GENERALIZED STRESS
CONCENTRATION FACTORS

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Stress Concentration for Engineers

![Stress Concentration Diagram](image)

**FIGURE A9** Flat Plate with Holes. The tensile load is central. For \( h/d = 1.1 \), tensile and bending curves are very close together down to \( r/d = 0.04 \). (After R. E. Peterson)\(^{[4, 21]}\)
Generalized Stress Stress Concentration Factors:

- Assume a body $\Omega$ is given (open, regular with smooth boundary).
- Assume a force $F$ is given in terms of a body force $b$ and a surface force $t$ and let $\sigma$ be a stress field that is in equilibrium with $F$.
- The stress concentration factor associated with the pair $F, \sigma$ is

$$K_{F,\sigma} = \frac{\text{ess sup}_x \{ |\sigma(x)| \}}{\text{ess sup}_{x,y} \{ |b(x)|, |t(y)| \}}, \quad x \in \Omega, \ y \in \partial \Omega.$$

- Denote by $\Sigma_F$ the collection of all possible stress fields that are in equilibrium with $F$. (There are many such stress fields because material properties are not specified.)
• The **optimal stress concentration factor** for the force $F$ is defined by

$$K_F = \inf_{\sigma \in \Sigma_F} \left\{ K_{F,\sigma} \right\}.$$ 

• The **generalized stress concentration factor** $K$—a purely geometric property of $\Omega$—is defined by

$$K = \sup_F \left\{ K_F \right\} = \sup_F \inf_{\sigma \in \Sigma_F} \left\{ \frac{\text{ess sup}_x \{ ||\sigma(x)|| \}}{\text{ess sup}_{x,y} \{ ||b(x)||, ||t(y)|| \}} \right\}.$$ 

**Result:**

$$K = ||\delta||,$$

where, $\delta$ is a mapping that extends vector fields from the interior of the body to its closure and is defined on Sobolev spaces or the related $LD$-spaces.
First case: General mechanics – simpler mathematics

- Forces may have non-vanishing resultants and total torques.
- The stress object contains a tensor field $\sigma_{ij}$ and a self force field $\sigma_{0i}$.
- The principle of virtual work is of the form

$$\int_\Omega b_i w_i dV + \int_{\partial\Omega} t_i w_i dA = \int_\Omega \sigma_{0i} w_i dV + \int_\Omega \sigma_{ij} w_{i,j} dV.$$

- The stress tensor $\sigma_{ij}$ need not be symmetric.
- With the self force field

$$K = \sup_F \inf_{\sigma \in \Sigma_F} \left\{ \frac{\text{ess sup}_{i,j,k,x} \{|\sigma_{0i}(x)|, |\sigma_{jk}(x)|\}}{\text{ess sup}_{i,x} \{|b_i(x)|, |t_i(x)|\}} \right\}.$$
Forces and Stresses as Linear Functionals

A Force: A linear functional (power functional) on virtual velocity fields,

\[ F(w) = \int_{\Omega} b_i w_i \, dV + \int_{\partial \Omega} t_i w_i \, dA. \]

A Stress: A linear functional (power functional) on the space of tensor fields (gradients of virtual velocity fields)

\[ \sigma(\chi) = \int_{\Omega} \sigma_{ij} \chi_{ij} \, dV. \]

We will generalize stresses to include self-forces so

\[ \sigma(\chi) = \int_{\Omega} \sigma_{0i} \chi_i \, dV + \int_{\Omega} \sigma_{ij} \chi_{ij} \, dV. \]

Equilibrium: \( F(w) = \sigma(\chi) \) if \( \chi_i = w_i \) and \( \chi_{ij} = w_{i,j} \).
The $L^1$ and $L^\infty$-Norms and Their Duality

Objective: The maximal absolute value of a stress component will be the magnitude or norm of the stress

$$\| \sigma \|^{L^\infty} = \text{ess sup}_{i,j,k,x} \{ |\sigma_{0i}(x)| , |\sigma_{jk}(x)| \}.$$ 

Duality: If we use the $L^1$-norm on the space of “local deformations” 

$$\{ \chi = (\chi_i, \chi_{jk}) \},$$ 

$$\| \chi \|^{L^1} = \sum_i \int_\Omega |\chi_i| \, dV + \sum_{i,j} \int_\Omega |\chi_{ij}| \, dV,$$

then every stress with finite $L^\infty$-norm is continuous and

$$\| \sigma \|^{L^\infty} = \| \sigma \|^{L^1^*} = \sup_{\| \chi \|=1} \frac{|\sigma(\chi)|}{\| \chi \|^{L^1}} = \sup_{\| \chi \|=1} |\sigma(\chi)|.$$
The Measure $\mu$ and the Corresponding Norms

