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ORIGINAL RESEARCH

How does beta explain stochastic dominance efficiency?

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Abstract Stochastic dominance rules provide necessary and sufficient conditions for characterizing efficient portfolios that suit all expected utility maximizers. For the finance practitioner, though, these conditions are not easy to apply or interpret. Portfolio selection models like the mean–variance model offer intuitive investment rules that are easy to understand, as they are based on parameters of risk and return. We present stochastic dominance rules for portfolio choices that can be interpreted in terms of simple financial concepts of systematic risk and mean return. Stochastic dominance is expressed in terms of Lorenz curves, and systematic risk is expressed in terms of Gini. To accommodate for risk aversion differentials across investors, we expand the conditions using the extended Gini.

Keywords Systematic risk · Gini · Extended Gini · Marginal conditional stochastic dominance · Lorenz curves

JEL Classification G11

1 Introduction

The essence of portfolio optimization is to find a combination of a safe asset and risky assets that maximizes expected return while keeping risk at a bearable minimum. This is the rationale behind the mean-risk models and in particular the mean–variance (MV) model, which was originally derived as a special case of expected utility (EU) maximization. Although the conditions for which MV is analytically consistent with EU seldom hold in practice,[1] MV is widely accepted as the theory that makes sense from a practitioner

[1] For instance, multivariate normal probability distribution of returns or quadratic utility functions.

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point of view, because it captures two attributes: maximizing expected returns and minimizing risk.

One trade-off in its intuitive attractiveness is the dependence of mean–variance on a specific measure of risk. A more general approach that relies on expected utility theory without fully specifying a utility function is stochastic dominance that is expressed in terms of probability distributions rather than the usual parameters of risk and return used in MV. Second degree stochastic dominance (SSD) rules, which are defined in the next section, apply a general form of expected utility theory assuming risk-averse expected utility maximizers; the outcomes thus apply to a wider group of investors.

A problem now is that generalizing the theory complicates the rules to the point that they seem intractable to most practitioners (see, for example, Thistle 1993). Moreover, when the rules are applied to portfolios of assets, which is the most relevant case for an investor, they cannot be reasonably explained and one must rely on faith in them and in the algorithm producing the optimal portfolios.

The aim of our work is to express SSD rules in terms of the traditional concepts used in portfolio theory. In other words, we will interpret SSD rules in terms of expected return and systematic risk (beta), so that portfolio managers can better grasp the rules. We do this by using absolute Lorenz curves in place of the typical cumulative probability functions. This lets us present SSD conditions expressed in terms of return and risk, and reconcile them with the capital asset pricing model.

Besides adjusting SSD rules for problems of interest to portfolio managers, we extend SSD to marginal conditional stochastic dominance (MCSD) rules. These rules state the conditions under which all risk-averse investors holding a specific portfolio will prefer to increase the share of one asset over the share of another. MCSD is a less demanding concept than SSD because it considers only marginal changes of holding risky assets in a given portfolio.[2]

MCSD conditions are expressed in terms of absolute concentrations curves (ACCs) which are the assets' cumulative expected returns conditional on holding the portfolio. ACCs are analogous to absolute or generalized Lorenz curves, which are the cumulative conditional expected returns of the portfolio. We provide SSD rules by using the necessary conditions for MCSD expressed in terms of means and risk-adjusted returns, where risk is measured in terms of beta.

We also consider risk aversion differentiation as formulated by the extended Gini coefficient and explain SSD and MCSD for a wider range of specific risk-averse agents. Indeed, when beta is estimated using the mean-extended Gini approach, risk aversion is explicitly expressed.

The structure of the paper is as follow: Section 2 defines the basic concepts of expected utility, stochastic dominance efficient set, and mean-Gini portfolios. Section 3 defines the concept of MCSD and introduces the notion of ACCs. Section 4 presents the extended Gini and derives the necessary conditions for MCSD including specific risk aversion. In Section 5, we briefly discuss the notion of beta in equilibrium and we conclude in Section 6.

2 Expected utility, stochastic dominance, and mean-Gini rules

To achieve portfolio efficiency under expected utility maximization we must use utility functions and know the probability distribution of returns of all assets. To alleviate the

[2] Yitzhaki and Mayshar (2002) have proven that the assumption of continuity in the portfolio space implies that, if there is no portfolio that dominates a given portfolio under MCSD, then there will be no other portfolio (among all of portfolios, not just marginal ones) that dominates the given portfolio.

need for specific utility functions in constructing optimal portfolios, we propose using the rules of stochastic dominance, which are expressed in terms of cumulative probability distributions. If we confine the discussion to the class of all risk-averse expected utility maximizers, an appropriate mechanism would be second-degree stochastic dominance (SSD) theory that states the necessary and sufficient conditions under which a portfolio is preferred to another by all risk-averse expected utility maximizers.

