maturity s, riskless r_f, are priced according to the Black–Scholes option model. In other words, the prices of European calls and puts on the ith asset at time t are given by:

$$
\begin{aligned}
\text{call}_{i,t} &= P_{i,t}(\Phi(d_1) - b_i \mathrm{e}^{-r_f(s-t)}\Phi(d_2)), \\
\text{put}_{i,t} &= P_{i,t}(b_i \mathrm{e}^{-r_f(s-t)}\Phi(-d_2) - \Phi(-d_1)),
\end{aligned}
$$

where $b_i = \frac{K_i}{P_{i,t}}$, K_i is the exercise prices of the ith asset, $P_{i,t}$ is the spot price of the underlying ith asset at time t, σ its volatility, $d_1 = \frac{-\ln(b_i)+(r_f+\sigma^2/2)(s-t)}{\sigma\sqrt{s-t}}$, $d_2 = d_1 - \sigma\sqrt{s-t}$, r_f is the riskless borrowing and lending rate, and $\Phi(\cdot)$ is the cumulative standard normal N(0, 1). Then, the gross returns on the period $[t, t+1]$ on European calls and puts are functions of the gross returns of the underlying $\frac{P_{i,t+1}}{P_{i,t}}$, i.e.:

$$
\begin{aligned}
\frac{\text{call}_{i,t+1}}{\text{call}_{i,t}} &= g_c\left(\frac{P_{i,t+1}}{P_{i,t}}\right) \\
&= \frac{(\frac{P_{i,t+1}}{P_{i,t}}\Phi(D_1(\frac{P_{i,t+1}}{P_{i,t}})) - b_i \mathrm{e}^{-r_f(s-t-1)}\Phi(D_2(\frac{P_{i,t+1}}{P_{i,t}})))}{(\Phi(d_1) - b_i \mathrm{e}^{-r_f(s-t)}\Phi(d_2))},
\end{aligned}
$$

$$\frac{\mathrm{put}_{i,t+1}}{\mathrm{put}_{i,t}} = g_p\left(\frac{P_{i,t+1}}{P_{i,t}}\right)$$
$$= \frac{(b_i \mathrm{e}^{-r_f(s-t-1)}\Phi(-D_2(\frac{P_{i,t+1}}{P_{i,t}})) - \frac{P_{i,t+1}}{P_{i,t}}\Phi(-D_1(\frac{P_{i,t+1}}{P_{i,t}})))}{(b_i \mathrm{e}^{-r_f(s-t)}\Phi(-d_2) - \Phi(-d_1))},$$

where $D_1(\frac{P_{(i,t+1)}}{P_{(i,t)}}) = \frac{-\ln(b_i)+\ln(\frac{P_{(i,t+1)}}{P_{(i,t)}})+(r_f+\sigma^2/2)(s-t-1)}{\sigma\sqrt{s-t-1}}$ and $D_2(\frac{P_{(i,t+1)}}{P_{(i,t)}}) = D_1(\frac{P_{(i,t+1)}}{P_{(i,t)}}) - \sigma\sqrt{s-t-1}$.

Hence, if we have T i.i.d. observations of log-returns $z_{(t)} = [z_{1,t},\dots,z_{n,t}]'$, $(t = 1,\dots,T)$ and we assume they are constant over the time b_i (for any $i = 1,\dots,n$), then, by the above transformation, we get T i.i.d. outcomes of option returns $R_{(t)} = [R_{1,t},\dots,R_{n,t}]'$ (in a Black–Scholes world).

Since option returns are distributed asymmetrically with heavy tails, we can control them by considering their first four moments.[9] Hence, investors determine the optimal choices isotonic with α_1,α_2-R–S order (consistent with deriving an α_1,α_2 convex type ordering) of log-returns by solving the following optimization for some values m, q_i; $i = 1, 2$ and t:

$$\max_x \frac{b}{T}\sum_{k=1}^{T}(t - x'z_{(k)})^{\alpha_1-1}I_{[t-x'z_{(k)}\geq 0]} + \frac{1-b}{T}\sum_{k=1}^{T}(x'z_{(k)} - t)^{\alpha_2-1}I_{[x'z_{(k)}-t\geq 0]}$$

$$\text{subject to } \frac{\sum_{s=1}^{T}(x'R_{(s)} - \frac{1}{T}\sum_{k=1}^{T}x'R_{(k)})^3}{T(x'Q_Rx)^{3/2}} \geq q_1;$$

$$\frac{\sum_{s=1}^{T}(x'R_{(s)} - \frac{1}{T}\sum_{k=1}^{T}x'R_{(k)})^4}{T(x'Q_Rx)^2} \leq q_2$$

$$\frac{1}{T}\sum_{k=1}^{T}x'R_{(k)} \geq m(x'Q_Rx)^{1/2};\quad \sum_{j=1}^{n}x_j = 1;\;\; x_j \geq 0;\;\; j = 1,\dots,n, \tag{3.4}$$

where Q_R is the variance-covariance matrix of the vector of option returns $R = [R_1,\dots,R_n]'$. For this problem, we maximize the variability of the log returns portfolio subject to the Sharpe ratio of option returns being equal to or greater than m for some appropriate skewness and kurtosis levels q_i; $i = 1, 2$. Alternatively, we can solve the following problem associated with the α_1,α_2-inverse R–S order $(\alpha_1,\alpha_2 > 2)$.

In many portfolio selection problems some concentration measures have been used to measure the variability in choices. The classic example is Gini's mean difference (GMD) and its extensions related to the fundamental work of Gini [34]. GMD is twice the area between the absolute Lorenz curve and the line of safe asset

[9] As many authors have pointed out, there is a strong connection between moments and stochastic orders (see, among others [1, 27] and the references therein).

joining the origin with the mean located on the right boundary vertical. The most frequently used representations are:

$$\begin{aligned}\Gamma_X(2) &= E(X) - 2\int_0^1 L_X(u)du = 1/2E(|X_1 - X_2|)\\ &= E(X) - E(\min(X_1, X_2)) = -2\mathrm{cov}(X, (1 - F_X(X))),\end{aligned}$$

where X_1 and X_2 are two independent copies of variate X. GMD depends on the spread of the observations among themselves and not on the deviations from some central value. Consequently, the measure relates location to variability, two properties that Gini himself argued are distinct and do not depend on each other. Although the Gini index, i.e. the ratio $\Gamma_X(2)/E(X)$,[10] has been used for the past 80 years as a measure of income inequality, the interest in GMD as a measure of risk in portfolio selection has only recently emerged [34].

In addition to GMD, we consider the extended Gini's mean difference [35, 36] that takes into account the degree of risk aversion as reflected by the parameter v. This index can also be derived from the Lorenz curve as follows:

$$\begin{aligned}\Gamma_X(v) &= E(X) - v(v-1)\int_0^1 (1-u)^{v-2} L_X(u)du\\ &= -v\mathrm{cov}(X, (1 - F_X(X))^{v-1}). \end{aligned}\tag{3.5}$$

From this definition, it follows that $\Gamma_X(v) - E(X) = -\Gamma(v+1)F_X^{(-(v+1))}$ (2.1) characterizes the previous Gini orderings. Thus, as a consequence of Remark 2.1 the extended GMD is a measure of spread associated with the expected bounded coherent risk measure $\Gamma_X(v) - E(X)$ for every $v > 1$. Interest in the potential applications of GMD and its extension to portfolio theory was fostered by [34], who explained the financial insights of these measures. The use of Gini measures can be extended to the Gini "*tail*" measures (for a given β) that are associated with a dilation order [37]:

$$\Gamma_{X,\beta}(1) = E(X) - \left(\int_0^\beta F_X^{-1}(u)du\right)\Big/\beta.$$

These measures can also be extended using $v > 1$ and the tail measure:

$$\begin{aligned}\Gamma_{X,\beta}(v) &= E(X) - v\int_0^\beta (\beta - u)^{v-1}F_X^{-1}(u)du/\beta^v\\ &= E(X) - v(v-1)\int_0^\beta (\beta - u)^{v-2}L_X(u)du/\beta^v \end{aligned}\tag{3.6}$$

for some $\beta \in [0, 1]$. In this case, $\Gamma_{X,\beta}(v) = E(X) - \Gamma(v+1)F_X^{(-(v+1))}(\beta)/\beta^v$ (for every $v > 1$) is the deviation measure associated with the expected bounded coherent risk measure $\Gamma_{X,\beta}(v) - E(X)$. Moreover, if $X\,RSD\,Y$ then $E(X) = E(Y)$ and

[10] In the income inequality literature, the Gini index is the area between the relative Lorenz curve and the 45° line expressing complete equality.

