GENERALIZED STRESS CONCENTRATION FACTORS FOR EQUILIBRATED FORCES AND STRESSES

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ABSTRACT. As a sequel to a recent work we consider the generalized stress concentration factor, a purely geometric property of a body that for the various loadings, indicates the ratio between the maximum of the optimal stress and maximum of the loading fields. The optimal stress concentration factor pertains to a stress field that satisfies the principle of virtual work and for which the stress concentration factor is minimal. Unlike the previous work, we require that the external loading be equilibrated and that the stress field be a symmetric tensor field.

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1. INTRODUCTION

Stress concentration is a technical notion used by engineers in order to indicate that under given loading the stresses in a body increase if its geometry is somewhat different than an idealized geometry for which they can use the simple formulae of strength of materials. Specifically, the stress concentration factor is the ratio between the maximal stress obtained for the elastic stress field in the true body and the maximal stress obtained using the formulae of strength of materials for the simplified geometry. Stress concentration factors (see [6]) are evaluated by solving the equations of the theory of elasticity for the given geometry, by approximating these solutions numerically or by experimental methods (e.g., photoelasticity). A typical example is a bar undergoing tension T where the bar comprises of two parts having distinct uniform cross section areas connected by a region where the cross section tapers. In this case the maximal stress for the 3-dimensional solution is related to the simplified nominal value $\sigma_{\max} = T/A_{\min}$. Regarding the nominal stresses as traction boundary conditions applied away from the taper we wish to formalize the notion of stress concentration and regard it as the ratio between the maximal value of the stress and the maximal value of the applied traction. In fact, we also consider below body forces and regard the stress concentration factor as the ratio between the maximal stress and the maximum of the applied load (either body force or traction).

Using this point of view, we introduced in a a recent article [8] the notion of a generalized stress concentration factor as a quantitative measure of how bad is the geometry of a body in terms of the ratio between the maximal stresses and the maximum of the applied loads. Specifically, generalized stress concentration

factors may be described as follows. Let F be a force on a body Ω that is given in terms of a body force field b and a surface force field t and let σ be any stress field that is in equilibrium with F. Then, the stress concentration factor for the pair F, σ is given by

$$K_{F,\sigma} = \frac{\sup_x \{|\sigma(x)|\}}{\sup_{x,y} \{|b(x)|, |t(y)|\}}, \quad x \in \Omega, \ y \in \partial\Omega. \tag{1.1}$$

Here, for $|\sigma(x)|$ we use some norm $|\cdot|$ on the space of stresses at a point—a finite dimensional space. Similarly, $|b(x)|$ and $|t(y)|$ are the norms in $\mathbb{R}^3$ of the values of body force field and the surface force fields. The value of $K_{F,\sigma}$ depends on the norms chosen for stresses and external loadings and the same it true for the other objects defined below.

We can interpret the foregoing definition in terms of notions from plasticity. Failure criteria (e.g., the Tresca and von Mises yield criteria) are usually semi-norms rather then norms on the space of stress matrices. If we overlook this fact momentarily, and regard them as norms, it is not surprising that $K_{F,\sigma}$ depends on the norm chosen—distinct norms may be thought of as distinct yield criteria. If $\sup_x\{|\sigma(x)|\} = Y$ is regarded as the condition that some region of the body becomes plastic, $Y/K_{F,\sigma}$ is the value of the supremum of the external force that will initiate plasticity. (In the usual case where the yield condition is given by a seminorm, $K_{F,\sigma}$ can only give a bound on the supremum of forces that will initiate plasticity.)

Returning to the definition of the generalized stress concentration factor, we note that since we do not specify a constitutive relation, for each force F there is a class Σ_F of stress fields σ that are in equilibrium with F. The optimal stress concentration factor for the force F is defined by

$$K_F = \inf_{\sigma \in \Sigma_F} \{K_{F,\sigma}\}, \tag{1.2}$$

i.e., it is the least stress concentration factor when we allow the stress field to vary over all fields that are in equilibrium with F. Finally, the generalized stress concentration factor K—a purely geometric property of Ω—is defined by

$$K = \sup_F \{K_F\} = \sup_F \frac{\inf_{\sigma \in \Sigma_F} \{\sup_x \{|\sigma(x)|\}\}}{\sup_{x,y} \{|b(x)|, |t(y)|\}}, \tag{1.3}$$

where F varies over all forces that may be applied to the body. Thus, the generalized stress concentration factor reflects the worst case of loading of the body.

