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Using *OLS* to test for normality

Haim Shalit *

Department of Economics, Ben Gurion University of the Negev, Beer-Sheva, Israel

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ABSTRACT

The *OLS* estimator is a weighted average of the slopes delineated by adjacent observations. These weights depend only on the independent variable. Equal weights are obtained if and only if the independent variable is normally distributed. This feature is used to develop a new test for normality which is compared to standard tests and provides better power for testing normality.

The purpose of this work is to present a different technique for testing for normality, which is a main issue in econometrics. Indeed since Madansky (1988) reviewed the various tests, more procedures have emerged: most notoriously the Jarque and Bera (1980) test and, recently, the *GMM* approach of Bontemps and Meddahi (2005). The major advantage of the proposed test for normality is that it is derived from a mathematical observation and therefore, entirely distribution-free and less sensitive to outliers.

In a major paper, Yitzhaki (1996) presented the following results regarding the ordinary least squares (*OLS*) estimator of a simple regression coefficient[1]:

1. The *OLS* estimator of the slope coefficient is a weighted average of the slopes delineated by adjacent observations.
2. The weights used in averaging the slopes depend solely on the distribution of the observations of the independent variable.
3. In particular, if the independent variable is normally distributed, the weights are equal to the normal density. Hence, equal proportions of the distribution receive equal weights for all the slopes of adjacent observations.

The main implication of Yitzhaki's results is that unless the independent variable is normally distributed, the *OLS* estimator is in fact a weighted regression estimator that attributes most of the weight to the more extreme observations. Usually, *OLS* is silent about the distribution of the independent variable, which is assumed to be non-random. Indeed, the issue of the distribution of the independent variable is rarely addressed in econometric texts. At most, normality of error terms is required as a prerequisite for statistical inference.

In a subsequent paper, Shalit and Yitzhaki (2002) showed that for observations characterized by fat tails such as financial data, so called "outliers" are receiving most of the explanatory power of the regression, thus yielding non-robust results. Removal of outliers is not a desirable practice as this eliminates valid information on the behavior of economic variables. Indeed, in light of the recent high volatility of security prices, extreme observations are the ones that contribute most in explaining price behavior. Hence independently of whether or not statistical inference is undertaken, a major prerequisite for the practitioner is testing whether or not the data are normally distributed in order to obtain robust results using *OLS*.

* Tel.: +972 8 647 2299; fax: +972 8 647 2941.
E-mail address: shalit@bgu.ac.il.

[1] The properties of the simple regression *OLS* estimator carry through to multiple regression. See also Heckman et al. (2006).

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In this work, I extend Yitzhaki's (1996) results by demonstrating that equal weights attributed to equal proportions of the distribution can be obtained if and only if the independent variable is normally distributed. The proof only requires solving a differential equation equating the regression weights to a density function, the only solution being the Gaussian probability distribution. The main implication of this result is that it serves as the basis for a new test for normality without having to specify the sampling distribution. In fact, since equal weights are necessary and sufficient for a distribution to be normal, the regression weights of *OLS* are the only data needed to check for normality.

The test involves only the independent variable of the *OLS* regression estimator since only this variate is required to calculate the regression weights. The main advantage of the test is that it does not require categorizing the mean and the variance of the normal distribution for the null hypothesis and, therefore, it is robust and insensitive to sample outliers. Hence, this new test is entirely distribution-free. In essence, the procedure consists of checking for the equality of the weights used in averaging the slopes delineated by adjacent observations in order to calculate the coefficient of the *OLS* regression. I devise the statistical test and compare its results with the performance of standard tests for normality such as Shapiro and Wilk (1965) and Shapiro and Francia (1972), and the most popular Jarque–Bera test (from 1980). I examine the new test by first exploring its size and its power and then applying it to using financial time-series data.

1. The *OLS* regression estimator

For convenience, I summarize Yitzhaki's results which claim that the *OLS* regression estimator of the slope coefficient is a weighted average of the slopes of the lines joining all pairs of adjacent observations.

Consider a simple regression model where variables are continuously random with a joint density function $f(X, Y)$, and where X is the independent variable and Y is the dependent variable. Let f_X, F_X, μ_X, and $\sigma^2 X$ denote, respectively, the marginal density, the marginal cumulative distribution, the expected value and the variance of X. Assume the existence of first and second moments.