$X \underset{-2}{\geq} Y$. Thus, $-\Gamma(v+1)F_X^{(-(v+1))}(\beta)/\beta^v \leq \Gamma_{Y,\beta}(v) - E(Y)$ for every $v \geq 1$ and for every $\beta \in [0,1]$, i.e. $\Gamma_{X,\beta}(v) \leq \Gamma_{Y,\beta}(v)$. This means that $\Gamma_{X,\beta}(v)$ is a variability measure consistent with the R–S order. In addition, $\Gamma_{X,\beta}(v) = \Gamma_{Y,\beta}(v)$ for any $\beta \in [0,1]$ if and only if $E(X) - E(Y) = \Gamma(v+1)\int_0^\beta (1-\frac{u}{\beta})^v d(F_X^{-1}(u) - F_Y^{-1}(u))$ for any $\beta \in [0,1]$, and if and only if $F_X = F_Y$, as remarked here:

Remark 3.1. For every $v \geq 1$ and for every $\beta \in (0,1)$ the Gini tail measure $\Gamma_{X,\beta}(v) = E(X) - \Gamma(v+1)F_X^{(-(v+1))}(\beta)/\beta^v$ is consistent with the R–S order. Moreover, if $\Gamma_{X,\beta}(v) = \Gamma_{Y,\beta}(v)$ for any $\beta \in [0,1]$, then $F_X = F_Y$.

From Remark 3.1 we can deduce that for any $v \geq 1$, a new variability ordering induced by a R–S ordering will be introduced. That is, we say that X dominates Y *in the sense of v-Gini variability ordering* if and only if $\Gamma_{X,\beta}(v) \leq \Gamma_{Y,\beta}(v)$ for any $\beta \in [0,1]$. These new variability orderings generalize the dilation order that holds when $v = 1$. Although we can minimize the Gini tail measure $\Gamma_{X,\beta}(v)$ by solving the optimization problem with a linear objective function, we cannot easily maximize $\Gamma_{X,\beta}(v)$ with the same linear optimization because the concentration curve appears with a negative sign in the optimization problem.

Therefore, if investors want to optimize a portfolio of European options as suggested in Sec. 1, they could maximize the variability with respect to the v-Gini ordering by solving the following optimization problem for some values m, q_i; $i = 1, 2$ and $\beta = s/T$ (with an integer $s \leq T$):

$$\begin{gathered}\max_x \frac{1}{T}\sum_{k=1}^{T} x'z_{(k)} - \frac{v}{\beta^v T}\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{v-1}(x'z)_{i:T} \quad \text{subject to} \\ \frac{\sum_{t=1}^{T}(x'R_{(t)} - \frac{1}{T}\sum_{k=1}^{T}x'R_{(k)})^3}{T(x'Q_R x)^{3/2}} \geq q_1; \quad \frac{\sum_{t=1}^{T}(x'R_{(t)} - \frac{1}{T}\sum_{k=1}^{T}x'R_{(k)})^4}{T(x'Q_R x)^2} \leq q_2 \\ \frac{1}{T}\sum_{k=1}^{T}x'R_{(k)} \geq m(x'Q_R x)^{1/2}; \quad \sum_{j=1}^{n} x_j = 1; \;\; x_j \geq 0; \;\; j = 1,\ldots,n, \end{gathered} \tag{3.7}$$

where $(x'z)_{k:T}$ is the kth of T ordered observations of the underlying log returns portfolio $x'z$. The portfolio optimization problems (3.4), and (3.7) are not as computationally simple as those problems from Sec. 2, since we generally get local solutions (which are not necessarily global optima) if we maximize convex variability measures (such as the functionals (3.1), (3.2), and (3.6)) on a compact set of admissible choices. Instead, if we minimize variability, we can linearize all the functionals (3.1), (3.2), and (3.6) using the arguments from Sec. 2. Moreover, Bäuerle and Müller in [23] have proved that the convexity property characterizes most of the variability measures consistent with the R–S ordering. Thus we have to solve generally computational complex portfolio problems when we maximize variability measures.

3.2. *Practical portfolio problems consistent with tracking-error orderings*

As shown by [38], there is a strong connection between probability metric theory and portfolio theory. In this section, we recall some of the basic properties of probability distances that under suitable assumptions can be used as variability measures (concentration/dispersion measures) and we introduce the concept of tracking error orderings and measures. Thus, we propose portfolio problems where a benchmark asset with return r_Y and n risky assets with returns $r = [r_1, \ldots, r_n]'$ are traded. In particular, we examine portfolio problems consistent with some particular tracking-error orderings using T i.i.d. observations of the vector of returns $r_{(k)} = [r_{1,k}, \ldots, r_{n,k}]'$ and of the benchmark return $r_{Y,k}$, $k = 1, \ldots, T$.

As explained in Sec. 1, investors want to reduce the distance to a given benchmark. Any probability functional μ is called a *probability distance* with parameter K if it is *positive* and satisfies the following additional properties:

(1) *Identity* $f(X) = f(Y) \Leftrightarrow \mu(X, Y) = 0$,
(2) *Symmetry* $\mu(X, Y) = \mu(Y, X)$,
(3) *Triangular inequality* $\mu(X, Z) \leq K(\mu(X, Y) + \mu(Y, Z))$ for all admissible random variables X, Y, and Z,

where $f(X)$ identifies some characteristics of the random variable X. If the parameter K equals 1, we have a *probability metric*. We can always define the alternative finite distance $\mu_H(X, Y) = H(\mu(X, Y))$, where $H : [0, +\infty) \to [0, +\infty)$ is a nondecreasing positive continuous function such that $H(0) = 0$ and $K_H = \sup_{t>0} \frac{H(2t)}{H(t)} < +\infty$.[11] Therefore, for any probability metric μ, μ_H is a probability distance with parameter K_H. In this case, we distinguish between primary, simple, and compound probability distances/metrics that depend on certain modifications of the identity property (see [31]). Compound probability functionals identify the random variable almost surely i.e. $\mu(X, Y) = 0 \Leftrightarrow \Pr(X = Y) = 1$. Simple probability functionals identify the distribution (i.e. $\mu(X, Y) = 0 \Leftrightarrow F_X = F_Y$). Primary probability functionals determine only some random variable characteristics. Often we can associate a *distance ordering* (on the space of random variables Λ) to compound (or simple) distances μ defined between a random variable belonging to Λ and a fixed benchmark $Z \in \Lambda = V$. The ordering depends on the distance/metric definition and is related to a functional $\rho : U \times B \to R$ (where (B, M_B) is a measurable space) that serves to order distances between distributions or random distances as explained in the following:

[11] See [31].

Definition 3.1. We say that X is preferred to Y with respect to the μ-compound (simple) distance from Z (namely, $X \succ_{\mu_Z} Y$) if and only if there exists a probability functional $\rho : \Lambda \times Z \times B \to R$ dependent on μ such that for any $t \in B$ and $X, Y \in \Lambda$, $\rho_X(t) \leq \rho_Y(t)$. In this case, the equality $\rho_X = \rho_Y$ implies a distributional equality $F_{g(X,Z)} = F_{g(Y,Z)}$ for compound distances μ and a distance equality $g(F_X, F_Z) = g(F_Y, F_Z)$ for simple distances μ, where $g(x, z)$ is a distance in R. We call ρ_X the (tail) tracking-error measure (functional) associated with the μ tracking error ordering.