It was shown in [8] that the generalized stress concentration factor is equal to the norm of a mapping associated with the trace operator of Sobolev mappings. Specifically, it was shown that when suprema in the expressions above are replaced by essential suprema, then,

$$K = \sup_{\phi \in W_1^1(\Omega,\mathbb{R}^3)} \frac{\int\limits_\Omega |\phi| \ \mathrm{d}V + \int\limits_{\partial\Omega} |\hat{\phi}| \ \mathrm{d}A}{\int\limits_\Omega |\phi| \ \mathrm{d}V + \int\limits_\Omega |\nabla\phi| \ \mathrm{d}V}, \tag{1.4}$$

where $W_1^1(\Omega, \mathbb{R}^3)$ is the Sobolev space of integrable vector fields ϕ on Ω whose gradients $\nabla\phi$ are also integrable, and $\hat{\phi}$ is the trace of $\phi \in W_1^1(\Omega, \mathbb{R}^3)$ on $\partial\Omega$ (whose existence is a basic property of Sobolev spaces).

Consider the Radon measure μ on $\overline{\Omega}$ defined by

$$\mu(D) = V(D \cap \Omega) + A(D \cap \partial\Omega) \tag{1.5}$$

(V and A are the volume and area measures, respectively), and let $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ be the space of fields on $\overline{\Omega}$ that are integrable relative to μ equipped with the $L^{1,\mu}$-norm so

$$\|w\|_{L^{1,\mu}} = \int_{\Omega} |w| \, dV + \int_{\partial\Omega} |w| \, dA. \tag{1.6}$$

Then, the trace operator induces an extension mapping $\delta \colon W_1^1(\Omega, \mathbb{R}^3) \to L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ and the expression for the generalized stress concentration factor above may be written in the form

$$K = \|\delta\| \tag{1.7}$$

—the basic result of [8].

The treatment in [8] allows stresses and forces that are more general than those treated usually in continuum mechanics. In addition to the usual stress tensor σ_{im} the stress object contains a self force field σ_i. Furthermore, the stress field need not be symmetric and the resultants and total torques due to the forces F need not vanish. The generalized form of the equilibrium equations between the forces and stresses was taken in the form

$$\int_{\Omega} b_i w_i \, dV + \int_{\partial\Omega} t_i w_i \, dA = \int_{\Omega} \sigma_i w_i \, dV + \int_{\Omega} \sigma_{ik} w_{i,k} \, dV. \tag{1.8}$$

Thus, the infimum in the definition of the optimal stress concentration factor may be attained for a stress field that is not admissible physically.

In the present work we restrict the admissible stress fields to symmetric tensor fields and the forces are required to have zero resultants and total torques. These requirements are well known to be equivalent to the requirements that the power produced by the forces and stresses on rigid velocity fields vanishes.

The expression for the generalized stress concentration factor we obtain here for the rigid velocity invariant forces and stresses may be written as

$$K = \|\delta/\mathscr{R}\|, \tag{1.9}$$

where $\mathscr{R}$ denotes the collection of rigid velocity fields, a subspace of the function-spaces we are considering. The extension mapping

$$\delta/\mathscr{R} \colon LD(\Omega)/\mathscr{R} \longrightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R} \tag{1.10}$$

between the corresponding quotient spaces is given by $\delta/\mathscr{R}([w]) = [\delta(w)]$. It is well defined for elements of the space $LD(\Omega)$ containing the vector fields w of integrable stretchings

$$\varepsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^T).$$

The space $LD(\Omega)$ and its properties (see [12, 13, 9, 11, 1, 2], and [4] for nonlinear strains) are the main technical tools we use in this work.

For a projection mapping that gives an approximating rigid velocity field to any vector field w and a corresponding w_0 that has zero rigid component, this result

may be written more specifically as

$$K = \|\delta_0\| = \sup_{w_0 \in LD(\Omega)_0} \frac{\inf_{r \in \mathscr{R}} \left\{ \int_\Omega \sum_i |w_{0i} - r_i| \, \mathrm{d}V + \int_{\partial\Omega} \sum_i |w_{0i} - r_i| \, \mathrm{d}A \right\}}{\frac{1}{2} \int_\Omega \sum_{i,m} |w_{0i,m} + w_{0m,i}| \, \mathrm{d}V} .$$

(1.11)

Here δ_0 is the extension mapping for vector fields having zero rigid components and $LD(\Omega)_0$ is the space of vector fields in $LD(\Omega)$ having zero rigid components.