Objective: Set

$$\| F \| = \max \left\{ \text{ess sup}_{i,x} \{|b_i(x)|\}, \text{ess sup}_{j,y} \{|t_j(y)|\} \right\}.$$ 

This will be the dual norm of a force if we use the norm

$$\| w \|^{L^{1,\mu}} = \sum_i \int_{\Omega} |w_i| \, dV + \sum_i \int_{\partial \Omega} |w_i| \, dA = \| w \|^{L^1} + \| w \|_{\partial \Omega}^{L^1}.$$ 

This is the $L^1$-norm relative to the measure $\mu$, denoted $L^{1,\mu}$, such that

$$\mu(D) = V(D \cap \Omega) + A(D \cap \partial \Omega).$$

Conclusion: We want to find a relation between the $L^\infty$-norm of the stress field and the $L^{\infty,\mu}$-norm $\| F \|$ of the force.
The Relation Between $L^{1, \mu}(\overline{\Omega}, \mathbb{R}^3)$ and $L^1(\Omega, \mathbb{R}^{12})$

\[
\begin{align*}
L^{1, \mu}(\overline{\Omega}, \mathbb{R}^3) & \xleftarrow{\delta} L^1(\Omega, \mathbb{R}^{12}) \\
L^{1, \mu}(\overline{\Omega}, \mathbb{R}^3)^* & \xrightarrow{\delta^*} L^1(\Omega, \mathbb{R}^{12})^* \\
L^{\infty, \mu}(\overline{\Omega}, \mathbb{R}^3) & \xrightarrow{\parallel} L^\infty(\Omega, \mathbb{R}^{12})
\end{align*}
\]
Sobolev mappings, traces and extensions

- We consider the Sobolev space $W^1_1(\Omega, \mathbb{R}^3)$ of $L^1$-mappings whose distributional derivatives are also $L^1$-mappings. The Sobolev space is a Banach space under the norm

$$
\| \phi \|_{W^1_1} = \sum_i \| \phi_i \|_{L^1} + \sum_{j,k} \| \phi_{j,k} \|_{L^1}.
$$

- There is a continuous linear mapping, the trace mapping

$$
\gamma : W^1_1(\Omega, \mathbb{R}^3) \rightarrow L^1(\partial \Omega, \mathbb{R}^3),
$$

$$
\gamma(u|_{\partial \Omega})(y) = u(y), \quad y \in \partial \Omega, \ u \in C(\overline{\Omega}, \mathbb{R}^3).
$$

Thus, there is a $K_\partial > 0$ such that

$$
\| \gamma(w) \|_{L^1} \leq K_\partial \| w \|_{W^1_1}.
$$
Implications to the Present Situation

- Clearly, we have a continuous inclusion mapping

\[ \iota_0 : W^1_1(\Omega, \mathbb{R}^3) \rightarrow L^1(\Omega, \mathbb{R}^3), \]

satisfying \( \|\iota_0(w)\|^{L^1} \leq \|w\|^{W^1_1} \).

- Hence, there is a linear injection—the extension to the boundary—

\[ \delta : W^1_1(\Omega, \mathbb{R}^3) \rightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3), \]

\[ \delta(w)(x) = w(x), \quad x \in \Omega, \quad \delta(w)(y) = \gamma(w)(y), \quad y \in \partial \Omega. \]

- The extension to the boundary is continuous and its norm is

\[ \|\delta\| = \sup_{w} \frac{\|\delta(w)\|^{L^{1,\mu}}}{\|w\|^{W^1_1}} = \sup_{w \in W^1_1(\Omega, \mathbb{R}^3)} \frac{\int_{\Omega} |w| \, dV + \int_{\partial \Omega} |\hat{w}| \, dA}{\int_{\Omega} |w| \, dV + \int_{\Omega} |\nabla w| \, dV}. \]
The Relation Between the $L^\infty,\mu$ and $W_1^{1,*}$-Norms of Forces

- As $\delta$ is a linear continuous injection, the dual mapping

$$\delta^*: L^1,\mu(\Omega, \mathbb{R}^3)^* = L^\infty,\mu(\Omega, \mathbb{R}^3) \rightarrow W_1^1(\Omega, \mathbb{R}^3)^*,$$

$$\delta^*(F)(w) = F(\delta(w)),$$

for all $w \in W_1^1(\Omega, \mathbb{R}^3)$, is continuous.