Theorem 1 (*Yitzhaki, 1996*)**.** *Let $E(Y|X) = \alpha + \beta X$ be the best linear predictor of Y, given X. Then β_{OLS} is the weighted average of the slopes of the regression curve $g(x) = E(Y|X = x)$, namely*

$$\beta_{OLS} = \int_X w(x)\ \delta(x) dx \tag{1}$$

where $\delta(x) = g'(x)$ and $w(x) > 0$, $w(x)dx = 1$ and the weights are given as

$$\begin{aligned} w(x) &= (1/\sigma_X^2)\left[\mu_X F_X(x) - \int_{-\infty}^{x} t\ f_X\ (t)\ dt\right] \\ &= \int_{-\infty}^{x} (\mu_X - t)\ f_X\ (t)\ dt/\sigma_X^2 \\ &= F_X(x)\frac{[\mu_X - \mu(x)]}{\sigma_X^2}. \end{aligned} \tag{2}$$

Proof. See the Appendix □

Theorem 1 presents the *OLS* regression coefficient as a weighted average of the dependent variable differences, conditional on the independent variable differences. The weighting scheme depends solely upon the cumulative distribution of the independent variable.

Specifically for the normal distribution with expected value μ_X and variance $\sigma^2{}_X$, the weighting scheme from Eq. (2) becomes (Yitzhaki, 1996)

$$w(x) = -\frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} te^{-(t-\mu_X)^2 2\sigma_X^2}\ dt = \frac{1}{\sqrt{2\pi\ \sigma_X^2}} e^{-(x-\mu_X)^2/2\sigma_X^2}. \tag{3}$$

The weights equal the density function of the normal distribution of X. Hence, equal shares of the distribution receive equal weights and the explanatory power of the regression is distributed evenly among the observations.

2. The normal distribution of the independent variable

I now demonstrate that the only valid occurrence for which equal weights are ensured is when the independent variable of the *OLS* regression is normally distributed. For any other probability distribution of the explanatory variable, we obtain uneven weights, implying that some observations provide higher explanatory power and others less. Henceforth, robust *OLS* results can be obtained solely for a normally distributed independent variable.

Theorem 2. *The weights from Eq. (2) equal the probability density function if and only if the independent variable is normally distributed.*

Proof. [2] In Eq. (2) I substitute $w(x) = f_X(x)$ for all x and obtain a differential equation that is solved by integrating by parts to yield

$$\sigma_X^2 f(x) = (\mu_X - x) F_X(x) + \int_{-\infty}^{x} F_X(t)\, dt \tag{4}$$

where $dF(x) = f(x) \equiv F'(x)$. If one differentiates Eq. (4) with respect to x one obtains

$$\sigma_X^2 F_X''(x) - (\mu_X - x) F_X'(x) = 0. \tag{5}$$

The general solution to the differential equation (5) is given as

$$F_X(x) = C \int_{-\infty}^{x} e^{-(t^2 - 2\mu_X t)/2\sigma_X^2} dt, \tag{6}$$

where the constant C is found for $F(\infty) = 1$ as

$$C = \frac{1}{\sqrt{2\pi\sigma_X^2}}\; e^{-\mu_X^2/2\sigma_X^2}. \tag{7}$$

After substituting for C, the solution for Eq. (5) which is the normal probability distribution function becomes

$$F_X(x) = \frac{1}{\sqrt{2\,\pi\,\sigma_X^2}} \int_{-\infty}^{x} e^{-(t-\mu_X)^2/2\sigma_X^2} dt. \quad \square \tag{8}$$

3. Testing for normality

3.1. The test

From Theorem 2 we are able to derive a new test for normality for any set of observations of a random variable. The test is entirely distribution-free as it depends only on the weights expressed by Eq. (2). Indeed if all the weights are equal, the random variable is normally distributed.

The test is composed of two steps. The first step consists of calculating the weights, and the second checks whether the weights are equal. I consider the sample X_i, $i = 1, \dots, n$, and test whether it is drawn from a normal distribution.

In the first step, I rank the sample in increasing order, $x_{i+1} \geq x_i \geq x_{i-1}$, $i = 2, \dots, n-1$. An estimator for the weights $w(x)$ is obtained by substituting:

(i) the sample average $\bar{X}_n$ for the mean μ_X,
(ii) the sample mean up to observation i $\bar{X}_i = \sum_{j=1}^{i} X_j/i$ for the conditional mean $\mu(x)$,
(iii) the sum of squares S_X^2 for the variance σ_X^2, and
(iv) the relative rank $\frac{i}{n}$ for the cumulative distribution function $F_X(x)$.