SSD conditions have been developed independently by Hanoch and Levy (1969), Hadar and Russell (1969), and Rothschild and Stiglitz (1970). SSD rules are typically obtained by comparing the areas under the cumulative distributions of portfolio returns as follows: (see Levy 1992, 2006).

We define SSD rules as follows: Consider two risky portfolios A and B with cumulative probability F_A and G_B. For all risk-averse investors with non-decreasing concave utility functions U with $U' \geq 0$ and $U'' \leq 0$, SSD states that A dominates B if $E_F U(A) \geq E_G U(B)$ where E_F and E_G are the expectations using F_A and G_B. SSD rules state that A dominates B if and only if $\int_{-\infty}^{z} [G_B(x) - F_A(x)]\mathrm{d}x \geq 0$ for all z, which belong to the range of returns on A and B. The SSD efficient set is defined by the set of all the portfolios which are not dominated by other portfolios according to SSD rules.

These necessary and sufficient conditions calculate the areas under the respective cumulative probability distributions. The rules are to compare these areas so that, for all returns, the area under the cumulative distribution for the preferred portfolio is always smaller than the area under the cumulative distribution for the dominated portfolio.

SSD rules are not easy to interpret specially by capital market practitioners who are used to evaluate risk and return (see Best et al. 2006). This is further enhanced when constructing portfolios of risky and safe assets, because it is difficult to evaluate cumulative distributions of portfolios whose composition is changing constantly. Hence, linear programming and numerical optimization methods are commonly used to build efficient SSD portfolios, most of them relying on discrete distributions (see, for example, Chow et al. 1992; Ogryczak and Ruszczyński 2002; Post 2003; Ruszczyński and Vanderbei 2003; Dentcheva and Ruszczyński 2006). As these techniques are based on numerical optimization methods, it is virtually impossible to check and interpret intuitively the results in terms used by the practitioners.

Financial economists and practitioners are used to visualize the analysis as done in the classroom or the conference room. Hence, we suggest an easier way: presenting SSD conditions by means of absolute Lorenz curves, following formulations by Shorrocks (1983) and Yitzhaki and Olkin (1991).[3] These curves enable us to see the contribution of every asset to the expected return and the risk of the portfolio.

The Lorenz curve expresses the cumulative return on the portfolio as a function of the cumulative probability distribution. Given a portfolio with cumulative distribution *F(x)*, the absolute Lorenz curve (the *Lorenz*) is defined as:

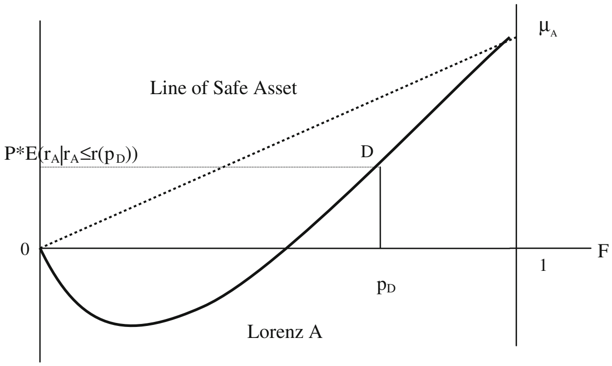
$$L(p) = \int_{-\infty}^{x_p} x f(x)\mathrm{d}x \quad \text{for } -\infty \leq x_p < \infty; \text{ where } x_p \text{ is defined by } p = \int_{-\infty}^{x_p} f(x)\mathrm{d}x \qquad (1)$$

where f is the density function of the portfolio.[4]

[3] Shorrocks (1983) calls these curves *generalized Lorenz curves*.

[4] We use densities for the random variables for the ease of presentation, although as pointed out by an anonymous referee, the results could be formulated for general distributions.

Fig. 1 SSD and absolute Lorenz curves



We can now use the *Lorenz* to compare portfolios. According to SSD rules, portfolio *A* dominates portfolio *B* if and only if:

$$L_A(p) \geq L_B(p) \quad \text{for all } 1 \geq p \geq 0 \tag{2}$$

The rationale for using absolute Lorenz curves to describe the properties of risky portfolios can be seen in Fig. 1. The *Lorenz* of a portfolio enables us to represent the expected return and the risk of the portfolio geometrically. As returns for a risky portfolio are ranked in increasing order, the shape of the *Lorenz* is convex, with the lowest returns being at the left of the given return which is also the slope of the *Lorenz*. The curve starts at (0, 0) and ends at (μ, 1), where μ is the expected return on the portfolio.