Section 2 presents some properties of rigid velocity fields, stretchings and the approximations of velocity fields by rigid ones. Section 3 outlines the definitions and results pertaining to the space $LD(\Omega)$ and is based on [12]. Section 4 applies the properties of LD-fields to the problem under consideration and Section 5 presents additional comments and observations. Some details regarding the notation we use and results on normed spaces and their normed dual spaces are available in [8].

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2. PRELIMINARIES ON STRETCHINGS AND RIGID VELOCITIES

2.1. **Basic definitions.** Let Ω be an open and bounded 3-dimensional submanifold of $\mathbb{R}^3$ with volume $|\Omega|$ having a differentiable boundary and w a vector field over Ω. We set $\varepsilon(w)$ to be the tensor field

$$\varepsilon(w)_{im} = \frac{1}{2}(w_{i,m} + w_{m,i}),$$

(2.1)

i.e., the symmetric part of the gradient. As w is interpreted physically as a velocity field over the body, $\varepsilon(w)$ is interpreted as the stretching. Alternatively, if w is interpreted as an infinitesimal displacement field, $\varepsilon(w)$ is the corresponding linear strain. In the sequel we will refer to $\varepsilon(w)$ as the *stretching* associated with w. Here, the partial derivatives are interpreted as the distributional derivatives so one need not care about the regularity of w.

We identify the space of symmetric 3×3 matrices with $\mathbb{R}^6$. For a symmetric tensor field ε whose components are integrable functions we use the L^1-norm

$$\|\varepsilon\| = \sum_{i,m} \|\varepsilon_{im}\|_{L^1} .$$

(2.2)

This norm maybe replaced by other equivalent norms (possibly norms invariant under coordinate transformations). Thus, the space of L^1-stretching fields is represented by $L^1(\Omega, \mathbb{R}^6)$ with the L^1-norm as above.

A vector field w on Ω is *of integrable stretching* if its components are integrable and if each component $\varepsilon(w)_{im} \in L^1(\Omega)$. It can be shown that this definition is coordinate independent. The vector space of velocity fields having integrable stretchings will be denoted by $LD(\Omega)$. This space is normed by

$$\|w\|_{LD} = \sum_i \|w_i\|_{L^1} + \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1} .$$

(2.3)

Clearly, we have a continuous linear inclusion $LD(\Omega, \mathbb{R}^3) \longrightarrow L^1(\Omega, \mathbb{R}^3)$. In addition, $w \mapsto \varepsilon(w)$ is given by a continuous linear mapping

$$\varepsilon \colon LD(\Omega) \longrightarrow L^1(\Omega, \mathbb{R}^6). \tag{2.4}$$

2.2. **The subspace of rigid velocities.** A *rigid* velocity (or displacement) field is of the form

$$w(x) = a + \omega \times x, \quad x \in \Omega \tag{2.5}$$

where a and ω are fixed in $\mathbb{R}^3$ and $\omega \times x$ is the vector product. We can replace $\omega \times x$ with $\tilde{\omega}(x)$ where $\tilde{\omega}$ is the associated skew symmetric matrix so $w(x) = a + \tilde{\omega}(x)$. We will denote the 6-dimensional space of rigid body velocities by $\mathscr{R}$. For a rigid motion

$$\tilde{\omega}_{im} = \frac{1}{2}(w_{i,m} - w_{m,i}), \tag{2.6}$$

an expression that is extended to the non-rigid situation and defines the vorticity vector field so $w_{i,m} = \varepsilon(w)_{im} + \tilde{\omega}_{im}$.

Considering the kernel of the stretching mapping $\varepsilon \colon LD(\Omega) \longrightarrow L^1(\Omega, \mathbb{R}^6)$, a theorem whose classical version is due to Liouville states (see [12, pp. 18–19]) that Kernel $\varepsilon = \mathscr{R}$.

2.3. **Approximation by rigid velocities.** We now wish to consider the approximation of a velocity field by a rigid velocity field. Let ρ be a Radon measure on $\overline{\Omega}$ and $1 \leqslant p \leqslant \infty$. For a given $w \in L^{p,\rho}(\overline{\Omega}, \mathbb{R}^3)$, we wish to find the rigid velocity r for

$$\inf_{r' \in \mathscr{R}} \left(\left\| w - r' \right\|_{L^{p,\rho}} \right)^p = \inf_{r' \in \mathscr{R}} \int_{\overline{\Omega}} \sum_i \left| w_i - r_i' \right|^p \, \mathrm{d}\rho \tag{2.7}$$

is attained. Thus we are looking for vectors a and b that minimize

$$e = \int_{\overline{\Omega}} \sum_i \left| w_i - a_i - \varepsilon_{ijk} b_j x_k \right|^p \, \mathrm{d}\rho. \tag{2.8}$$