The weight of the segment $\Delta x_i = x_{i+1} - x_i$ then becomes

$$w_i = \frac{i}{n}\left[\frac{\bar{X}_n - \bar{X}_i}{S_X^2}\right]\Delta x_i. \tag{9}$$

From the ranked sample, I compute the weights for $n-1$ segments to examine whether these weights are even. This is the second step of the test where I check whether the weights follow a uniform distribution. For this purpose, I use the standard Kolmogorov–Smirnov (KS) test on the cumulative probability distribution of the uniform distribution. Let the empirical distribution be the cumulative sum of the weights $F_w(i) = \sum_{j=1}^{i} w_j$ computed from Eq. (9) and let the theoretical distribution be $F_e(i) = \frac{i}{n-1}$. The KS statistic is obtained as

$$KS = \max_i |F_w(i) - F_e(i)|\,. \tag{10}$$

A possible alternative for the significance points for the KS statistic can be the standard critical values that are provided in Table 1. Another is the significance values obtained by sizing the sample as is done below.

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[2] In this proof, I consider probability distributions that are absolutely continuous with respect to the Lebesgue measure. Another proof that uses central moments was given by Preminger and Shalit (1999).

Table 1
Kolmogorov–Smirnov critical values.

Percentages	.10	.05	.01	.005
Critical values	$1.22/\sqrt{n}$	$1.36/\sqrt{n}$	$1.63/\sqrt{n}$	$1.731/\sqrt{n}$

To illustrate the entire testing procedure, I present in Table 2 the results of the test for 50 observations that were drawn from a gamma distribution.[3] The weights are calculated in the fifth column of the table. In the ninth column, the absolute difference between $F_w(i)$ and $F_e(i)$ is shown. The *KS* statistic for this sample is 0.221774 which exceeds the critical values for $\alpha = .1$ and $\alpha = .05$, thereby rejecting the null hypothesis that the sample is drawn from a normal distribution. The Jarque–Bera (*JB*) statistic for this sample is 5.58184, which also rejects normality for $\alpha = .1$ and $\alpha = .05$. Also using the Shapiro–Wilk (*SW*) statistic for this sample, we reject normality since $SW = 0.934$ which is less than the critical values 0.947 for 5% and 0.955 for 10%.

3.2. The size of the test

To calculate the significance points for the test, a Monte Carlo simulation is performed for a normally distributed set of observations drawn by using a random generator. For the simulation, I first use 100,000 and then 500,000 repetitions. For each replication, *N* observations are created using a normal random number generator with mean 5 and standard deviation 5. The estimates of the 1%, 5% and 10% significance points of the test are shown in Table 3a for 100,000 and in Table 3b for 500,000 replications. There are no other very considerable differences between the two tables.

3.3. The power of the test

To compare the test with other normality tests, I investigate its power under various distribution alternatives, again using Monte Carlo simulations. The results are reported in Tables 4a and 4b for 100,000 and 500,000 replications. The other normality tests include the standard Jarque–Bera (*JB*) test, the Shapiro–Wilk (*SW*) for 20 and 50 observations, the Shapiro–Francia (*SF*) for 100 observations and 200 observations, the Lagrange multiplier test of Deb and Sefton (*LM*(*DS*)) which is essentially a *JB* test with new calculated critical values, and the Lilliefors (1967) test which is a modified version of the Kolmogorov–Smirnov test. All the simulations and alternative tests were run in MATLAB and are available from the author. The distributions used to investigate the power are: Normal(10, 25), Gamma(3, 1), Student-t(5), Beta(3, 2), Exponential(10), Chi-squared(5), and Weibull(3, 1).

Looking at Tables 4a and 4b, we can see that the new test has in general the best power for testing normality against most of the alternative distributions, mainly for the smaller numbers of observations. The new test follows the power of the standard Shapiro–Wilk and Shapiro–Francia tests. In some simulations and for some distributions, the *OLS* test performs better; in others the Shapiro–Wilk has a better power. The *OLS* test however involves much easier computation as it does not require the use of specific tables and order statistics as required by Shapiro–Wilk and Shapiro–Francia. This result is quite promising when one compares the performance of the new test against the two *JB* benchmark tests. Furthermore, the result holds in particular when the alternatives are the Gamma, the Beta, the Exponential, the Chi-squared and the Weibull distributions. However, the two *JB* tests have a better power when the alternative is the Student-t distribution. But when the number of replications increases to 500,000 and the number of observations is greater than 100, the power of the new test is not as great as one would expect for a test for normality.