A safe asset with the same expected return μ will have a linear *Lorenz* that starts at (0, 0) and ends at (μ, 1). In Fig. 1, the *Lorenz* of this asset is drawn as the straight dotted line which we label the "line of safe asset" (LSA) as it represents the expected return multiplied by the probability *p*.[5] Now we can express the risk of a portfolio as the difference between the LSA that yields the same expected return and its *Lorenz*. Indeed, for every probability *p*, investing in the portfolio provides a cumulative expected return expressed by the *Lorenz* while investing in the riskless asset yields the same cumulative mean as given by the LSA.

The risk of the portfolio is a function of the vertical differences between the LSA and the *Lorenz*. Therefore, the farther the LSA is from the *Lorenz*, the greater the risk assumed by the portfolio. One possible measure of risk is the Gini's mean difference (GMD) of the portfolio which is obtained from the distances between the LSA and the *Lorenz*. Equation (3) shows that the area between the LSA and the *Lorenz* is actually one-fourth of GMD (Yitzhaki 2003, Eq. 4.4, p. 297).

$$\int_0^1 [\mu p - L(p)]\mathrm{d}p = \mathrm{cov}[r, F(r)] = \frac{1}{2}\Gamma \tag{3}$$

where $\mu\, p$ is the LSA, $L(p)$ is the Lorenz, and $\Gamma = 2\mathrm{cov}[r, F(r)]$ is half the Gini's mean difference of the portfolio. Other measures of risk, like the extended Gini and even the

[5] In the income inequality literature, this is called the *line of perfect equality*.

variance, can be obtained as functionals of the vertical difference between the LSA and the *Lorenz*.[6]

We can gain other insight from Fig. 1. The horizontal axis is defined as the probabilities ranked from those generating the lowest portfolio returns and yielding the highest marginal utility to those generating the highest returns with the lowest marginal utility. Thus, the (equal) probabilities on the horizontal axis are ranked according to declining marginal utility. Since utility is defined over wealth, ranking probabilities with respect to portfolio returns yields the same result as if the ranking were according to declining marginal utility for each investor. All investors concur with this ranking because it is based only on portfolio returns that are assumed to be their only wealth.

While investors, who hold the same portfolio, may not exhibit the same marginal utility from portfolio returns, they all agree upon the *ranking* of the marginal utility of these returns. Hence, ranking with respect to portfolio returns is the only information we need in order to rank portfolios with respect to marginal utility. The vertical axis in Fig. 1 shows the cumulative portfolio returns up to a specific state of nature, where states of nature are ordered according to the return associated with their occurrence. The vertical difference between the LSA and the *Lorenz* of the portfolio represents the returns that, multiplied by the marginal utility, make up the expected utility. In other words, the loss in expected utility due to riskiness is the sum (integral) of marginal utility multiplied by the distance between the LSA and the *Lorenz*. Different investors have different marginal utilities, so the loss due to riskiness differs among investors.

The connection between SSD and the non-intersection of Lorenz curves can be explained as follows. If one chooses to use a linear utility function, a necessary condition for the portfolio to be preferred by all expected utility maximizers is that it is preferred by the risk-neutral investor, whose marginal utility is a constant. In this case, one needs to look only at the last point on the *Lorenz*, which equals the portfolio expected return.

Another necessary condition is that the area below the *Lorenz* of the dominating portfolio be greater than the area below the *Lorenz* of the dominated portfolio. This area is one-half the expected returns minus one-fourth of the GMD ($\frac{1}{2}\Gamma = \mathrm{cov}[r, F(r)]$). This is the logic behind the mean-Gini (MG) necessary conditions for SSD (Yitzhaki 1982), which are expressed as:

$$\begin{aligned} \mu_A &\geq \mu_B \\ \mu_A - \Gamma_A &\geq \mu_B - \Gamma_B \end{aligned} \tag{4}$$

These conditions state that if portfolio A is SSD preferred to portfolio B, then the mean and the risk-adjusted mean return of A cannot be less than the mean and the risk-adjusted mean return of B when risk is measured by the Gini of the portfolio.[7]

3 Absolute concentration curves and marginal conditional stochastic dominance

Having described the necessary conditions for second-degree stochastic dominance in terms of risk-adjusted mean returns, treating each portfolio with a given composition of

[6] The variance is obtained from the area enclosed between the two curves if one uses returns instead of probabilities on the horizontal axis (Yitzhaki 1998). In this case the LSA ceases to be a line, which complicates the plotting.