Consequently, in passive strategies we minimize the tracking-error functional ρ_X associated with the μ-tracking-error ordering. This is generally different from active strategies where investors want to outperform the benchmark and optimize a particular nonsymmetric probability tracking-error functional [38]. For active strategies, we also require that optimal portfolio reward measures be greater than the reward measures of the benchmark.

In essence, probability metrics can be used as tracking-error measures. In solving the portfolio problem with a probability distance, we intend to "approach" the benchmark and change the perspective for different types of probability distances. Hence, if the goal is only to control the variability of an investor's portfolio or to limit its possible losses, mimicking the uncertainty or the losses of the benchmark can be done with a primary probability distance. When the objective for an investor's portfolio is to mimic the entire benchmark, a simple or compound probability distance should be used. In addition to its role of measuring tracking errors, a compound distance can be used as a measure of variability. If we apply any compound distance $\mu(X, Y)$ to X and $Y = X_1$ that are i.i.d., we obtain:

$$\mu(X, X_1) = 0 \Leftrightarrow \Pr(X = X_1) = 1 \Leftrightarrow X \text{ is a constant almost surely.}$$

For this reason, we refer to $\mu(X, X_1) = \mu_I(X)$ as a *concentration measure* derived by the compound distance μ. Similarly, if we apply any compound distance $\mu(X, Y)$ to X and $Y = E(X)$ (either $Y = M(X)$, i.e. the median or a percentile of X, if the first moment is not finite), we get:

$$\mu(X, E(X)) = 0 \Leftrightarrow \Pr(X = E(X)) = 1 \Leftrightarrow X \text{ is a constant almost surely.}$$

Hence, $\mu(X, E(X)) = \mu_{E(X)}(X)$ can be referred to as a *dispersion measure* derived by the compound distance μ. Following are some examples of compound metrics, the associated concentrations, the dispersion measures, the tracking-error orderings, and the associated practical portfolio problems.

3.2.1. *Examples of probability compound metrics*

For each probability compound metric we can generate a probability compound distance $\mu_H(X, Y) = H(\mu(X, Y))$ with parameter K_H.

(i) ***L^p-metrics:*** For every $p \geq 0$ we recall the L^p-metrics: $\mu_p(X, Y) = E(|X - Y|^p)^{\min(1,1/p)}$; the associated concentration measures are $\mu_{I,p}(X) =$

$E(|X - X_1|^p)^{\min(1,1/p)}$, where X_1 is an i.i.d. copy of X; and the associated dispersion measures are the central moments $\mu_{E(X),p}(X) = E(|X - E(X)|^p)^{\min(1,1/p)}$. The dispersion and concentration measures $\mu_{E(X),p}(X)$ and $\mu_{I,p}(X)$ are variability measures consistent with the $(p+1)$ R–S order for any $p \geq 1$. We can consider for L^p metrics the tracking-error measures

$$\begin{aligned}\rho_{X,p}(t) &= (\mu_p(XI_{[|X-Z|\geq t]}, ZI_{[|X-Z|\geq t]}))^{1/\min(1,1/p)} - t^p \Pr(|X - Z| \geq t)\\ &= E((|X - Z|^p - t^p)_+)\end{aligned}$$

for any $t \in [0, +\infty)$ associated with a μ_p tracking-error ordering. Moreover, $\rho_{X,p} = \rho_{Y,p}$ implies that $F_{|X-Z|} = F_{|Y-Z|}$. Thus, all investors who choose portfolios that are consistent with this μ_p tracking-error ordering solve the following portfolio selection problem for some given $t > 0$:

$$\begin{aligned}&\min_{x,\nu_k,u_k} \frac{1}{T}\sum_{k=1}^{T} u_k \quad \text{subject to}\\ &\sum_{j=1}^{n} x_j = 1; \quad x_j \geq 0;\ \ j = 1,\ldots,n;\ \ v_k \geq x' r_{(k)} - r_{Y,k}\\ &v_k \geq r_{Y,k} - x' r_{(k)}; \quad u_k \geq v_k^p - t^p;\ \ u_k \geq 0;\ \ k = 1,\ldots,T\end{aligned} \tag{3.8}$$

which is linear when $p = 1$ and convex when $p > 1$.

(ii) ***Ky Fan metrics*****:** $k_1(X,Y) = \inf\{\varepsilon > 0/\Pr(|X - Y| > \varepsilon) < \varepsilon\}$ and $k_2(X,Y) = E(\frac{|X-Y|}{1+|X-Y|})$, the respective concentration measures are $k_{1,I}(X) = \inf\{\varepsilon > 0/\Pr(|X - X_1| > \varepsilon) < \varepsilon\}$, $k_{2,I}(X) = E(\frac{|X-X_1|}{1+|X-X_1|})$, while the associated dispersion measures are $k_{1,E(X)}(X) = \inf\{\varepsilon > 0/\Pr(|X - E(X)| > \varepsilon) < \varepsilon\}$, $k_{2,E(X)}(X) = E(\frac{|X-E(X)|}{1+|X-E(X)|})$.

For Ky Fan metrics we consider the tracking-error measures $\rho_{X,i}(t) = k_i(XI_{[|X-Z|\geq t]}, ZI_{[|X-Z|\geq t]})$ associated with the k_i-tracking-error orderings for any $i = 1, 2$, where $\rho_{X,i} = \rho_{Y,i}$ implies $F_{|X-Z|} = F_{|Y-Z|}$. Therefore, we obtain choices consistent with k_1 tracking-error order by solving the following optimization problem for some $t \geq 0$

$$\begin{aligned}&\min_x u \quad \text{subject to}\\ &\sum_{j=1}^{n} x_j = 1; \quad x_j \geq 0;\ \ u \geq 0;\\ &\frac{1}{T}\sum_{k=1}^{T} I_{[|x' r_{(k)} - r_{Y,k}| > \max(u,t)]} < u.\end{aligned} \tag{3.9}$$

On the other hand, investors who choose portfolios consistent with this k_2 tracking-error ordering should minimize the consistent estimator $\hat{\rho}_{X,2}(t) = \frac{1}{T}\sum_{k=1}^{T} \frac{|x' r_{(k)} - r_{Y,k}| I_{[|x' r_{(k)} - r_{Y,k}| \geq t]}}{1+|x' r_{(k)} - r_{Y,k}|}$ for some $t \geq 0$.

Generally, when we use the compound metric/distance as dispersion/ concentration measure $\mu(X, f(X))$ (where $f(X)$ is either a functional of X or an independent copy of X), we obtain a tracking-error measure between X and Z using $\mu(X-Z, f(X-Z))$, which have been used in the portfolio literature [38].

Moreover, even simple probability distances can be used as dispersion measures and tracking-error measures, but, generally not as concentration measures. As a matter of fact, when we apply any simple distance $\mu(X,Y)$ to X and $Y = E(X)$ (either $Y = M(X)$, i.e. median or a percentile of X, if the first moment is not finite), we obtain

$$\mu(X, E(X)) = 0 \Leftrightarrow F_X = F_{E(X)} \Leftrightarrow X \text{ is a constant almost surely.}$$

Thus, we refer to $\mu(X, E(X)) = \mu_{E(X)}(X)$ as a *dispersion measure* derived by the simple distance μ. As for compound metrics, we can generate a simple probability distance $\mu_H(X,Y) = H(\mu(X,Y))$ with parameter K_H for any simple probability metric $\mu(X,Y)$.