- A basic implication of the Hahn-Banach theorem: $\|\delta^*\| = \|\delta\|$.

Thus,

$$\sup_F \frac{\|\delta^*(F)\|_{W_1^1,*}}{\|F\|_{L^\infty,\mu}} = \|\delta\|, \quad F \in L^\infty,\mu(\Omega, \mathbb{R}^3).$$
The Representation of $W^1_1$-Forces by Stresses in $L^\infty(\Omega, \mathbb{R}^{12})$

- Consider the injection

$$j : W^1_1(\Omega, \mathbb{R}^3) \longrightarrow L^1(\Omega, \mathbb{R}^{12}), \quad j(\phi) = (\phi_i, \phi_{l,m}).$$

- We note that $\|\phi\|_{W^1_1} = \|j(\phi)\|_{L^1}$.

- It follows that every $W^1_1$-force $S$ may be represented (non-uniquely) by some stress $\sigma$ in $L^\infty(\Omega, \mathbb{R}^{12})$ in the form

$$S = j^*(\sigma).$$

- In addition,

$$\|S\|_{W^1_1^*} = \inf_{S = j^*(\sigma)} \|\sigma\|_{L^\infty}.$$
Result

The situation so far

\[ L^{1,\mu}(\Omega, \mathbb{R}^3) \xleftarrow{\delta} W^{1}(\Omega, \mathbb{R}^3) \xrightarrow{j} L^{1}(\Omega, \mathbb{R}^{12}) \]

\[ L^{1,\mu}(\Omega, \mathbb{R}^3)^* \xrightarrow{\delta^*} W^{1}(\Omega, \mathbb{R}^3)^* \xleftarrow{j^*} L^{1}(\Omega, \mathbb{R}^{12})^* \]

\[ \parallel L^{\infty,\mu}(\Omega, \mathbb{R}^3) \parallel \parallel L^{\infty}(\Omega, \mathbb{R}^{12}) \parallel \]
\[ \| \delta \| = \| \delta^* \| = \sup_{F \in L^\infty,\mu(\Omega,\mathbb{R}^3)} \frac{\| \delta^*(F) \|_{W_1^*}}{\| F \|_{L^\infty,\mu}} \]

\[ = \sup_{F \in L^\infty,\mu(\Omega,\mathbb{R}^3)} \inf_{\sigma, \delta^*(F) = j^*(\sigma)} \left\{ \text{ess sup}_{i,j,k,x} \left\{ |\sigma_{0i}(x)|, |\sigma_{jk}(x)| \right\} \right\} \]

\[ \text{ess sup}_{i,x} \left\{ |b_i(x)|, |t_i(x)| \right\} \]

We recall that \( \delta^*(F) = j^*(\sigma) \) means

\[ \delta^*(F)(w) = j^*(\sigma)(w). \]

It follows that \( \sigma \in \Sigma_F \) because

\[ \int_{\Omega} b_i w_i dV + \int_{\partial\Omega} t_i w_i dA = \int_{\Omega} \sigma_{0i} w_i dV + \int_{\Omega} \sigma_{ij} w_{i,j} dV. \]

Hence,

\[ \| \delta \| = K \]
Adaptation to Equilibrated Forces and Stresses

**Basic idea:** consider forces in the various dual spaces that do not perform power on rigid velocity fields (infinitesimal displacement fields).
Stretchings and Rigid Velocity Fields

- For a velocity field $w \in W$, the associated stretching (strain, deformation) $\varepsilon(w)$ is the tensor field

  $$\varepsilon(w)_{im} = \frac{1}{2}(w_{i,m} + w_{m,i}).$$

- A rigid velocity (or displacement) field is of the form

  $$w(x) = a + \omega \times x, \quad x \in \Omega.$$ 

- $\mathcal{R}$ – the collection of rigid velocity fields—a 6-dimensional subspace of the spaces of velocity fields.

- Considering the kernel of the stretching mapping $\varepsilon : w \mapsto \varepsilon(w)$, a theorem whose classical version is due to Liouville states that Kernel $\varepsilon = \mathcal{R}$. In particular, $\varepsilon(w + r) = \varepsilon(w)$. 