3.4. The Jarque–Bera test, its validity, and its modifications

Although most professional statisticians preconize the use of the Shapiro–Wilk test as well as its improvements by Shapiro and Francia (1972) and by Royston (1982), the Jarque–Bera test remains the most popular test for normality in the field of economics and finance. This is because it is easy to calculate and is a combination of two moments, namely, skewness and kurtosis. The Jarque and Bera (1980) statistic is also known as the D'Agostino and Pearson (1973) and the Bowman and Shelton (1975) test and, and under the JB test name, it has become a standard feature in many econometric packages. The standard *JB* statistic is given by

$$JB = \frac{N}{6}\left\{\left[\sqrt{b_1}\right]^2 + \frac{1}{4}\left[b_2 - 3\right]^2\right\} \tag{11}$$

where $\sqrt{b_1}$ is the sample skewness and b_2 is the sample kurtosis, which, under the null hypothesis of a normal distribution, are asymptotically independent and normally distributed. Hence, the *JB* statistic follows a χ^2 distribution with two degrees of freedom.

[3] The observations are taken from Madansky (1988, Table 4).

Table 2
Test procedure for 50 observations drawn from a gamma distributed sample.

Variate X	Rank i	$\Delta X = X_{i+1} - X_i$	$E_i(X) = \sum_{j=1}^{i} X_j/i$	w_i weight	$F_w(i) = \sum_{j=1}^{i} w_j$	$f_e(i) = 1/n-1$	$F_e(i) = \sum_{j=1}^{i} f_e(j)$	$\lvert F_w - F_e \rvert$
6.48875	1	1.21929	6.48875	0.00383	0.00383	0.020408	0.020408	0.016579
7.70804	2	0.75346	7.098395	0.004577	0.008407	0.020408	0.040816	0.032409
8.4615	3	4.29524	7.552763	0.038148	0.046555	0.020408	0.061224	0.01467
12.75674	4	0.32381	8.853758	0.003549	0.050103	0.020408	0.081633	0.031529
13.08055	5	0.35226	9.699116	0.004573	0.054677	0.020408	0.102041	0.047364
13.43281	6	0.506	10.3214	0.007563	0.062239	0.020408	0.122449	0.06021
13.93881	7	1.09636	10.83817	0.018445	0.080684	0.020408	0.142857	0.062173
15.03517	8	0.40584	11.3628	0.007514	0.088198	0.020408	0.163265	0.075067
15.44101	9	0.31254	11.81593	0.006294	0.094492	0.020408	0.183673	0.089181
15.75355	10	0.03146	12.20969	0.000683	0.095175	0.020408	0.204082	0.108907
15.78501	11	0.87417	12.53472	0.020344	0.115519	0.020408	0.22449	0.10897
16.65918	12	0.05509	12.87843	0.00136	0.116879	0.020408	0.244898	0.128019
16.71427	13	0.71471	13.17349	0.018651	0.13553	0.020408	0.265306	0.129776
17.42898	14	2.19712	13.47746	0.060161	0.195691	0.020408	0.285714	0.090023
19.6261	15	0.16341	13.88736	0.004624	0.200315	0.020408	0.306122	0.105807