We have

$$\frac{\partial e}{\partial a_l} = \int_{\overline{\Omega}} p \sum_i \left| w_i - a_i - \varepsilon_{ijk} b_j x_k \right|^{p-1} \frac{\left(w_i - a_i - \varepsilon_{ijk} b_j x_k \right)}{\left| w_i - a_i - \varepsilon_{ijk} b_j x_k \right|} (-\delta_{il}) \, \mathrm{d}\rho,$$

$$\frac{\partial e}{\partial b_l} = \int_{\overline{\Omega}} p \sum_i \left| w_i - a_i - \varepsilon_{ijk} b_j x_k \right|^{p-1} \frac{\left(w_i - a_i - \varepsilon_{ijk} b_j x_k \right)}{\left| w_i - a_i - \varepsilon_{ijk} b_j x_k \right|} (-\varepsilon_{ijk} \delta_{jl} x_k) \, \mathrm{d}\rho, \tag{2.9}$$

and we obtain the 6 equations for the minimum with the 6 unknowns a_l, b_m

$$0 = \int_{\overline{\Omega}} \left| w_l - a_l - \varepsilon_{ljk} b_j x_k \right|^{p-2} \left(w_l - a_l - \varepsilon_{ljk} b_j x_k \right) \mathrm{d}\rho,$$

$$0 = \int_{\overline{\Omega}} \sum_i \left| w_i - a_i - \varepsilon_{ijk} b_j x_k \right|^{p-2} \left(w_i - a_i - \varepsilon_{ijk} b_j x_k \right) \varepsilon_{ilk} x_k \, \mathrm{d}\rho. \tag{2.10}$$

Particularly simple are the equations for $p = 2$. In this case we obtain

$$\int_{\overline{\Omega}} w \, d\rho = \int_{\overline{\Omega}} r \, d\rho, \quad \text{and} \quad \int_{\overline{\Omega}} x \times w \, d\rho = \int_{\overline{\Omega}} x \times r \, d\rho. \tag{2.11}$$

If we interpret ρ as a mass distribution on $\overline{\Omega}$, these two conditions simply state that the best rigid velocity approximations should give the same momentum and angular momentum as the original field.

Of particular interest (see [12, p. 120]) is the case where ρ is the volume measure on Ω. Set $\bar{x}$ to be the center of volume of Ω, i.e.,

$$\bar{x} = \frac{1}{|\Omega|} \int_{\Omega} x \, dV. \tag{2.12}$$

Without loss of generality we will assume that $\bar{x} = 0$ (for else we may replace x by $x - \bar{x}$ in the sequel).

Let $\bar{w}$ be the mean of the field w and I the inertia matrix relative to the center of volume, so

$$\bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w \, dV, \quad I_{im} = \int_{\Omega} (x_k x_k \delta_{im} - x_i x_m) \, dV \tag{2.13}$$

and

$$I(\omega) = \int_{\Omega} x \times (\omega \times x) \, dV. \tag{2.14}$$

The inertia matrix is symmetric and positive definite and so the solution for r gives

$$r = \bar{w} + \omega \times x \tag{2.15}$$

with $\bar{w}$ as above and

$$\omega = I^{-1} \left(\int_{\Omega} x \times w \, dV \right). \tag{2.16}$$

Thus, $w \mapsto (\bar{w} + \omega \times x)$, with $\bar{w}$ and ω as above, is well defined for integrable velocity fields and we obtain a mapping

$$\pi_{\mathscr{R}} : L^1(\Omega, \mathbb{R}^3) \longrightarrow \mathscr{R}. \tag{2.17}$$

It is straightforward to show that $\pi_{\mathscr{R}}$ is indeed a linear projection onto $\mathscr{R}$.

Also of interest below will be the case where $p = 1$ and and the measure ρ is given by

$$\rho(D) = \mu(D) = V(D \cap \Omega) + A(D \cap \partial\Omega), \tag{2.18}$$

as in Section 1. The conditions for best approximations $r = a + b \times x$ assume the form

$$\int_{\Omega} \frac{\left(w_l - a_l - \varepsilon_{ljk}b_j x_k\right)}{\left|w_l - a_l - \varepsilon_{ljk}b_j x_k\right|} \, dV + \int_{\partial\Omega} \frac{\left(w_l - a_l - \varepsilon_{ljk}b_j x_k\right)}{\left|w_l - a_l - \varepsilon_{ljk}b_j x_k\right|} \, dA = 0, \tag{2.19}$$