As an example of simple metrics, with their associated dispersion measures, and their tracking-error orderings we consider the Zolotarev–Rachev metric (ZRM): For every $q \geq 0$, the ZRM among variates X, Y with support on the interval $[a,b]$ is given by $\mathrm{ZRM}_{q,\alpha}(X,Y) = (\int_a^b (|F_X^{(\alpha)}(t) - F_Y^{(\alpha)}(t)|^q dt))^{\min(1/q,1)}$. This metric was introduced by Zolotarev for $q = 1$ and extended by [31] for $q \neq 1$. The associated dispersion measure is

$$\begin{aligned}
&\mathrm{ZRM}_{q,\alpha}(X, E(X)) \\
&\quad = \left(\int_a^{E(X)} |F_X^{(\alpha)}(t)|^q dt + \int_{E(X)}^b \left| F_X^{(\alpha)}(t) - \frac{(t - E(X))^{\alpha-1}}{\Gamma(\alpha)} \right|^q dt \right)^{\min(1/q,1)}.
\end{aligned}$$

Associated to a $\mathrm{ZRM}_{p,\alpha}$ tracking-error ordering, we provide the tracking-error measure $\rho_X(t) = \mathrm{ZRM}_{p,\alpha}(XI_{[X\leq t]}, ZI_{[Z\leq t]})^{1/\min(1/p,1)}$ where $\rho_X = \rho_Y$ implies that $|F_X^{(\alpha)} - F_Z^{(\alpha)}| = |F_Y^{(\alpha)} - F_Z^{(\alpha)}|$. Then, for any $t \geq c = \min_{1\leq k\leq T} r_{Y,k}$, a consistent estimator of $\mathrm{ZRM}_{p,\alpha}(x'rI_{[x'r\leq t]}, r_Y I_{[r_Y\leq t]})^{1/\min(1/p,1)}$ for a large M is the functional

$$\begin{aligned}
\tilde{G}_{x'r,r_Y}^{(\alpha,p)}(t) = \Bigg(\frac{t-c}{M} \sum_{i=0}^{M} \Bigg| \frac{1}{T\Gamma(\alpha)} \sum_{k=1}^{T} \left(c + i\frac{t-c}{M} - x'r_{(k)} \right)^{\alpha-1} I_{[c+i\frac{t-c}{M} > x'r_{(k)}]} \\
- \left(c + i\frac{t-c}{M} - r_{Y,k} \right)^{\alpha-1} I_{[c+i\frac{t-c}{M} > r_{Y,k}]} \Bigg|^p \Bigg),
\end{aligned}$$

since $\tilde{G}_{x'r,r_Y}^{(\alpha,p)}(t) \overset{M,T\to\infty}{\longrightarrow} \mathrm{ZRM}_{p,\alpha}(x'rI_{[x'r\leq t]}, r_Y I_{[r_Y\leq t]})^{1/\min(1/p,1)}$. Thus, all investors who choose portfolios consistent with this $\mathrm{ZRM}_{p,\alpha}$ tracking-error ordering would solve the following portfolio selection problem for some given

$t \geq c = \min_{1 \leq k \leq T} r_{Y,k}$:

$$\min_x \tilde{G}^{(\alpha,p)}_{x'r,r_Y}(t) \quad \text{subject to}$$
$$\sum_{j=1}^{n} x_j = 1; \quad x_j \geq 0; \;\; j = 1, \ldots, n. \tag{3.10}$$

4. An Empirical Comparison Among Different Portfolio Models

To analyze the ex-post wealth obtained from the classic portfolio strategy based on the maximization of the Sharpe ratio, we use 2983 firms from the NYSE, NASDAQ, and AMEX markets whose shares have been traded actively from January 1, 2007 until September 25, 2010 (939 daily observations). The data are taken from Thomson-Reuters Datastream. Using a lagged window of six months of daily log-returns, we compute the average basic statistics every 20 trading days from June 23, 2007 to September 25, 2010. Table 1 reports the average values for the basic statistics: minimum, maximum, mean, standard deviation, skewness, and kurtosis.

4.1. *The dataset and the methodology for large-scale selection problems*

To test whether stock returns follow a normal distribution, we compute the Jarque–Bera statistic for normality (with a 95% confidence level). The Gaussian assumption is often justified in terms of its asymptotic approximation. This can be only a partial justification however, because the Central Limit Theorem for normalized sums of i.i.d. random variables determines the domain of attraction of each stable law $S_\alpha(\sigma, \beta, \mu)$, which depends on four parameters: the index of stability $\alpha \in (0, 2]$, the asymmetry parameter $\beta \in [-1, 1]$, the dispersion parameter $\sigma > 0$, and the location parameter μ [39]. Table 1 reports the average values for the maximum likelihood estimates of stable Paretian parameters and percentage of rejection using

Table 1. Average values of the mean, standard deviation, skewness, kurtosis, maximum, minimum Jarque–Bera (JB) test for normality (with a 95% confidence level), MLE of stable Paretian parameters and Kolmogorov–Smirnoff (K–S) test (with a 95% confidence level) computed on the daily log returns of the global dataset and on preselected assets.

Sample	Mean	St. dev	Skewness	Kurtosis	Max	Min	JB 95%
All assets	0.0000378	0.02732	0.04036	7.6279	0.10738	−0.10909	0.7443
Preselected	0.0028444	0.03212	1.17879	10.3726	0.16618	−0.09593	0.8585
			Stable paretian parameters				
Sample	α	β	Dispersion σ	Location μ	K–S 95%		
All assets	1.6644	0.07016	0.01498	0.0000333	0.165116		
Preselected	1.6322	0.26528	0.01649	0.0034222	0.170692		

the Kolmogorov–Smirnov statistic to test the stable Paretian with a 95% confidence level. The average results show that the Gaussian distributional hypothesis is rejected for about the 74% of the stocks in our study and the stable Paretian hypothesis rejected for about 16% of the stocks. The average values of the other parameters suggest the presence of heavy tails since the average kurtosis exceeds three and the stability parameter α is less than two. The average values for the other parameters do not suggest the presence of skewness, since the average of the asymmetry parameter β and the skewness are around zero and the average maximum and minimum are almost equal in absolute value.

These casual empirical findings indicate a very strong impact of heavy tails in this dataset. Moreover, since the number of stocks exceeds the number of observations, to get a good approximation of the portfolio input measures, it is necessary to find the right tradeoff between the number of observations and a statistical approximation of the historical series [28, 29]. In particular, we use two techniques to reduce the dimensionality of large scale portfolio problems: preselection and PCA. With preselection, only a limited number of stocks is chosen before optimizing the portfolio. For each optimization problem, we preselect the first 150 stocks that present the highest Rachev ratio for probability $p = 0.05$. The Rachev ratio is measured as follows [30]:

$$RR_p(X) = \frac{\mathrm{CVaR}_p(-X)}{\mathrm{CVaR}_p(X)},$$

where X is the variate and $\mathrm{CVaR}_p(X) = \inf_u\{u + \frac{1}{p}E((-X-u)_+)\}$ is the conditional value-at-risk or expected shortfall [10]. We employ the Rachev ratio because the portfolios that maximize this reward-risk performance measure generally present higher earnings, a positive skewness, and lower losses [30]. Moreover, the Rachev ratio is based on the values of the return distributional tails and it has been often used to preselect stocks in momentum portfolio strategies [40].

We compute the statistics of the log-returns of the 150 preselected stocks using a moving window of data, where the preselected stocks can change every 20 trading days. In particular, we compute the average statistics using the last six months of daily observations for every 20 trading days from June 23, 2007 to September 25, 2010. Since the 150 preselected stocks are not always the same for all the observation periods, then the whole number of preselected stocks is 1991 from among the 2983. Hence, we guarantee a substantial portfolio turnover since on average, for every 20 days, there are more than 48 new preselected stocks classified as having the best Rachev ratio. For any preselection, the log returns of the preselected stocks present on average higher skewness, higher earnings (the maximum), and lower losses (the minimum) than those of the entire dataset. This is observed in Table 1 where we report for the preselected stocks and the entire dataset, the average statistics over time.

When we compare the statistics of the entire dataset and the statistics of the preselected stocks, we can observe that the mean of the preselected log-returns on average is about 100 times that of the entire dataset even if the preselected log-returns exhibit heavier tails. Thus, it is not surprising that for the preselected log-returns normality is rejected for about 85% of the stocks, whereas the stable Paretian hypothesis is rejected for about 17%. Furthermore, both percentages observed for the preselected log-returns are higher than those observed for the entire dataset (that include even all the assets which are not preselected). Therefore, our analysis serves as a form of stress testing of alternative portfolio selection models for non-Gaussian leptokurtic returns with positive skewness.