19.78951	16	0.1705	14.25625	0.004975	0.20529	0.020408	0.326531	0.12124
19.96001	17	0.08532	14.59176	0.002563	0.207853	0.020408	0.346939	0.139086
20.04533	18	0.10388	14.89474	0.003208	0.211061	0.020408	0.367347	0.156286
20.14921	19	0.47839	15.17129	0.015167	0.226228	0.020408	0.387755	0.161527
20.6276	20	0.37236	15.44411	0.012082	0.23831	0.020408	0.408163	0.169854
20.99996	21	0.00935	15.70867	0.00031	0.238619	0.020408	0.428571	0.189952
21.00931	22	1.32929	15.94961	0.044938	0.283558	0.020408	0.44898	0.165422
22.3386	23	0.53382	16.22739	0.018288	0.301846	0.020408	0.469388	0.167542
22.87242	24	0.06833	16.50427	0.002366	0.304212	0.020408	0.489796	0.185584
22.94075	25	0.29846	16.76173	0.010438	0.31465	0.020408	0.510204	0.195554
23.23921	26	0.57502	17.01086	0.020283	0.334933	0.020408	0.530612	0.195679
23.81423	27	0.33701	17.26284	0.011956	0.34689	0.020408	0.55102	0.204131
24.15124	28	0.10757	17.50885	0.003832	0.350722	0.020408	0.571429	0.220707
24.25881	29	0.98166	17.74161	0.035095	0.385817	0.020408	0.591837	0.20602
25.24047	30	0.13033	17.99157	0.004654	0.390471	0.020408	0.612245	0.221774
25.3708	31	1.11016	18.22961	0.039579	0.43005	0.020408	0.632653	0.202603
26.48096	32	0.51117	18.48747	0.018096	0.448146	0.020408	0.653061	0.204915
26.99213	33	1.29893	18.74518	0.045549	0.493695	0.020408	0.673469	0.179775
28.29106	34	0.90931	19.02594	0.031381	0.525075	0.020408	0.693878	0.168802
29.20037	35	0.36193	19.31664	0.012233	0.537308	0.020408	0.714286	0.176977
29.5623	36	0.7441	19.60124	0.024576	0.561885	0.020408	0.734694	0.172809
30.3064	37	0.26785	19.89057	0.008606	0.570491	0.020408	0.755102	0.184611
30.57425	38	1.27841	20.17172	0.03987	0.610362	0.020408	0.77551	0.165149
31.85266	39	0.853	20.47123	0.025613	0.635975	0.020408	0.795918	0.159943
32.70566	40	0.60638	20.77709	0.017417	0.653392	0.020408	0.816327	0.162934
33.31204	41	0.03803	21.08282	0.001039	0.654431	0.020408	0.836735	0.182304
33.35007	42	0.08208	21.3749	0.002126	0.656557	0.020408	0.857143	0.200586
33.43215	43	0.61991	21.6553	0.015172	0.671729	0.020408	0.877551	0.205823
34.05206	44	6.4137	21.93705	0.147137	0.818865	0.020408	0.897959	0.079094
40.46576	45	5.33323	22.3488	0.108374	0.927239	0.020408	0.918367	0.008872
45.79899	46	0.73737	22.85858	0.012385	0.939624	0.020408	0.938776	0.000848
46.53636	47	0.55198	23.36236	0.007256	0.94688	0.020408	0.959184	0.012303
47.08834	48	5.54644	23.85666	0.052151	0.999031	0.020408	0.979592	0.019439
52.63478	49	0.20533	24.44396	0.000969	1	0.020408	1	3.55E − 15
52.84011	50							