[7] Yitzhaki (1982) also show that the mean-Gini conditions for SSD are sufficient whenever cumulative probability distributions intersect at most once.

assets, the next step is to measure the relative dominance of assets in and out of the portfolio. At the core of portfolio theory is that diversification of asset holdings reduces an investor's exposure to risk. SSD in a portfolio must be applied in an environment where investors can change the choice of assets. For this purpose, we rely on absolute concentration curves (ACCs). Since SSD rules are much more complex in a portfolio context than in application to individual assets, one must recognize its limitations as we note in Shalit and Yitzhaki (1994), and formulate a more simple question.

Rather than define rules for dominance, one might ask whether a given portfolio A belongs to the SSD efficient set. This inquiry proceeds in several steps:

(a) First, is it possible to find an alternative portfolio B in the neighborhood of A that differs from A by changing the shares of only two assets and then SSD dominates portfolio A?
(b) If it is impossible to find such a portfolio, is it possible to find an alternative portfolio B in the neighborhood of A that differs from A by more than two assets and SSD dominates A?
(c) Finally, provided that we have failed to find portfolios that dominate A according to (a) and (b), is it possible to find an alternative portfolio B that SSD dominates A?

A portfolio that is not dominated by another portfolio according to these conditions belongs to the SSD efficient set. We address each question separately.

The first problem is answered using the concept of marginal conditional stochastic dominance (MCSD) as defined by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994). MCSD states the conditions under which all risk-averse investors, holding a given portfolio A, prefer to increase the share of one asset over another. MCSD is more confining than SSD because it considers only marginal changes in holding risky assets in a given portfolio, and restricts the change to involve two assets only.[8]

To make MCSD operational, we develop the concept of ACC as follow:

Consider a portfolio of n risky assets $\{\alpha|\sum_{i=1}^{n}\alpha_i=1\}$ whose returns r_α are defined by $r_\alpha=\sum_{i=1}^{n}\alpha_i r_i$, where r_i are the returns on asset i, and $f_{\boldsymbol{\alpha}}$ is the density function of the portfolio. Let $\mu_i(t)=E(r_i|r_\alpha=t)$ be the conditional expected return on asset i, given the portfolio return t. The ACC of asset i with respect to portfolio $\{\boldsymbol{\alpha}\}$ is defined as the cumulative conditional expected returns on asset i as a function of the portfolio cumulative distribution $p=F_{\boldsymbol{\alpha}}(r_\alpha)$:

$$ACC_i(p)=\int_{-\infty}^{r_\alpha}\mu_i(t)f_\alpha(t)\mathrm{d}t \quad \text{for } \infty\geq r_\alpha\geq-\infty \tag{5}$$

where r_α is the p-quantile of the return distribution f_α defined as: $p=\int_{-\infty}^{r_\alpha}f_\alpha(t)\mathrm{d}t$.

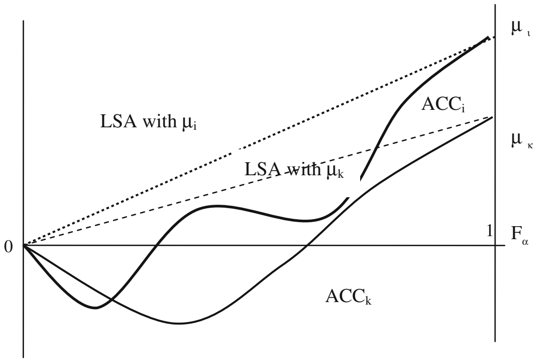
Similarly, from Equation (1), the *Lorenz* of portfolio $\{\boldsymbol{\alpha}\}$ is:

$$L_\alpha(p)=\int_{-\infty}^{r_\alpha}tf_\alpha(t)\mathrm{d}t \quad \text{for } \infty\geq r_\alpha\geq-\infty \tag{6}$$

Following the definition of the portfolio, its *Lorenz* can then be written as the weighted sum of the asset ACCs held in the portfolio, which is expressed as:

[8] The restriction to a marginal change can be interpreted as a search for the direction to move in. Then, one has to evaluate the size of the step in that direction.