The portfolio problems we are considering have a benchmark with return r_Y and n risky stocks with returns $r = [r_1, \dots, r_n]'$ being traded. No short selling is allowed and, to guarantee enough diversification for all portfolio problems, no single stock may comprise 5% or more of the portfolio. Clearly, the selection among 150 stocks still represents a large scale portfolio selection problem. For the ex-post comparison, we use a window of 120 daily observations and thus we need to further reduce the dimension of the portfolio problem.[12] Therefore, we perform a PCA of the returns of the preselected 150 stock returns in order to identify what are the few factors f_i with the highest variability.

For each optimization problem, we apply a PCA to the correlation matrix of the 150 preselected stocks to identify the first 20 components that explain the majority of the global variance. Subsequently, each series $r_i (i = 1, \dots, 150)$ can be represented as a linear combination of 20 factors plus a small uncorrelated noise. Using a factor model, we approximate the preselected returns r_i as follows:

$$r_{i,t} = \alpha_i + \sum_{j=1}^{20} \beta_{i,j} f_{j,t} + e_{i,t} \quad i = 1, \dots, 150,$$

where $r_{i,t}$ and $e_{i,t}$ are, respectively, returns and errors in the approximation of ith asset at time t and $\alpha_i; \beta_{i,j}$ are the coefficients of the factor model. The dimension of the portfolio problem depends now on only 20 factors. In the empirical comparison, we value ex-post the sample paths of the wealth obtained using different models. We optimize the portfolio every month from June 23, 2007 to September 25, 2010 and then recalibrate daily the portfolio by keeping the proportions invested in each stock constant. We start with an initial wealth of one (i.e. $W_{t_0} = \sum_{i=1}^{150} x_i = 1$) and for each strategy we compute the optimal portfolio composition 43 times including the last observation day. At the kth optimization ($k = 0, 1, 2, \dots, 42$), the following three steps are performed to compute the ex-post final wealth:

Step 1. Preselect the first 150 stocks with the highest Rachev ratio. Apply the PCA component to the correlation matrix of the preselected stocks. Then apply the factor model using the first 20 principal components to approximate the variability of the preselected returns.

[12]The classic way to reduce the dimensionality of the problem is to approximate the return series with a regression-type model (such as a k-fund separation model, see [41]) that depends on an adequate number (not too large) of parameters.

Step 2. Determine the optimal portfolio $\bar{x}_{(k)}$ that has the proportions invested in each of the preselected stocks for the period $[t_k,\, t_{k+1}]$.

Step 3. During the period $[t_k,\, t_{k+1}]$ (where $t_{k+1} = t_{k+20}$) recalibrate the portfolio daily by maintaining the proportions invested in each asset that are equal to those in the optimal portfolio $\bar{x}_{(k)}$. The ex-post final wealth is given by:

$$W_{t_{k+1}} = W_{t_k} \prod_{i=1}^{20} \bar{x}_{(k)}{}' z_{t_k+i}^{(\text{ex-post})},$$

where $z_{t_k+i}^{(\text{ex-post})} = [z_{1,t_k+i}^{(\text{ex-post})}, \ldots, z_{150,t_k+i}^{(\text{ex-post})}]'$ is the vector of observed daily gross returns for the period $[t_k + i - 1,\, t_k + i]$. These returns are given as $z_{m,t_k+i}^{(\text{ex-post})} = P_{m,t_k+i}^{(\text{ex-post})}/P_{m,t_k+i-1}^{(\text{ex-post})}$ where $P_{m,t_k+i}^{(\text{ex-post})}$ is the price of the mth asset observed at time $t_{k+1} + i$.

The three steps are repeated for all the optimization problems for all available observations. To evaluate the impact of preselection and the classic portfolio strategy, we compare the ex-post wealth from the two strategies on preselection: One is a "Uniform" strategy where at each time we invest 1/150 in each of the 150 preselected stocks and the second is a "Sharpe" strategy that maximizes the Sharpe ratio for every month. There are several papers [42, 43] that provide a justification for the use of the "Uniform strategy" (also called "naïve strategy"). For the Sharpe strategy, we solve the following problem for every month:

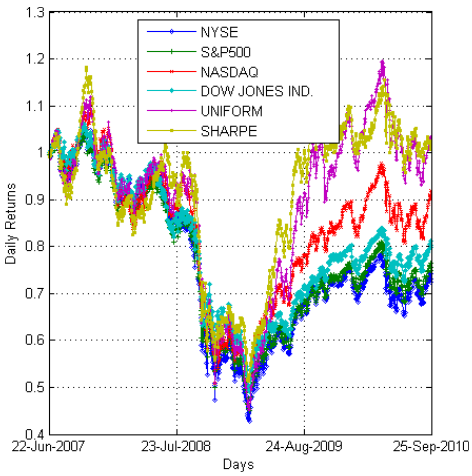
$$\max_x \frac{\sum_{i=1}^{150} x_i E(r_i - r_Y)}{\text{st.dev}(x'r)}$$

$$\text{s.t.} \sum_{i=1}^{150} x_i = 1; \quad x_i \geq 0; \;\; x_i \leq 0.05,$$

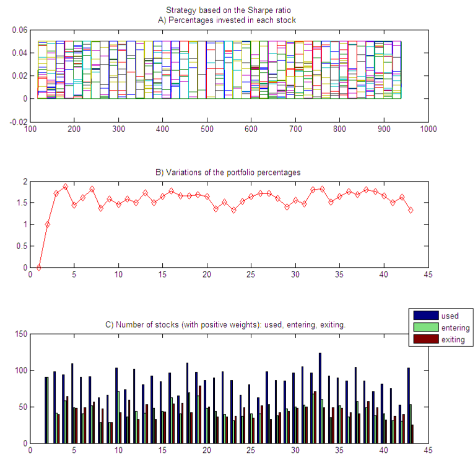
where $\text{st.dev}(x'r)$ is the standard deviation of the portfolio returns and r_Y is return of the benchmark asset that we assume it is not included in this optimization problem.

Figure 1(a) compares the performance of several US market indices (NYSE Composite, S&P500 Index, NASDAQ Composite, and Dow Jones Industrial Average, DJIA) with the ex-post wealth obtained from the "Uniform" and "Sharpe" strategies. Figure 1(b) consists of three sub-figures, A, B, and C, that offer a graphical analysis of the portfolio turnover and diversification time evolution for the "Sharpe" strategy. We observe the following:

(1) The ex-post period of analysis accounts for the US sub-prime mortgage crisis period (September 2008 to March 2009) and the US credit crisis beginning period (May 2010 to September 2010). These two crises have a strong impact on the preselected stocks. The US sub-prime mortgage crisis appears to have had a stronger impact than the US credit crisis.
(2) There are no substantial differences between the two strategies (Uniform and Sharpe). At the end of the period, these strategies yield practically the same



(a) Portfolio strategy comparison



(b) Portfolio turnover and diversification

Fig. 1. (a) Comparing the performance of US market indexes (NYSE, S&P500, NASDAQ, DJI), the ex-post wealth from the "Uniform" strategy and the ex-post wealth from the "Sharpe" strategy. (b) Portfolio turnover and diversification of the "Sharpe" strategy.

initial capital (1.02 and 1.03 for the Sharpe and Uniform strategy, respectively). With these strategies, a little loss would have been realized had transaction costs been considered.

(3) At the end period, all the US market indices show large losses. In particular, from an initial wealth of 1, we get 0.74 for the NYSE Composite, 0.76 for the S&P500 Index, 0.92 for the NASDAQ Composite, and 0.81 for the DJIA. Since the two portfolio selection strategies outperform the market indices, we have a further justification for searching for better performing portfolio strategies.

(4) Figure 1(b)-A shows the changes to the portfolio composition for the Sharpe strategy. As can be seen, these proportions change significantly over time and thus there is good diversification for each optimal portfolio.