Distortions and Approximations by Rigid Velocities

- For a space $W$ of velocity fields, the associated space of *distortions* is $W/\mathbb{R}$. On $W/\mathbb{R}$ the stretching map $\varepsilon/\mathbb{R}([w])$ is well defined. We have the natural projection $\pi : W \rightarrow W/\mathbb{R}$.

- A norm is induced on $W/\mathbb{R}$ by $\|[w]\| = \inf_{r \in \mathbb{R}} \|w - r\|$ – the error of the best approximation by rigid motion.

- For the $L^2$-norm, the best approximation is the rigid motion that gives the same linear and angular momentum as $w$. This gives a projection $\pi_{\mathbb{R}} : W \rightarrow \mathbb{R}$.
  - Setting $W_0 = \text{Kernel } \pi_{\mathbb{R}} \subset W$, we have an isomorphism $W/\mathbb{R} \cong W_0$ and
    
    $\begin{array}{c}
    0 \rightarrow \mathbb{R} \xrightarrow{\iota_{\mathbb{R}}} W \xrightarrow{\pi} W/\mathbb{R} \rightarrow 0 \\
    0 \leftarrow \mathbb{R} \xleftarrow{\pi_{\mathbb{R}}} W \xleftarrow{\iota} W_0 \leftarrow 0.
    \end{array}$
  - Thus, $W \cong W_0 \oplus \mathbb{R}$. 

Equilibrated Forces

- \( W \) – a vector space of velocities (we assume that it contains the rigid velocities).

- A force \( F \in W^* \) is equilibrated if \( F(r) = 0 \) for all \( r \in \mathbb{R} \).

- As the quotient projection is surjective, the dual mapping \( \pi^*: (W/R)^* \rightarrow W^* \) is injective and its image is the collection of equilibrated forces. *Equilibrated forces \( \cong (W/R)^* \)*

- \( \pi^* \) is norm preserving. Thus, we may identify the collection of equilibrated forces in \( W^* \) with \( (W/R)^* \).

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \mathbb{R} & \overset{\iota_R}{\longrightarrow} & W & \overset{\pi}{\longrightarrow} & W/R & \longrightarrow & 0 \\
0 & \longleftarrow & \mathbb{R}^* & \overset{\iota^*_R}{\longleftarrow} & W^* & \overset{\pi^*}{\longleftarrow} & (W/R)^* & \longleftarrow & 0.
\end{array}
\]

- Using the projection \( \pi_R \) and the Whitney sum structure it induces we have a Whitney sum structure \( W^* = W_0^* \oplus \mathbb{R}^* \).
General Setting

We had:

\[ \text{Velocities: } L^1,\mu(\Omega, \mathbb{R}^3) \xleftarrow{\delta} W^1_1(\Omega, \mathbb{R}^3) \xrightarrow{j} L^1(\Omega, \mathbb{R}^{12}) \]

\[ \text{Forces: } L^1,\mu(\Omega, \mathbb{R}^3)^* \xrightarrow{\delta^*} W^1_1(\Omega, \mathbb{R}^3)^* \xleftarrow{j^*} L^1(\Omega, \mathbb{R}^{12})^* \]

Replace spaces of velocities \( W \) by the corresponding spaces of distortions \( W/\mathcal{R} \) and replace the spaces of forces \( W^* \) by the corresponding \( (W/\mathcal{R})^* \).

\[ L^1,\mu(\Omega, \mathbb{R}^3)/\mathcal{R} \xleftarrow{(\delta/\mathcal{R})^*} (L^1,\mu(\Omega, \mathbb{R}^3)/\mathcal{R})^* \xrightarrow{(\varepsilon/\mathcal{R})^*} L^1(\Omega, \mathbb{R}^6) \]

\[ (L^1,\mu(\Omega, \mathbb{R}^3)/\mathcal{R})^* \xrightarrow{(\delta/\mathcal{R})} (L^1,\mu(\Omega, \mathbb{R}^3)/\mathcal{R}) \xrightarrow{(\varepsilon/\mathcal{R})} L^\infty(\Omega, \mathbb{R}^6) \]

\( ? \) should satisfy the following conditions:

- \( \varepsilon/\mathcal{R} \) should be a norm preserving injection.
- \( \delta \) and \( \delta/\mathcal{R} \) should be well defined and in particular, the trace theorem should hold.
Bad News

$W^1_1(\Omega, \mathbb{R}^3)$ is not suitable.

Good News

$\square = LD(\Omega)$ has the required properties.