Fig. 2 Absolute concentration curves



$$L_\alpha(p) = \sum_{i=1}^{n} \alpha_i ACC_i(p) \quad \text{for } \infty \geq r_\alpha \geq -\infty \tag{7}$$

Figure 2 depicts the ACC of asset i. The horizontal axis represents the cumulative distribution of the portfolio's return and the vertical axis measures the cumulative expected returns. The ACC of asset i, which is an asset that does not need to be included in the portfolio, relates the cumulative expected return on that asset to the cumulative probability distribution of the portfolio. The ACC of asset i is the solid curve. The dashed straight line is the line of safe asset (LSA) that connects the origin (0, 0) with the point (1, μ_i), where μ_i is the unconditional expected return of asset i. The LSA represents an asset whose returns are independent of the performance of the portfolio and that has the same unconditional expected return as asset i.[9]

We now state the main theorem to determine MCSD using ACCs:

MCSD THEOREM: (Shalit and Yitzhaki 1994): Given portfolio $\{\boldsymbol{\alpha}\}$, asset k dominates asset j for all concave utility functions if and only if:

$$ACC_k^\alpha(p) \geq ACC_j^\alpha(p) \quad \text{for all } 1 \geq p \geq 0 \tag{8}$$

with at least one strong inequality.

Intuitive Proof: Equation (7) provides a very simple proof for the theorem. Given the shares of each asset in the portfolio, the ACC is the derivative of the *Lorenz* of the portfolio. To increase the share of one asset on behalf of another in order for the new portfolio to SSD-dominate the given portfolio, the derivative of the *Lorenz* of the portfolio with respect to the dominating asset has to be greater everywhere than the derivative of the dominated asset.

To derive the necessary conditions for MSCD and relate them to the fundamental ideas in finance, we describe the ACCs basic properties:

(1) The ACC of asset i passes through the points (0, 0) and (1, μ_i).

(2) The derivative of the ACC of asset i with respect to p is $\mu_i(t) = E_i(r_i | r_\alpha = t)$. Consequently, the ACC increases if and only if $\mu_i(t) > 0$.

(3) The ACC is convex, straight, or concave if and only if $\partial \mu_i(t)/\partial t \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$.

(4) When returns r_α and r_i are independent, the ACC_i [p_α] coincides with the LSA.

[9] LSA coincides with the Yitzhaki and Olkin (1991) line of independence (LOI). Samuelson (1967) shows that independent assets that are not included in the portfolio would be added to it if they have the same expected returns.

(5) The area between the LSA and the ACC is equal to $cov[r_i, F_\alpha(r_\alpha)]$, the covariance of the return on asset i and the cumulative probability distribution of portfolio $\{\alpha\}$. That is:

$$\int_0^1 ACC_i(p)\mathrm{d}F_\alpha = \frac{1}{2}\{\mu_i - 2\mathrm{cov}[r_i, F_\alpha(r_\alpha)]\} = \frac{1}{2}(\mu_i - \beta_i^\Gamma \Gamma_\alpha)$$

where $\beta_i^\Gamma = \frac{2\mathrm{cov}[r_i, F_\alpha(r_\alpha)]}{\Gamma_\alpha}$ is the Gini regression coefficient of asset i on the portfolio $\{\alpha\}$ and $\Gamma_\alpha = 2\mathrm{cov}[r_\alpha, F_\alpha(r_\alpha)]$ is one-half of the GMD of the portfolio.[10]

These properties allow us to state the necessary conditions for MCSD, namely, that if asset j dominates asset k conditional of holding portfolio $\{\alpha\}$:

$$\begin{aligned} \mu_j &\geq \mu_k \\ \mu_j - \beta_j^\Gamma \Gamma_\alpha &\geq \mu_k - \beta_k^\Gamma \Gamma_\alpha \end{aligned} \tag{9}$$

The first condition implies that a dominating asset has a higher expected return than the dominated asset, regardless of the risk involved. The second necessary condition is more meaningful, as it states that a preferred asset has a higher risk-adjusted expected return than the risk-adjusted expected return of the less favored asset. Indeed as β_j^Γ expresses systematic risk in the mean-Gini model (MG-CAPM),[11] then $(\mu_j - \beta_j^\Gamma \Gamma_p)$ is the risk-adjusted expected return, which is defined as the mean less the beta calculated in Gini terms.

From the second necessary condition one can evaluate:

$$\frac{(\mu_k - \mu_j)}{\Gamma_\alpha} \geq \beta_k^\Gamma - \beta_j^\Gamma \tag{10}$$

i.e., when a security dominates another by MCSD, the difference between the two securities' expected returns per unit of portfolio risk must be greater than the difference in their systematic risks defined in terms of MG–CAPM.

Using the mean and the risk-adjusted mean return, this result allows for a complete ordering of investment alternatives. MCSD criteria using ACCs establish only a partial ordering. A complete ordering is an advantage when no dominance can be assessed by using ACCs, but a decision maker nevertheless wants to rank investment alternatives. In that case, the mean-Gini necessary conditions for MCSD provide an investment ranking that does not necessarily satisfy the sufficient conditions.