(5) Figure 1(b)-B identifies the percentages ϕ_k $(k = 1, \ldots, 43)$ of the portfolio that changes every 20 days using the formula:

$$\phi_k = \sum_{i=1}^{150} |\bar{x}_{(k),i} - \bar{x}_{(k-1),i}|.$$

In particular, ϕ_k belongs to the interval $[0, 2]$, where 0 means the portfolio has not changed during the period $[t_{k-1}, t_k]$, while 2 means the portfolio is sold and a different portfolio is purchased. Since the values are very close to 2, this means that the Sharpe strategy requires a significant turnover in the portfolio composition.

(6) Figure 1(b)-C shows for each optimization, the number of stocks with positive proportions (i.e. long positions), the new stocks in the optimal portfolio, and

the stocks removed from the optimal portfolio. This figure confirms the good diversification and turnover of the Sharpe strategy. In particular, we observe that the number of stocks with positive proportions is on average about 80.

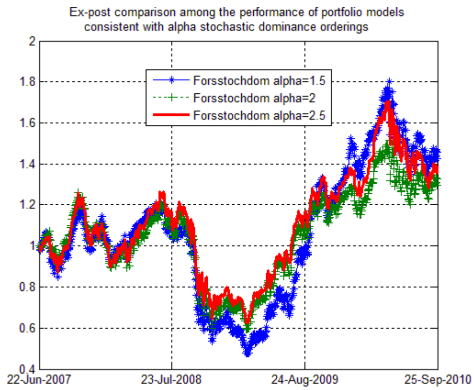
4.2. *Empirical comparison of portfolio selection problems that are consistent with risk orderings*

To compare the different models that are consistent with risk orderings, we use the methodology and the dataset from Sec. 4.1 and evaluate the portfolio problems from Secs. 2.1–2.3. As in Sec. 4.1, we assume that the benchmark r_Y is not allowed. To solve the portfolio problems that could exhibit more local solutions which are not necessarily global, we adopt the heuristic approach proposed by [44].

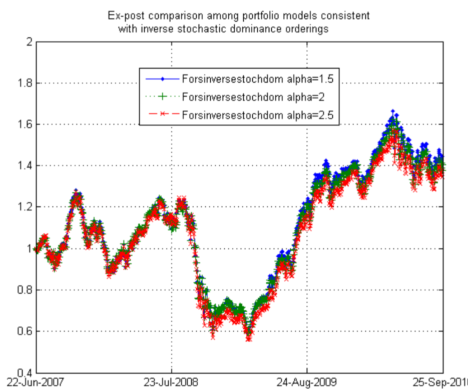
We use a window of six months of daily observations and optimize the portfolio every month from June 23, 2007 until September 25, 2010. Then we recalibrate the portfolio daily by keeping the proportions invested in each stock constant. We assume that no short sales are allowed and no single stock carries more than 5% (i.e. $0 \leq x_i \leq 0.05$). Since the portfolio is optimized monthly, we constrain the empirical mean not to be lower than 10% (i.e. in the previous problems $m = 0.1$) as there are always more than 20 stocks among the preselected ones that satisfy this requirement and the optimization is always feasible.

For the portfolio problems from Secs. 2.1 and 2.2 we consider portfolio selections that are consistent to α stochastic dominance and α inverse stochastic dominance orderings with values $\alpha = 1.5; 2; 2.5$. For stochastic dominance orderings, we minimize the functionals $E((t-X)_+^{\alpha-1})$ with $t = 0.005$ and $-\int_0^p (p-u)^{\alpha-2} F_X^{-1}(u)du$ with $p = 0.04$ for inverse stochastic dominance orderings. For $\alpha = 2; 2.5$ we observe that choices are consistent with nonsatiable and risk-averse investors which allows us to solve linear optimization problems (2.3) and (2.6). For $\alpha = 1.5$, the choices are consistent with nonsatiable investors who are not necessarily risk-averse, thus, we solve optimization problems (2.1) and (2.4). Figures 2(a) and 2(b) present the ex-post comparison between these strategies, revealing a final wealth that is higher than the final wealth obtained from the Sharpe ratio maximization strategy. The three strategies based on inverse stochastic dominance orderings ($\underset{-\alpha}{\geq}$ orderings) do not offer substantial differences but appear to be more conservative (as they lose less) than those strategies that are consistent with stochastic dominance orderings ($\underset{\alpha}{\geq}$ orderings). As expected, the strategy with $\alpha = 1.5$ appears to be the most aggressive that loses more during the crisis but recovers speedily. This is why this strategy yields the highest final wealth. Figure 2(a) also shows that the most prudent behavior of the strategy with $\alpha = 2.5$ gives a higher wealth during the market crisis (October 2008–March 2009).

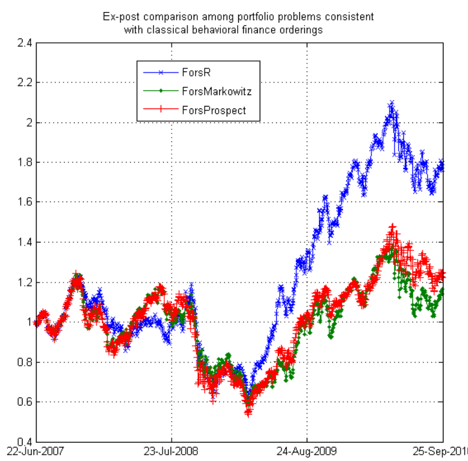
Among the portfolio problems discussed in Sec. 2.3, we now analyze the selection models consistent with Markowitz stochastic orderings, prospect stochastic orderings, and a particular aggressive-coherent stochastic ordering. We minimize



(a) Consistency with stochastic dominance orderings

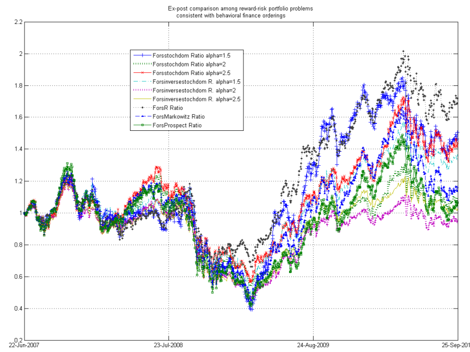


(b) Consistency with inverse stochastic dominance orderings



(c) Consistency with classic behavioral finance orderings

Fig. 2. Ex-post comparison among portfolio problems consistent with risk orderings.



(d) Consistency with reward-risk behavioral finance orderings

Fig. 2. (*Continued*)

the functionals (defined in Sec. 2.3) as follows:

$$m_X(a,y) = aE((y-X)_+) - (1-a)E((X+y)_+)$$
$$\text{for } a = 0.6 \text{ and } y = -0.003,$$

$$g_X(a,y) = aE((-y-X)I_{[X\in(0,-y]]}) - yF_X(0) - (1-a)E((X-y)I_{[X\in(y,0]]})$$
$$\text{for } a = 0.6 \text{ and } y = -0.003$$

and $S_X(a,t_1,t_2) = a\rho_{1,X}(t_2) - (1-a)\rho_{2,-X}(t_1)$, where $\rho_{1,X}(t) = \rho_{2,X}(t) = \mathrm{CVaR}_t(X)$ for $a = 0.5$ and $t_1 = t_2 = 0.04$.

For these portfolio problems we optimize functionals that admit more local solutions which are not necessarily global. Hence, in order to obtain any single optimum (with the heuristic for global optimization), we need at the same time to preselect 150 stocks and to apply the PCA. Figure 2(c) depicts the ex-post comparison between these strategies. Even if the Markowitz and prospect-type portfolio strategies (labeled ForsMarkowitz and ForsProspect, respectively) indicate a better performance than the classic strategy that maximizes the Sharpe ratio, they perform worse than strategies consistent with $\underset{\alpha}{\geq}$ and $\underset{-\alpha}{\geq}$ orderings. In contrast, the strategy consistent with the aggressive-coherent stochastic ordering (labeled *ForsR*) presents the best behavior during all the observed periods with a final wealth increase of about 80% and about 20% a year.