Table 3a
Critical values for the test obtained using 100,000 repetitions.

Observations	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
20	0.2736	0.3093	0.3801
50	0.1816	0.2037	0.2485
100	0.1308	0.147	0.1784
200	0.0937	0.1051	0.1277
500	0.0599	0.067	0.0813

The *JB* test is not without its skeptics. The main concern is that its asymptotic distribution is problematic and its convergence is very slow, both of which result in an undersized test. Various alternative versions have been suggested by Deb and Sefton (1996), Poitras (2006), and Gel and Gastwirth (2008), but none have managed to supplant it. In the present work, I also evaluated the new *OLS* test against the standard Shapiro–Wilk and Shapiro–Francia normality tests. These tests, although having a better power than the popular *JB* test, are less prevalent in econometrics. From Tables 3a, 3b, 4a and 4b one can assert the marginal superiority of the new *OLS* test relative to the *JB* test. On the basis of our results, we can conclude

Table 3b
Critical values for the test obtained using 500,000 repetitions.

Observations	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
20	0.2741	0.309	0.3795
50	0.1815	0.2039	0.2486
100	0.1308	0.1468	0.1784
200	0.0938	0.1051	0.1278

Table 4a
Powers of normality tests for 100,000 replications.

Observations	Test	Distributions					
		$N(10, 25)$	$\Gamma(3, 1)$	$t(5)$	$\beta(3, 2)$	$\chi(5)$	$Wei(3, 1)$
$\alpha = 5\%$							
20	*OLS*	0.04896	0.35013	0.18026	0.05497	0.40249	0.74770
	JB	0.05006	0.30804	0.22972	0.02181	0.34809	0.62736
	SW	0.05031	0.38068	0.18883	0.07499	0.44415	0.83752
	LM(*DS*)	0.05067	0.31006	0.23068	0.02213	0.34998	0.62998
	Lilliefors	0.04911	0.22889	0.13122	0.06493	0.26854	0.57924
50	*OLS*	0.05006	0.76999	0.28939	0.11792	0.83546	0.99452
	JB	0.04946	0.67436	0.42886	0.01810	0.74412	0.97585
	SW	0.04191	0.81445	0.26763	0.25995	0.88613	0.99952
	LM(*DS*)	0.04900	0.67207	0.42766	0.01767	0.74184	0.97534
	Lilliefors	0.05027	0.51584	0.21225	0.11710	0.59527	0.96017
100	*OLS*	0.05853	0.988094	0.43345	0.31327	0.99319	1
	JB	0.04989	0.95921	0.64753	0.09557	0.98183	1
	SF	0.06194	0.98839	0.64886	0.39129	0.99101	1
	LM(*DS*)	0.04983	0.95908	0.64730	0.09507	0.98174	0.99999
	Lilliefors	0.04859	0.82618	0.32800	0.23222	0.89159	0.99997
200	*OLS*	0.05079	0.99998	0.59461	0.64134	1	1
	JB	0.05024	0.99999	0.86743	0.70886	1	1
	SF	0.0515	1.00000	0.85087	0.8507	1	1
	LM(*DS*)	0.04967	0.99999	0.86658	0.70265	1	1
	Lilliefors	0.04888	0.98887	0.54151	0.48726	0.99711	1
$\alpha = 10\%$							
20	*OLS*	0.10014	0.47469	0.25842	0.11461	0.53045	0.84240
	JB	0.09907	0.45055	0.31232	0.06465	0.50208	0.79862
	SW	0.09939	0.49587	0.25592	0.15206	0.56200	0.90411
	LM(*DS*)	0.10015	0.45314	0.31395	0.06573	0.50522	0.80141
	Lilliefors	0.09726	0.33946	0.20607	0.12684	0.38359	0.70503
50	*OLS*	0.09780	0.85606	0.37745	0.21485	0.90356	0.99827
	JB	0.09893	0.83840	0.51618	0.12996	0.89014	0.99746
	SW	0.09699	0.89574	0.34226	0.44048	0.94246	0.99989
	LM(*DS*)	0.09869	0.83783	0.51590	0.12928	0.88970	0.99745
	Lilliefors	0.10100	0.64880	0.30598	0.20774	0.72035	0.98398
100	*OLS*	0.09820	0.99014	0.51230	0.43335	0.99699	1
	JB	0.09896	0.99115	0.71789	0.46421	0.99750	1
	SF	0.11768	0.99534	0.72447	0.57410	0.99912	1
	LM(*DS*)	0.09907	0.99116	0.71801	0.46804	0.99750	1
	Lilliefors	0.09791	0.90215	0.44464	0.35916	0.94638	1
200	*OLS*	0.10162	1	0.70152	0.79499	1	1
	JB	0.10033	1	0.90336	0.93781	1	1
	SF	0.10516	1	0.89808	0.94627	1	1
	LM(*DS*)	0.09961	1	0.90305	0.93664	1	1
	Lilliefors	0.09964	0.99641	0.66196	0.64326	0.99929	1

that not only is the size of the proposed test comparable to that of the *JB* test, but also the new test has a better power that is similar to those of the Shapiro–Wilk and Shapiro–Francia tests. It was suggested by a colleague that the standard tests for normality calculate parameters that might conceal non-normality in the aggregation process. On the other hand, the new *OLS* test that is based on calculating the weights for each couple of adjacent observations may be the source of its superior statistical power.

Table 4b
Powers of normality tests for 500,000 replications.