Given definition of the SSD criteria in a portfolio context by changing only two assets, we can extend it to several assets in a relatively simple manner.

(b) Is it possible to find an alternative portfolio B, in the neighborhood of A, that SSD dominates A and differs from it in more than two assets?

According to Equation (7), a combination of ACCs of several assets defines a new ACC that is a linear combination of individual ACCs. Hence, to address MCSD involving more than two assets we need to search for a linear combination of assets whose ACC is not below a linear combination of other assets. This can be solved numerically as in Shalit and Yitzhaki (2003), and then the optimal ACC can be delineated.

[10] See Olkin and Yitzhaki (1992) and Shalit and Yitzhaki (2002) for the definition of the Gini regression coefficient and Carroll et al. (1992) for its use.

[11] Whenever the CAPM is mentioned, we interpret it as the reference portfolio held by the investor, and not necessarily the market portfolio.

(c) Is it possible to find an alternative portfolio B that SSD dominates A?

Yitzhaki and Mayshar (2002) have shown that if a portfolio is not MCSD-dominated by another portfolio it is also not SSD-dominated by any other portfolio. To understand the intuition to prove this, let us consider two portfolios, *A* and *B*, where *B* SSD-dominates *A*. In that case, for all risk-averse utility functions:

$$E[U(B)] \geq E[U(A)] \tag{11}$$

Hence, to prove the argument it must be shown that if Equation (11) holds, there is also a portfolio in the neighborhood of *A* that SSD dominates *A*. First note that:

$$\lambda E[U(B)] + (1-\lambda)E[U(A)] \geq E[U(A)] \quad \text{for } 1 \geq \lambda \geq 0 \tag{12}$$

Because U is concave, we know that:

$$E\{U[(1-\lambda)A + \lambda B]\} \geq \lambda E[U(B)] + (1-\lambda)E[U(A)] \tag{13}$$

Combining (12) and (13) we get:

$$E\{U[(1-\lambda)A + \lambda B]\} \geq E[U(A)] \quad \text{for } 1 \geq \lambda \geq 0 \tag{14}$$

We now apply (14) for $\lambda \to 0$ and $\lambda > 0$, by which we find a portfolio in the neighborhood of *A* that SSD-dominates *A*. Therefore, it is impossible to have an SSD portfolio that dominates *A*, without having also a portfolio, in the neighborhood of *A*, that SSD dominates *A*. Thus, we may conclude that if *A* is not MCSD-dominated it is also true that neither is *A* SSD-dominated.

4 Risk aversion, extended Gini, and MCSD

With an additional parameter, the extended Gini enables us to analyze risk aversion differentiation when we calculate the measure of dispersion. Indeed, with the parameter v, which represents risk aversion, the extended Gini coefficient characterizes risk-averse investors ranging from risk-neutral ($v = 1$) to highly risk-averse maxi-min individuals ($v = \infty$). Other necessary conditions for MCSD that are specific to risk-averse agents can then be derived using the mean and systematic risk. The MCSD-dominating asset has to have a higher risk-adjusted expected return than the dominated asset, for every risk-averse investor. We adjust the expected return using the mean-extended Gini CAPM. For each asset and risk aversion coefficient, the extended Gini beta is calculated and used to adjust the expected return for risk.

The extended Gini specifies increasing risk aversion by stressing the lower segments of the distribution of portfolio returns. Similarly to the standard Gini, which is defined as the vertical difference between the LSA and the *Lorenz* of the portfolio, the extended-Gini is the *weighted* vertical difference between the LSA and the *Lorenz*. Using the parameter v to adjust the area definition, we define the extended Gini for asset X as:

$$\Gamma_X(v) = v(v-1)\int_0^1 (1-p)^{v-2}(p\mu_X - L_X(p))\mathrm{d}p \tag{15}$$

where $L_X(p) = \int_{-\infty}^{X_p} x f_X(x)\mathrm{d}x$ is the *Lorenz*, X_p is indirectly determined by $p = \int_{-\infty}^{X_P} f_X(x)\mathrm{d}x$, $v(v-1)(1-p)^{v-2}$ is the weight associated with each portion of the area, and $p\mu_X$ is the LSA. The parameter v (>0) is being established by researchers.[12]

[12] See Aaberge (2000), and Kleiber and Kotz (2002) on additional connections between the Lorenz curve and extended Gini.

There are some special cases of interest for the extended Gini:

For $v = 2$ Equation (15) becomes one-half of Gini's mean difference.