This empirical analysis compares the effect of various portfolio problems that are consistent with different orderings when choices are represented in the mean-risk space. To account for the impact of a different reward measure we propose to maximize the reward-risk ratios where the reward measure is $E((X-t)_+)$ with $t = 0.005$ and the risk measure is the one used above. Therefore, we maximize a reward/risk ratio with the same constraints as in the previous analysis except for the monthly mean returns constraint (i.e. $0 \leq x_i \leq 0.05$ and $\sum_{i=1}^{150} x_i = 1$). The reward functional $E((X - t)_+)$ is convex and the portfolio problem based on its maximization is consistent with choices of nonsatiable and risk-lovers. Thus, by optimizing reward/risk ratios of this type, we obtain choices that are consistent with possible behavioral finance orderings where investors are neither risk-averse nor risk-loving. Figure 2(d) shows that the nine reward/risk ratios strategies are more aggressive than the analogous ones in the mean-risk space. Hence, most of the strategies lose more wealth during the crisis. At the horizon, all strategies except the three risk measures consistent with stochastic dominance produce lower wealth than the corresponding strategy in the mean-risk space. By analyzing the portfolio turnover and diversification we see that the strategies based on the measure $S_X(a, t_1, t_2)$ do not diversify as much between the preselected stocks. Even if a higher final wealth is reached, the number of preselected assets is small and, in several cases, is the minimum allowed (20 stocks). Except for this strategy, all other strategies diversify well between the preselected assets and exhibit a significant portfolio turnover.

4.3. *Comparing portfolios consistent either with convex-type or tracking-error orderings*

We now compare between the different models presented in Secs. 3.1 and 3.2. In particular, implementing the algorithm and the dataset used in Sec. 4.1, we evaluate the impact of portfolios that are consistent with convex-type orderings and tracking-error orderings. Since all the problems in this section have many local solutions which are not necessarily global, we solve the optimizations with an heuristic for global optimum [44]. As in Sec. 4.2, we use a window of six months of daily observations and optimize the portfolio every month by maintaining constant the proportions invested in each asset. We assume no short sales and that the allocation to any one asset will not exceed 5% (i.e. $0 \leq x_i \leq 0.05$).

To compare the portfolio problems of Sec. 3.1, we implicitly assume that the hypothesis of Black and Scholes is verified. The prices of European options are computed using the Black–Scholes model and the returns on these derivatives are computed as in Sec. 3.1. We preselect the "best" 150 derivatives (European calls and puts) whose returns present the highest Rachev ratio. Thus, every month the agent invests in a portfolio of European options. For each optimization, we use calls and puts that have a maturity of two months and have out-of-the-money exercise prices that are 1.03 and 0.97 times the actual prices respectively for calls and puts. We assume an annual riskless rate of 1% (the daily riskless rate is 0.01/250). This analysis serves only as a trial to value the impact of the models of typology 2 based on different concepts of variability for the following reasons:

(1) We do not use the option prices of the market but we do compute the European option prices using the Black–Scholes model and considering months consisting of 20 trading days.

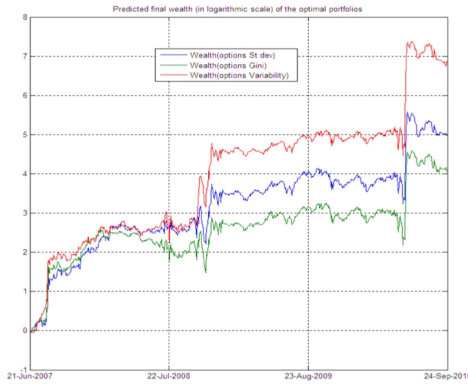
(2) We can easily find European options for indexes but not for individual stocks. So, even if we use real stock prices in our computation, we cannot guarantee similar results as those obtained with European options on indexes.
(3) We use a large number of European options having the same following characteristics:

  (a) They have same maturity of two months (40 trading days) and the same riskless rate over a long period;
  (b) They are out-of-the money (3% more or less) options that are never exercised because every month (20 trading days) the portfolio is re-optimized using new options.
  (c) There are no transaction costs and no liquidity constraints.

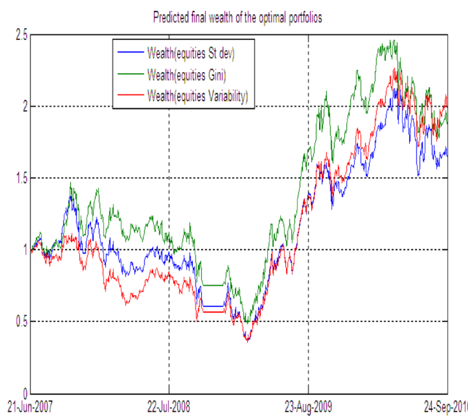
For the problem discussed in Sec. 3.1, we maximize a variability measure of the underlying stock returns. Thus, to reduce the dimensionality of the portfolio problem we apply a PCA to the correlation matrix of the underlying equities and approximate the stock returns that are used to compute the variability measure. We consider portfolio selection problems that maximize one of the following variability measures: the standard deviation; the tail Gini measure $\Gamma_{X,\beta}(v) = E(X) - \frac{v}{\beta^v}\int_0^\beta (\beta-u)^{v-1}F_X^{-1}(u)du$ with $v = 1.5$ and $\beta = 0.04$; the dispersion measure $F_X^{(\alpha_1,\alpha_2)}(b,t) = bF_X^{(\alpha_1)}(t) + (1-b)\bar{F}_X^{(\alpha_2)}(t)$ with $b = 0.5$, $t = 0.005$ and $\alpha_1 = \alpha_2 = 2.5$.

We solve the optimization problems similarly to (3.4) and (3.7) with the additional constraints on the weights ($x_i \le 0.05$). As parameters q_1, q_2, and m of problems (3.4) and (3.7) we use respectively the skewness, the kurtosis, and the Sharpe ratio of the derivative returns for a portfolio with equal weights (i.e. we compute these statistics on the uniform portfolio $\frac{\sum_{i=1}^{150} R_i}{150}$ where R_i is the return on the ith option). We recalibrate the portfolio daily, maintaining constant the proportions during the period $[t_k, t_{k+1}]$. The ex-post wealth is still obtained with the same formula in Sec. 4.1, namely, $W_{t_{k+1}} = W_{t_k}\prod_{i=1}^{20} \bar{x}_{(k)}{}' z_{t_k+i}^{(\text{ex-post})}$ where the vector of the ex-post gross returns $z_{t_k+i}^{(\text{ex-post})} = [z_{1,t_k+i}^{(\text{ex-post})}, \ldots, z_{150,t_k+i}^{(\text{ex-post})}]'$ (in the period $[t_k+i-1, t_k+i]$) is obtained using the Black–Scholes model. That is, $z_{m,t_k+i}^{(\text{ex-post})} = \frac{V_{m,t_k+i}^{(\text{ex-post})}}{V_{m,t_k+i-1}^{(\text{ex-post})}}$ where $V_{m,t_k+i}^{(\text{ex-post})}$ is the Black–Scholes model's theoretical price for the mth derivative at time $t_{k+1}+i$.

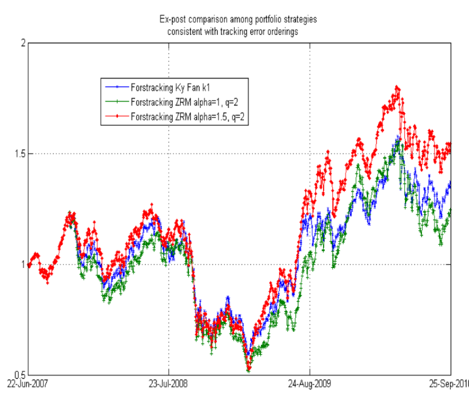
Figure 3(a) shows the ex-post comparison of the wealth obtained with these strategies. As wealth is expressed in base 10 logarithms, we see that using puts and calls options even during the crises periods increases a thousand times the initial investment. In particular the strategy based on the maximization of the dispersion measure $F_X^{(\alpha_1,\alpha_2)}(b,t)$ shows the highest ex-post wealth. The portfolio diversification is different from all the other portfolio strategies. While the portfolio turnover is completely achieved after one month, the number of optimal calls and puts is generally small and this number changes with respect to the variability



(a) Consistency with convex type orderings: portfolio wealth obtained investing in European options

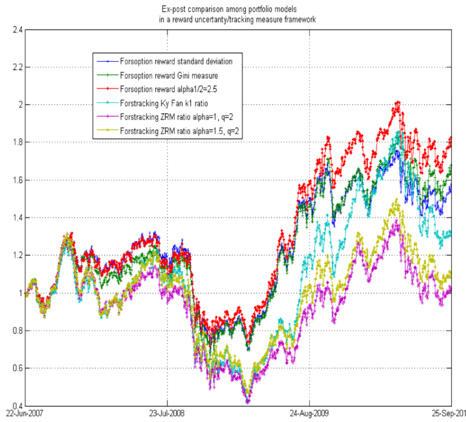


(b) Consistency with convex type orderings: portfolio wealth obtained investing in stock assets



(c) Consistency with tracking-error orderings

Fig. 3. Ex-post comparison of portfolios consistent with convex-type and tracking-error orderings.