Observations	Test	Distributions					
		$N(10, 25)$	$\Gamma(3, 1)$	$t(5)$	$\beta(3, 2)$	$\chi(5)$	$Wei(3, 1)$
$\alpha = 5\%$							
20	*OLS*	0.05017	0.34985	0.18036	0.05501	0.40333	0.75013
	JB	0.05009	0.30748	0.2304	0.02152	0.34988	0.62827
	SW	0.05196	0.38024	0.18846	0.07418	0.44413	0.83856
	LM(DS)	0.05061	0.30925	0.23053	0.02194	0.35182	0.63075
	Lilliefors	0.05012	0.22913	0.13123	0.06558	0.26688	0.58239
50	*OLS*	0.05023	0.76854	0.28956	0.11963	0.83576	0.99500
	JB	0.05005	0.67398	0.43001	0.01803	0.74466	0.97620
	SW	0.04307	0.81298	0.26760	0.26013	0.88576	0.99954
	LM(DS)	0.04954	0.67165	0.42882	0.01761	0.74238	0.97559
	Lilliefors	0.04991	0.51472	0.21185	0.11853	0.59520	0.96151
100	*OLS*	0.06019	0.98070	0.43538	0.31587	0.99301	1.0000
	JB	0.05000	0.95888	0.64569	0.09591	0.98193	1.0000
	SF	0.06283	0.98837	0.64759	0.39453	0.99694	1.0000
	LM(DS)	0.04994	0.95873	0.64550	0.09538	0.98186	1.0000
	Lilliefors	0.05050	0.82471	0.32945	0.23165	0.89335	0.9999
200	*OLS*	0.05064	0.99992	0.59396	0.64275	1.0000	1.0000
	JB	0.05039	0.99998	0.86622	0.70909	1.0000	1.0000
	SF	0.05132	0.99998	0.85119	0.85915	1.0000	1.0000
	LM(DS)	0.04972	0.99993	0.86550	0.70247	1.0000	1.0000
	Lilliefors	0.04995	0.98843	0.54176	0.48647	0.99711	1.0000
$\alpha = 10\%$							
20	*OLS*	0.10147	0.47331	0.25911	0.11429	0.53035	0.84377
	JB	0.09986	0.45054	0.31294	0.06337	0.50235	0.79922
	SW	0.10018	0.49598	0.25604	0.15101	0.56273	0.90447
	LM(DS)	0.10109	0.45333	0.31444	0.06470	0.50521	0.80199
	Lilliefors	0.10060	0.33972	0.20632	0.12692	0.38486	0.70668
50	*OLS*	0.09900	0.85447	0.37823	0.21597	0.90361	0.99840
	JB	0.09995	0.83832	0.51765	0.13103	0.89147	0.99762
	SW	0.09644	0.89538	0.34272	0.44141	0.94264	0.99989
	LM(DS)	0.09973	0.83787	0.51738	0.13024	0.89103	0.99760
	Lilliefors	0.10046	0.64692	0.30673	0.20893	0.71991	0.98429
100	*OLS*	0.10051	0.99020	0.51356	0.43508	0.99678	1.0000
	JB	0.10016	0.99099	0.71717	0.46963	0.99723	1.0000
	SF	0.11930	0.99534	0.72409	0.57737	0.99896	1.0000
	LM(DS)	0.10031	0.99100	0.71728	0.47045	0.99724	1.0000
	Lilliefors	0.10000	0.90159	0.44515	0.36178	0.94644	1.0000
200	*OLS*	0.10134	1.0000	0.70203	0.79575	1.0000	1.0000
	JB	0.10022	1.0000	0.90311	0.93682	1.0000	1.0000
	SF	0.10425	1.0000	0.89776	0.94531	1.0000	1.0000
	LM(DS)	0.09962	1.0000	0.90279	0.93578	1.0000	1.0000
	Lilliefors	0.10005	0.99613	0.66220	0.64203	0.99926	1.0000

4. Applying the test to financial time series

In the empirical part of the work I apply the test to financial time series. Normality of returns is a theoretical prerequisite for using the mean–variance model in portfolio analysis as well as for using the capital assets pricing model (*CAPM*). For this reason, I test the normality of returns on five US financial markets indices and five European financial markets indices using both monthly returns and daily returns.

The US indices are the Dow–Jones Industrial Average (DJIA), Standard and Poor's 500 (S&P500), Standard and Poor's 100 (S&P100), the Nasdaq composite index (NASDAQ), and the Russell 2000 (RUSS2000). The European financial indices are: the London Financial Times shares exchange (FTSE), the Frankfurt German stock market index (DAX), the Paris French stock market index (CAC40), the Zurich Swiss market index (SMI) and the Amsterdam Dutch stock exchange index (AEX). For the monthly data, I collected 120 returns from October 1999 until October 2009. For the daily data, I collected 250 daily returns from October 7, 2008, until October 2, 2009.

The results are displayed in Table 5. For most of the monthly series, the *JB* statistic follows the new *OLS* statistic. According to the *JB* test and the *OLS* test, normality is rejected for most of the monthly financial returns. However, the test results for the Russell 2000 index, the S&P100 index and the NASDAQ composite index are ambiguous. For instance, following the *OLS* weight test, normality is not rejected for the NASDAQ and the Russell 2000 whereas the *JB* test does not reject normality for the NASDAQ and the S&P100 indices.