For $v \to \infty$ the extended Gini reflects the attitude of a max–min decision maker who wants to express risk in terms of only the worst outcome.

For $v \to 1$ Equation (15) becomes the expected return, allowing a risk-neutral investor who does not use any measure of dispersion to evaluate risk.

For $0 < v < 1$ the extended Gini is negative and models a risk-loving investor. For ease of presentation and because we are dealing with risk-averse investors, we assume that $v > 1$, although many of the results we report can be applied without modification to risk-loving investors. In financial analysis, the covariance formula for the extended Gini is more convenient:

$$\Gamma_X(v) = -v\text{cov}\{x, [1 - F(x)]^{v-1}\} \tag{16}$$

Equation (16) is obtained by integrating Equation (15) by parts with the following elements:

$$U = (1-p)^{v-1}, \mathrm{d}U = -(v-1)(1-p)^{v-2}, V = p\mu_x - L_X(p), \mathrm{d}V = \mu_x - x(p)$$

where *x(p)* is the inverse of the cumulative distribution. This leads to:

$$v(v-1)\int_0^1 (1-p)^{v-2}[p\mu_x - L_X(p)]\mathrm{d}p = -v(1-p)^{v-1}|_0^1 \cdot [p\mu_x - L_X(p)]|_0^1 + v\int_0^1 [\mu_x - x(p)](1-p)^{v-1}\mathrm{d}p$$

The first term on the right-hand side is equal to zero and the second term becomes:

$$+v\int_0^1 [\mu_x - x(p)](1-p)^{v-1}\mathrm{d}p = -v\int_a^b (x-\mu_x)[1-F(x)]^{v-1}f(x)\mathrm{d}x = -v\text{cov}\{x, [1-F(x)]^{v-1}\}$$

.

In the first step above, we changed the integrant from p to x using $p = F(x)$, and in the second step we used the definition of the covariance. Additional insight into (15) can be gained by showing that the first term is simply the area under the diagonal. By twice integrating by parts, the weighted area under the diagonal is

$$v(v-1)\int_0^1 (1-p)^{v-2}p\mu_x\mathrm{d}p = v\mu\int_0^1 (1-p)^{v-1}\mathrm{d}p = -\mu(1-p)^v|_0^1 = \mu \tag{17}$$

Hence, the weighted area under the Lorenz curve is equal to:

$$\mu - v\text{cov}\{r, [1-F(r)]^{v-1}\} \tag{18}$$

We refer to Equation (18) as RAR(v)—the risk-adjusted expected return of an asset using the extended Gini $\Gamma(v)$.[13]

One can introduce risk aversion differentiation into the SSD and MCSD necessary conditions and make them specific to various investors. A necessary condition for SSD is that the RAR(v) of the dominating portfolio will be not lower than the RAR(v) of the dominated portfolio. Hence, the conditions for the portfolios shown in Equation (4) become:

$$\begin{aligned} \mu_A &\geq \mu_B \\ \mu_A - \Gamma_A(v) &\geq \mu_B - \Gamma_B(v) \end{aligned} \tag{19}$$

The necessary conditions for MCSD developed in Equation (9) can be replicated with the extended Gini to become:

If asset j MCSD dominates asset k conditional on holding portfolio $\{\alpha\}$:

$$\begin{aligned} \mu_j &\geq \mu_k \quad \text{and} \\ \mu_j - \beta_j^{\Gamma}(v)\Gamma_\alpha(v) &\geq \mu_k - \beta_k^{\Gamma}(v)\Gamma_\alpha(v) \end{aligned} \tag{20}$$

Except that this time, $\beta_j^{\Gamma}(v)$ is defined in terms of the extended Gini as follows:

$$\beta_j^{\Gamma}(v) = \frac{\operatorname{cov}\{r_j, [1 - F_\alpha(r_\alpha)]^{v-1}\}}{\operatorname{cov}\{r_\alpha, [1 - F_\alpha(r_\alpha)]^{v-1}\}} \tag{21}$$

and $\Gamma(v)$ is the extended Gini as shown by Equation (16).

Interpretation of Equation (20) remains the same as for Equation (9), except that the necessary conditions depend on the investor's specific coefficient of risk aversion. This is the main point of our work: If asset j dominates asset k according to MCSD, then it must be that the risk-adjusted expected return of j is higher than the risk-adjusted expected return of k, where risk is measured by extended Gini betas for all possible risk aversion coefficients v. In other words, if asset j MCSD dominates asset k for a given portfolio α, there is no extended Gini beta for k for all possible v that will increase the RAR(v) of k more than the RAR(v) of j. These conditions, however, are merely necessary and not sufficient, because the family of extended-Gini utility functions does not cover all possible risk-averse utility functions. For example, they do not include a change in the coefficient of risk aversion v on a given point along the distribution of returns.