$$\int_{\Omega} \sum_i \frac{\left(w_i - a_i - \varepsilon_{ijk}b_j x_k\right)}{\left|w_i - a_i - \varepsilon_{ijk}b_j x_k\right|} \varepsilon_{ilk}x_k \, dV + \int_{\partial\Omega} \sum_i \frac{\left(w_i - a_i - \varepsilon_{ijk}b_j x_k\right)}{\left|w_i - a_i - \varepsilon_{ijk}b_j x_k\right|} \varepsilon_{ilk}x_k \, dA = 0,$$

$$\tag{2.20}$$

where $z/|z|$ is taken as 0 for $z = 0$. (For an analysis of L^1-approximations see [7] and reference cited therein.)

2.4. **Distortions.** Let **W** be a vector space of velocities on Ω containing the rigid velocities $\mathscr{R}$ and let w_1 and w_2 be two velocity fields in **W**. We will say that the two have the same *distortion* if $w_2 = w_1 + r$ for some rigid motion $r \in \mathscr{R}$. This clearly generates an equivalence relation on **W** and the corresponding quotient space $\mathbf{W}/\mathscr{R}$ will be referred to as the space of distortions. If χ is an element of $\mathbf{W}/\mathscr{R}$ then $\varepsilon(w)$ is the same for all members of $w \in \chi$. The natural projection

$$\pi \colon \mathbf{W} \longrightarrow \mathbf{W}/\mathscr{R} \tag{2.21}$$

associates with each element $w \in \mathbf{W}$ its equivalence class $[w] = \{w + r | r \in \mathscr{R}\}$.

If **W** is a normed space, then, the induced norm on $\mathbf{W}/\mathscr{R}$ is given by (see Appendix A)

$$\|[w]\| = \inf_{w' \in [w]} \|w'\| = \inf_{r \in \mathscr{R}} \|w - r\|. \tag{2.22}$$

Thus, the evaluation of the norm of a distortion, is given by the best approximation by a rigid velocity as described above.

Let **W** be a vector space of velocities contained in $L^1(\Omega, \mathbb{R}^3)$, then, $\pi_{\mathscr{R}}$ defined above induces an additional projection

$$\pi_0(w) = w - \pi_{\mathscr{R}}(w). \tag{2.23}$$

The image of π_0 is the kernel $\mathbf{W}_0$ of $\pi_{\mathscr{R}}$ and it is the subspace of **W** containing velocity fields having zero approximating rigid velocities. Clearly, we have a bijection $\beta \colon \mathbf{W}/\mathscr{R} \to \mathbf{W}_0$. On $\mathbf{W}_0$ we have two equivalent norms: the norm it has as a subspace of **W** and the norm that makes the bijection $\beta \colon \mathbf{W}/\mathscr{R} \to \mathbf{W}_0$ an isometry.

With the projections π_0 and $\pi_{\mathscr{R}}$, **W** has a Whitney sum structure $\mathbf{W} = \mathbf{W}_0 \oplus \mathscr{R}$.

2.5. **Equilibrated forces.** Let **W** be a vector space of velocities (we assume that it contains the rigid velocities). A force $F \in \mathbf{W}^*$ is equilibrated if $F(r) = 0$ for all $r \in \mathscr{R}$. This is of course equivalent to $F(w) = F(w + r)$ for all $r \in \mathscr{R}$ so F induces a unique element of $(\mathbf{W}/\mathscr{R})^*$. Conversely, any element of $G \in (\mathbf{W}/\mathscr{R})^*$ induces an equilibrated force F by $F(w) = G([w])$, where $[w]$ is the equivalence class of w. In other words, as the quotient projection is surjective, the dual mapping $\pi^* \colon (\mathbf{W}/\mathscr{R})^* \to \mathbf{W}^*$ is injective and its image—the collection of equilibrated forces—is orthogonal to the kernel of π. Furthermore, as in Appendix A, π^* is norm preserving. Thus, we may identify the collection of equilibrated forces in $\mathbf{W}^*$ with $(\mathbf{W}/\mathscr{R})^*$.