Table 5
OLS normality test and *JB* normality test statistics for monthly and daily selected financial series.

INDICES	DJIA	SP500	SP100	RU2K	NASDAQ	FTSE	DAX	CAC40	SSMI	AEX
Monthly (120 obs)										
OLS	0.147634	0.181817	0.173951	0.118089	0.112189	0.189191	0.132139	0.149822	0.214265	0.229718
JB	7.90626	10.7069	3.93096	5.24512	1.90627	10.1059	15.0046	5.01005	7.50581	19.0402
Daily (250 obs)										
OLS	0.156797	0.132325	0.147046	0.070527	0.124226	0.152465	0.165372	0.16304	0.184832	0.174608
JB	142.577	89.1529	112.52	8.79569	55.8693	134.308	157.465	148.254	248.808	99.8212

5. Conclusion and implications

It was shown that if and only if the independent variable of an *OLS* regression is normally distributed, the regression weights attributed to the observations are equal to the density of the normal distribution. This implies that normality is a necessary and sufficient condition for each proportion of the population to receive an equal share of the weights used by the regression. Only then will all observations contribute evenly to the *OLS* estimation and yield robust estimators. This supports Shalit and Yitzhaki's (2002) claim that if observations are not normally distributed, other regression techniques should be used to ensure robustness.

This result on regression weights allowed us to derive a new test for normality for any sample of observations. The proposed test is an equality test using the *KS* procedure when the alternative distribution is the uniform distribution. Hence, besides the calculated regression weights, the only parameter that is required to estimate the expected distribution is the number of observations. The simulation results showed that, compared to standard tests for normality, the new test has the best power for testing normality against most of the classes of alternative continuous distributions.

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Appendix

Proof of Theorem 1 (*(Yitzhaki, 1996)*). Let $E(Y|X) = \alpha + \beta X$ be the best linear predictor of Y, given X. Then β_{OLS} is the weighted average of the slopes of the regression curve $g(x) = E(Y|X = x)$, namely

$$\beta_{OLS} = \int_X w(x)\ \delta(x)dx \tag{12}$$

where $\delta(x) = g'(x)$ and $w(x) > 0$, $\int w(x)dx = 1$ and the weights are given as

$$\begin{aligned} w(x) &= (1/\sigma_X^2)\ \left[\mu_X\ F_X(x) - \int_{-\infty}^{x} tf_X(t)dt\right] \\ &= \int_{-\infty}^{x} (\mu_X - t)f_X(t)dt/\sigma_X^2 \\ &= F_X(x)[\mu_X - \mu(x)]/\sigma_X^2, \end{aligned} \tag{13}$$

where $\mu(x)$ is defined as the conditional expectation $E(X|X \le x)$.

Proof. As $\beta = \mathrm{cov}(Y, X)/\sigma_X^2$, the numerator is expressed as

$$\mathrm{cov}(Y, X) = E_X E_Y[(X - \mu_X)Y] = E_X(X - \mu_X)E_Y(Y|X = x) = \int (x - \mu_X)g(x)f_x(x)dx,$$

where $g(x) = E_Y(Y|X = x)$ is the conditional expectation of Y given X. Integrating by parts with $V(x) = \int_{\infty}^{x}(t - \mu_X)f_X(t)dt$, $V'(t) = (t - \mu_X)f_X(t)$, $U(x) = g(x)$, and $U'(x) = g'(x)$ leads to

$$\mathrm{cov}(Y, X) = \left[\int_{-\infty}^{x}(t - \mu_X)f_X(t)dt\right] g(x)\,|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left[\int_{-\infty}^{x}(\mu_X - t)f_X(t)dt\right] g'(x)dx.$$

As second moments exist, the first term converges to zero. Hence,

$$\mathrm{cov}(Y, X) = \int_{-\infty}^{\infty} \left[\mu_X F_X(x) - \int_{-\infty}^{x} t f_X(t) dt \right] g'(x) dx.$$

The sum of weights equals 1 since the same procedure can be applied to the denominator σ_X^2 which equals $\sigma_X^2 = \int_{-\infty}^{\infty} [\mu_X F_X(x) - \int_{-\infty}^{x} t f_X(t) dt] \, dx$. □

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