If we can use the extended Gini to express the necessary conditions for SSD and MCSD, how then do we choose the risk aversion parameter v? Hence, the question to be asked is really how one can choose a utility function that represents a specific investor. By gathering information on investor decision making under risk, presumably one can estimate the parameter v specifically for a particular investor, but this is a question for further research.

5 Beta and capital market equilibrium

Stochastic dominance was developed in order to construct portfolios for specific investors' classes. As such, it ignores the notion of capital market equilibrium. On the other hand, the concept of beta emerged as the equilibrium price of non-diversifiable risk carried by an

[13] It can be shown that $\mu_X - \Gamma_X(v)$ is a special case of Yaari's (1987) dual utility function. Yaari's decision function can be written as $\int \Psi[1 - F(x)]\mathrm{d}x$. Substituting the general function $\Psi(.)$ by $[1 - F(x)]^v$ yields $\mu_X - \Gamma_X(v)$. For a proof, see Yitzhaki (1983).

asset in a competitive financial market situation. We would like to comment briefly on the gap between these two approaches.

Consider first a financial market of risky assets where all returns follow a multivariate normal distribution. The agents in this market may have a different level of risk aversion. In this case, investors will hold an identical portfolio of risky assets, i.e., the "market portfolio", and the only difference between them will come about in the allocation of their wealth between the risky portfolio and the risk-free asset. In this theoretical textbook case, the setting for deriving SD and beta will be identical.

Now, consider now a more complex financial market, where risky assets returns follow a more general distribution with finite higher moments and where agents are restricted to have mean–variance or mean-extended Gini preferences, as analyzed in Shalit and Yitzhaki (2009). In equilibrium, each type of risk-averse investor holds the same risky portfolio as the marginal rates of substitution between risk and return and between the risks carried by any two assets will be identical for all investors. This case is similar to the classical MV-CAPM except for two major differences. For one thing, investors need not necessarily hold the "market portfolio" and the other is that the market price for risk will depend on the income distribution among investors.

Our conclusion is that in this general model, beta continues to be relevant, although its interpretation is slightly different. In equilibrium, the ratios of betas for different risk-averse investors, who also hold different portfolios, are equal. Although investors do not see eye to eye on the definition of risk, at equilibrium the ratios of risks embodied by the assets, with respect to the portfolio they are holding, is identical. In a sense, the equilibrium in a financial market does not differ from the equilibrium in a commodities market. Although consumers may have different preferences for commodities, at equilibrium, the marginal rates of substitutions between every two consumers are identical. And it should be borne in mind that the consumption baskets of consumers may be different and the market price for the commodities is a function of the agents' income distribution.

6 Conclusion

We have shown how to use stochastic dominance rules in constructing portfolios. For an economist and a practitioner used to think in terms of risk and return, a major weakness of the models based on numerical optimization is their inability to express the results intuitively. Our remedy is to characterize the rules geometrically by using absolute Lorenz curves for second-degree stochastic dominance and absolute concentration curves for marginal conditional stochastic dominance. We can then interpret the rules in terms of risk-adjusted mean returns depending on different measures of risk aversion.

How does systematic risk explain stochastic dominance efficiency? Beta, which is used by practitioners in finance, measures systematic risk as the covariance between asset return and market return.[14] The concept is rooted in mean–variance theory as it prices security risk in capital market equilibrium. The measure is mainly dependent on the validity of MV and its compatibility to maximizing expected utility when returns are multivariate normally distributed or when the investor's utility function is quadratic. The presence of fat tails and skewness in financial data precludes normality of returns, and quadraticity of preferences leads to unwarranted results.

[14] In general the term market's return should be interpreted as the portfolio's return. See Shalit and Yitzhaki (2009) concerning CAPM with heterogeneous risk-averse investors.

Alternative measures of systematic risk have since emerged. Shalit and Yitzhaki (2002) have shown that the correct approach should be to look at the covariance between asset return and marginal utility to express undiversifiable risk correctly. Hence systematic risk depends upon the choice of the risk measure chosen by investors. In the case of Gini's mean difference and the extended Gini, the resulting betas are the mean-extended Gini betas used in the necessary conditions for stochastic dominance. Gregory-Allen and Shalit (1999) have shown that MEG betas, which depend upon the investor's degree of risk aversion, subside to the standard MV beta only when returns are normally distributed. As it is seldom the case that normality holds, we advocate MEG betas to be used for stochastic dominance.

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