If $\iota_{\mathscr{R}} \colon \mathscr{R} \to \mathbf{W}$ is the inclusion of the rigid velocities, then,

$$\iota_{\mathscr{R}}^* \colon \mathbf{W}^* \longrightarrow \mathscr{R}^*, \tag{2.24}$$

is a continuous and surjective mapping. The image $\iota_{\mathscr{R}}^*(F)$ will be referred to as the total of the force. In particular, its component dual to $\bar{w}$ will be referred to as the force resultant and the component dual to ω will be referred to as the resultant torque. Thus, in particular, the resultant force and torque vanish for an equilibrated force. This structure may be illustrated by the sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{R} & \xrightarrow{\iota_{\mathscr{R}}} & \mathbf{W} & \xrightarrow{\pi} & \mathbf{W}/\mathscr{R} & \longrightarrow & 0 \\[2mm]
0 & \longleftarrow & \mathscr{R}^* & \xleftarrow{\iota_{\mathscr{R}}^*} & \mathbf{W}^* & \xleftarrow{\pi^*} & (\mathbf{W}/\mathscr{R})^* & \longleftarrow & 0.
\end{array}
\tag{2.25}
$$

Using the projection $\pi_{\mathscr{R}}$ and the Whitney sum structure it induces we have a Whitney sum structure $\mathbf{W}^* = \mathbf{W}_0^* \oplus \mathscr{R}^*$ and it is noted that the norm on $\mathbf{W}_0^*$ is implied by the choice of norm on $\mathbf{W}_0$.

3. FIELDS OF INTEGRABLE STRETCHINGS

In this section we list the basic properties of vector fields of integrable stretching (or deformation) as in [12] (see also [13, 9, 11, 1, 2] and [4] for nonlinear strains). The presentation below is adapted to the application we consider and is not necessarily the most general.

If both w and $\varepsilon(w)$ are in L^p for $1 < p < \infty$, the Korn inequality (see [3]) implies that $w \in W_1^1(\Omega)$. This would imply in particular that w has a trace on the boundary of Ω. However, as shown by Ornstein [5], w need not necessarily be in $W_1^1(\Omega, \mathbb{R}^3)$ for the critical value $p = 1$. Nevertheless, the theory of integrable stretchings shows that the trace is well defined even for $p = 1$.

3.1. **Definition.** We recall that $LD(\Omega)$ is the vector space of fields with integrable stretchings. With the norm

$$\|w\|_{LD} = \sum_i \|w_i\|_{L^1} + \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}, \tag{3.1}$$

$LD(\Omega)$ is a Banach space.

3.2. **Approximation.** $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ is dense in $LD(\Omega)$.

3.3. **Traces.** The trace operator can be extended from $W_1^1(\Omega, \mathbb{R}^3)$ onto $LD(\Omega, \mathbb{R}^3)$. Thus, there is a unique continuous linear mapping

$$\gamma \colon LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3) \tag{3.2}$$

such that $\gamma(w) = w\big|_{\partial\Omega}$, for every field w of bounded stretching that is a restriction to Ω of a continuous field on the closure $\overline{\Omega}$. Thus, the norm of the trace mapping is given by

$$\|\gamma\| = \sup_{w \in LD(\Omega)} \frac{\|\gamma(w)\|_{L^1}}{\|w\|_{LD}}. \tag{3.3}$$

As a result of the approximation of fields of bounded stretchings by smooth vector fields on $\overline{\Omega}$, $\|\gamma\|$ may be evaluated using smooth vector fields in the expression above, i.e.,

$$\|\gamma\| = \sup_{w \in C^\infty(\overline{\Omega}, \mathbb{R}^3)} \frac{\left\|w\big|_{\partial\Omega}\right\|_{L^1}}{\|w\|_{LD}}. \tag{3.4}$$

3.4. **Extensions.** There is a continuous linear extension operator

$$E \colon LD(\Omega) \longrightarrow LD(\mathbb{R}^3)$$

such that $E(w)(x) = w(x)$ for almost all $x \in \Omega$.

3.5. **Regularity.** If w is any distribution on Ω whose corresponding stretching is L^1, then $w \in L^1(\Omega, \mathbb{R}^3)$.

3.6. **Distortions of integrable stretchings.** On the space of LD-distortions, $LD(\Omega)/\mathscr{R}$, we have a natural norm

$$\|\chi\| = \inf_{w \in \chi} \|w\|_{LD} . \tag{3.5}$$

This norm is equivalent to

$$\|\varepsilon(\chi)\| = \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1} \tag{3.6}$$

where w is any member of χ. Clearly, the value of this expression is the same for all members $w \in \chi$ and we can use any other equivalent norm on the space of symmetric tensor fields.