(d) Portfolio selection based in a reward variability tracking error framework

Fig. 3. (*Continued*)

measure being used. Investors diversify (on average) between 38, 27, 56 options for the models when we maximize either the standard deviation, or the Gini measure $\Gamma_{X,\beta}(v)$, or the dispersion measure $F_X^{(\alpha_1,\alpha_2)}(b,t)$, respectively. The model suggests that taking positions in calls and puts with the highest probability of being exercised provides the highest expected profit. In a sense, the model considered here for this typology of problems seems to work quite well. All the strategies yield a very high wealth that is not comparable with the wealth obtained by investing in the various stocks. However, these outstanding performances with a portfolio of options are not so surprising considering the limitations of our experiment and especially since no liquidity constraints are imposed [45]. To compare these choices in the stock market, we implicitly assume that whoever invests in a portfolio of call options expects that the portfolio of underlying stock prices will grow. Thus, for all call options chosen in the previous model we compare the ex-post wealth obtained by investing in those underlying stocks. By doing so, we obtain an alternative model for the portfolio selection in the stock market. Figure 3(b) reports the ex-post results for the three variability measures. As can be seen, there is a period during the subprime mortgage crisis where no calls have been preselected and investors lose less. Moreover, we observe very good performances of the models based on both the Gini measure and the dispersion measure $F_X^{(\alpha_1,\alpha_2)}(b,t)$.

To compare the portfolio problems described in Sec. 3.2, we consider as benchmark the upper market stochastic bound, that is $Y = \max_{x\epsilon\Delta} x'r$ where $\Delta = \{x \in R^n \,|\, \sum_{i=1}^n x_i = 1; 0 \leq x_i \leq 0.05\}$. This benchmark dominates the market in the sense that $Y \geq x'r$ for any $x \in \Delta$. Thus, any investor wants to minimize a distance from the upper market stochastic bound whose observations are simply derived from return observations (i.e. the observation at time t of Y is given by $Y_t = \max_{x\epsilon\Delta} x'r_t$ where r_t is the vector of returns at time t). For the portfolio problems of Sec. 4.2, we propose to minimize either the Ky Fan compound metric k_1 (solving a problem similar to (3.9) with $t = 0$) or the Zolotarev–Rachev simple metric $\mathrm{ZRM}_{q,\alpha}(X,Y) = (\int_a^b (|F_X^{(\alpha)}(t) - F_Y^{(\alpha)}(t)|^q dt))^{\min(1/q,1)}$ with $\alpha = 1; 1.5$ and $q = 2$ (solving a problem similar to (3.10)). Figures 3(c) reports the ex-post comparison of these strategies. The strategy based on the use of the Zolotarev–Rachev simple metric with $\alpha = 1.5$ presents the best performance. Moreover, we observe that for all these strategies, the portfolio composition shows a substantial turnover and diversification similar to that observed for the Sharpe ratio maximization.

In order to consider a reward measure different from the mean, we propose maximizing the reward-"risk" functional where the reward measure is $E((X - t)_+)$ with $t = 0.005$ (as in Sec. 4.2), and the "risk" is represented by one of the variability (or tracking-error) measures used in the previous analysis. We propose optimization problems where we maximize either:

(a) a functional given by the sum of the reward measure and the variability measure (i.e. we maximize $E((X - t)_+)$ + variability measure) for problems of typology 2, or;
(b) the reward/(tracking-error) ratio for problems of typology 3.

For problems of typology 2, we compare the ex-post wealth obtained from investing in the portfolio of stocks with the same weights of the optimal portfolio of call options chosen in the optimization model. Moreover, the reward functional $E((X - t)_+)$ is applied to the portfolio of the call returns for the portfolio problem of typology 2 and to the portfolio of the stock returns for the portfolio problem of typology 3. The computational complexity of the resulting optimizations is generally higher than the corresponding mean variability problems and more time is needed in the calculation of each optimal portfolio. At the end of the observed period the strategy based on the maximization of the functional $E((x'R - t_1)_+) + F_{x'r}^{(\alpha_1,\alpha_2)}(b, t_2)$ presents the highest final wealth (about 80% more). The other strategies such as reward variability (tracking-error) exhibit a lower wealth than the corresponding strategies valued in the mean-variability (tracking-error) framework. The results we get studying the portfolio turnover and diversification are analogous to the previous ones; in that we observe a substantial portfolio turnover and diversification for only the portfolio problems of typology 3.

5. Concluding Remarks

There exists a strong connection between portfolio theory and ordering theory. In portfolio theory, investors express their preferences and choices among several random variables. Therefore, portfolio selection choices should be consistent with proper stochastic orderings that better represent an investor's preferences. Starting with this simple fact, we deduce that at least three fundamental aspects characterize the portfolio selection problem and its link with ordering theory: (1) the preference ordering of the admissible portfolio choices; (2) the description of portfolio selection problem consistent with the investor's preferences (generally using proper probability functionals); and (3) the practical application of the portfolio selection problem and its computational complexity.

We have demonstrated how to consider these three aspects for different typologies of portfolio selection problems. This is the original idea and the unifying objective of the paper. In particular, we show that portfolio selection problems are consistent with various types of orderings that could be either a risk ordering, a variability ordering, or a distance ordering. Thus, we have proposed several practical portfolio selection applications of ordering theory. By doing so, we have introduced new risk and behavioral risk orderings, new Gini variability orderings, and new distance orderings that can be used to better classify investor choices. Furthermore, we analyzed the proposed new coherent risk measures, variability measures, and tracking-error measures based on probability functionals that are consistent with some stochastic orderings, as well as examining the computational applicability of the portfolio problems that arise from optimizing a risk measure, a variability measure, or a probability distance. We discussed the large scale portfolio problem analyzing both the computational complexity of portfolio selection problems and the techniques needed to reduce the dimensionality of the problem. Furthermore, for each appropriate ordering, we proposed several practical portfolio optimization problems that could be solved even for large portfolios when the choices used are consistent with risk orderings.

This paper's empirical contribution involves an ex-post comparison between most of the proposed models applied to the US stock market during the subprime mortgage crisis and the credit crisis that followed. In particular, we presented empirical evidence that suggests that the ex-post sample path of wealth obtained by applying the proposed portfolio problems outperforms the classic portfolio selection benchmark (the ex-post final wealth we get maximizing the Sharpe ratio) and different US market benchmarks (NYSE, Nasdaq, S&P500, and DJIA). Moreover, for several optimization problems we observed that the optimal choices of nonsatiable investors who are neither risk-averse nor risk-loving present a higher ex-post final wealth than the wealth obtained by nonsatiable risk-averse investors. On the other hand, the optimal choices of the most risk-averse investors produced a higher ex-post wealth during the two financial crisis periods in our study. Finally, the empirical experiments with derivative assets show that the Black and Scholes model can be used to propose alternative portfolio selection models in both the options market and the stock market.

Several new perspectives and problems arise from this analysis. Since we can better specify portfolio optimization by taking into account the attitude of investors toward risk, we have to consider the ideal characteristics of the associated statistics and their asymptotic behavior. By using the theory of probability metrics, we can explain and argue why a given metric must be used for a particular optimization problem.

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