$L D(\Omega)$ – the space of integrable stretchings (deformations, strains):

$$\| w \| = \| \pi_R (w) \| + \| \varepsilon (w) \|^{L^1}.$$
Properties of $LD(\Omega)$

**Definition:** The collection of fields for which

$$\|w\|^{LD} = \sum_i \|w_i\|_1 + \sum_{i,m} \|\varepsilon(w)_{im}\|_1$$

is finite and serves as a norm, $LD(\Omega)$ is a Banach space.

**Approximation:** $C^\infty(\Omega, \mathbb{R}^3)$ is dense in $LD(\Omega)$.

**Traces:** There is a unique continuous linear mapping

$$\gamma : LD(\Omega) \longrightarrow L^1(\partial \Omega, \mathbb{R}^3)$$

such that $\gamma(u|_{\Omega}) = u|_{\partial \Omega}$, $u \in C(\Omega, \mathbb{R}^3)$.

**Extension to the boundary:** There is a unique continuous extension to the boundary

$$\delta : LD(\Omega) \longrightarrow L^{1,\mu}(\Omega, \mathbb{R}^3),$$

such that $\delta(u|_{\Omega}) = u$, for every $u \in C(\Omega, \mathbb{R}^3)$. 
Distortions of integrable stretching: On the space of $LD$-distortions, $LD(\Omega)/R$, the norm

$$\|\chi\| = \inf_{w \in \chi} \|w\|_{LD}$$

is equivalent to $\|\varepsilon/R([w])\|_{L^1} = \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}$.

Equivalent norms:

$$\|w\|_{LD} = \sum_{i} \|w_i\|_{L^1} + \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}$$

is equivalent to

$$\|\pi_R(w)\| + \|\varepsilon(w)\|_{L^1} = \|\pi_R(w)\| + \|\varepsilon/R([w])\|_{L^1}$$. 
Resulting Structure

\[
\begin{align*}
L^1,\mu(\Omega, \mathbb{R}^3) & \xleftarrow{\delta} LD(\Omega) \xrightarrow{\varepsilon} L^1(\Omega, \mathbb{R}^6) \\
\downarrow \pi & \downarrow \pi \\
L^1,\mu(\Omega, \mathbb{R}^3)/\mathbb{R} & \xleftarrow{\delta/\mathbb{R}} LD(\Omega)/\mathbb{R} \xrightarrow{\varepsilon/\mathbb{R}} L^1(\Omega, \mathbb{R}^6) \\
\downarrow \pi & \\
L^\infty,\mu(\Omega, \mathbb{R}^3) & \xleftarrow{\delta^*} LD(\Omega)^* \xrightarrow{\varepsilon^*} L^\infty(\Omega, \mathbb{R}^6) \\
\uparrow \pi^* & \uparrow \pi^* \\
(L^1,\mu(\Omega, \mathbb{R}^3)/\mathbb{R})^* & \xleftarrow{(\delta/\mathbb{R})^*} (LD(\Omega)/\mathbb{R})^* \xrightarrow{(\varepsilon/\mathbb{R})^*} L^\infty(\Omega, \mathbb{R}^6) \\
\end{align*}
\]

\[K = \|\delta/\mathbb{R}\|\]
Using \((\pi_0, \pi_R)\): \(LD(\Omega) \iff LD(\Omega)_0 \oplus R\)

\[
\begin{array}{c}
L^1,\mu(\overline{\Omega}, \mathbb{R}^3) \\
\downarrow \pi_0 \\
L^1,\mu(\overline{\Omega}, \mathbb{R}^3)_0
\end{array}
\quad \begin{array}{c}
\delta \\
\downarrow \pi_0 \\
LD(\Omega)_0 \cong LD(\Omega)/R
\end{array}
\quad \begin{array}{c}
LD(\Omega) \\
\delta_0 \\
LD(\Omega)_0 \cong LD(\Omega)/R
\end{array}
\quad \begin{array}{c}
\varepsilon \\
\varepsilon \\
L^1(\Omega, \mathbb{R}^6)
\end{array}
\]

with quotient norm \(K = \|\delta_0\|\)

isometric

\[
(L^1,\mu(\overline{\Omega}, \mathbb{R}^3)^*_0 \\
\delta_0^* \\
LD(\Omega)^*_0 \cong (LD(\Omega)/R)^*
\]

\[
\varepsilon^* \\
L^\infty(\Omega, \mathbb{R}^6)
\]