Using the projection $\pi_{\mathscr{R}}$ as above we denote by $LD(\Omega)_0$ the kernel of $\pi_{\mathscr{R}}$ and by π_0 the projection onto $LD(\Omega)_0$ so

$$(\pi_0, \pi_{\mathscr{R}}) \colon LD(\Omega) \longrightarrow LD(\Omega)_0 \oplus \mathscr{R}.$$

Then, there is a constant C depending only on Ω such that

$$\|\pi_0(w)\|_{L^1} = \|w - \pi_{\mathscr{R}}(w)\|_{L^1} \leqslant C \|\varepsilon(w)\|_{L^1} . \tag{3.7}$$

3.7. **Equivalent norms.** Let p be a continuous seminorm on $LD(\Omega)$ which is a norm on $\mathscr{R}$. Then,

$$p(w) + \|\varepsilon(w)\|_{L^1} \tag{3.8}$$

is a norm on $LD(\Omega)$ which is equivalent to the original norm in 3.1. For example, using the fact that the trace mapping is continuous, we may use

$$p(w) = \|\gamma(w)\|_{L^1(\partial\Omega, \mathbb{R}^3)} \tag{3.9}$$

and the the following equivalent to the LD-norm

$$\|w\|_{\times} = \|\gamma(w)\|_{L^1(\partial\Omega, \mathbb{R}^3)} + \|\varepsilon(w)\|_{L^1} . \tag{3.10}$$

4. Application to Equilibrated Forces and Stresses

4.1. *LD-velocity fields and forces.* The central object we consider is $LD(\Omega)$ whose elements are referred to as LD-velocity fields. Elements of the dual space $LD(\Omega)^*$ will be referred to as LD-forces. Our objective is to represent LD-forces by stresses and by pairs containing body forces and surface forces.

Rather than the original norm of 3.1 it will be convenient to use an equivalent norm as asserted by 3.7 as follows. Let

$$\pi_{\mathscr{R}} \colon LD(\Omega) \longrightarrow \mathscr{R} \tag{4.1}$$

be the continuous linear projection defined in Paragraph 2.3 and let $q \colon \mathscr{R} \to \mathbb{R}$, be a norm on the finite dimensional $\mathscr{R}$. Then,

$$p = q \circ \pi_{\mathscr{R}} \colon LD(\Omega) \longrightarrow \mathbb{R} \tag{4.2}$$

is a continuous seminorm that is a norm on $\mathscr{R} \subset LD(\Omega)$. It follows from 3.7 that

$$\|w\|'_{LD} = q\left(\pi_{\mathscr{R}}(w)\right) + \|\varepsilon(w)\|_{L^1} \tag{4.3}$$

is a norm on $LD(\Omega)$ which is equivalent to the original norm defined in 3.1.

4.2. *LD*-**distortions.** With the norm $\|\cdot\|'_{LD}$, the induced norm on $LD(\Omega)/\mathscr{R}$ is given by

$$\|[w]\|'_{LD} = \inf_{r \in \mathscr{R}} \|w + r\|'_{LD}, \tag{4.4}$$

so using $\pi_{\mathscr{R}}(r) = r$, $\varepsilon(r) = 0$ and choosing $r = -\pi_{\mathscr{R}}(w)$, we have

$$\begin{aligned}
\|[w]\|'_{LD} &= \inf_{r \in \mathscr{R}} \{q(\pi_{\mathscr{R}}(w + r)) + \|\varepsilon(w + r)\|_{L^1}\} \\
&= \inf_{r \in \mathscr{R}} \{q(\pi_{\mathscr{R}}(w) + r) + \|\varepsilon(w)\|_{L^1}\} \\
&= \|\varepsilon(w)\|_{L^1}.
\end{aligned} \tag{4.5}$$

Let $\pi_0 \colon LD(\Omega) \to LD(\Omega)_0$ be the projection onto $LD(\Omega)_0 \subset LD(\Omega)$, the kernel of $\pi_{\mathscr{R}}$. Then,

$$\begin{aligned}
\|\pi_0(w)\|'_{LD} &= \|w - \pi_{\mathscr{R}}(w)\|'_{LD} \\
&= q(\pi_{\mathscr{R}}(w - \pi_{\mathscr{R}}(w))) + \|\varepsilon(w - \pi_{\mathscr{R}}(w))\|_{L^1} \\
&= \|\varepsilon(w)\|_{L^1}.
\end{aligned} \tag{4.6}$$

We conclude that with our choice of norm $\|\cdot\|'_{LD}$ on $LD(\Omega)$, the two norms in 3.6 are not only equivalent but are actually equal. Thus, this choice makes $LD(\Omega)_0$ isometrically isomorphic to $LD(\Omega)/\mathscr{R}$.

4.3. **Equilibrated *LD*-forces and their representations by stresses.** Summarizing the results of the previous sections we can draw the commutative diagram

$$\begin{array}{ccc}
LD(\Omega) & \xrightarrow{\ \varepsilon\ } & L^1(\Omega, \mathbb{R}^6) \\
\downarrow{\scriptstyle \pi} & & \| \\
LD(\Omega)/\mathscr{R} & \xrightarrow{\ \varepsilon/\mathscr{R}\ } & L^1(\Omega, \mathbb{R}^6).
\end{array} \tag{4.7}$$

Here, Liouville's rigidity theorem implies that the kernels of ε and π are identical, the rigid velocity fields, and $\varepsilon/\mathscr{R}$ given by $\varepsilon/\mathscr{R}(\chi) = \varepsilon(w)$, for some $w \in \chi$, is an isometric injection.

This allows us to represent *LD*-forces—elements of $LD(\Omega)^*$—using the dual diagram.

$$\begin{array}{ccc}
LD(\Omega)^* & \xleftarrow{\ \varepsilon^*\ } & L^\infty(\Omega, \mathbb{R}^6) \\
\uparrow{\scriptstyle \pi^*} & & \| \\
(LD(\Omega)/\mathscr{R})^* & \xleftarrow{\ (\varepsilon/\mathscr{R})^*\ } & L^\infty(\Omega, \mathbb{R}^6).
\end{array} \tag{4.8}$$

Now, $(\varepsilon/\mathscr{R})^*$ is surjective and as in [8] the Hahn-Banach Theorem implies that any $T \in (LD(\Omega)/\mathscr{R})^*$ may be represented in the form

$$T = (\varepsilon/\mathscr{R})^*(\sigma) \tag{4.9}$$

for some essentially bounded symmetric stress tensor field $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$. Furthermore, the dual norm of T is given by

$$\|T\| = \inf_{T=(\varepsilon/\mathscr{R})^*(\sigma)} \|\sigma\|_{L^\infty} = \inf_{T=(\varepsilon/\mathscr{R})^*(\sigma)} \left\{ \operatorname{ess\,sup}_{i,m,x\in\Omega} |\sigma_{im}(x)| \right\}. \tag{4.10}$$

In fact, the infimum is attainable so there is a stress tensor field $\hat{\sigma} \in L^\infty(\Omega, \mathbb{R}^6)$ such that $\|S\| = \|\hat{\sigma}\|_{L^\infty}$, with $S = \varepsilon^*(\hat{\sigma})$. As π^* is norm preserving (see Appendix A), the same holds for any equilibrated LD-force. That is, using the same argument for $(LD(\Omega)/\mathscr{R})^*$ and the fact that π^* is a norm-preserving injection, any equilibrated LD-force $S \in LD(\Omega)^*$ may be represented in the form

$$S = \varepsilon^*(\sigma) \tag{4.11}$$

for some stress field σ and

$$\|S\| = \inf_{S=\varepsilon^*(\sigma)} \|\sigma\|_{L^\infty} = \inf_{S=\varepsilon^*(\sigma)} \left\{ \text{ess} \sup_{i,m,x\in\Omega} |\sigma_{im}(x)| \right\}. \tag{4.12}$$

4.4. μ-integrable distortions and equilibrated forces on bodies. Following [8] we use $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ to denote the space of integrable vector fields on $\overline{\Omega}$ whose restrictions to $\partial\Omega$ are integrable relative to the area measure on $\partial\Omega$. On this space we use the norm

$$\|w\|_{L^{1,\mu}} = \int_\Omega |w| \, dV + \int_{\partial\Omega} |w| \, dA = \|w\|_{L^1(\Omega,\mathbb{R}^3)} + \|w\|_{L^1(\partial\Omega,\mathbb{R}^3)}. \tag{4.13}$$

Alternatively, the $L^{1,\mu}$-norm may be regarded as the L^1-norm relative to the Radon measure μ, defined above and hence the notation.

Forces, being elements of the dual space $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)^*$, may be identified with elements of $L^{\infty,\mu}(\overline{\Omega}, \mathbb{R}^3)$. A force F on a body, given in terms of a body force b and a surface force t, may be identified with a continuous linear functional relative to the $L^{1,\mu}$-norm if the body force components b_i and surface force components t_i (alternatively, $|b|$ and $|t|$) are essentially bounded relative to the volume and area measures, respectively. In this case, the representation is of the form

$$F(w) = \int_\Omega b_i w_i \, dV + \int_{\partial\Omega} t_i w_i \, dA. \tag{4.14}$$

Moreover, the dual norm of a force is the $L^{\infty,\mu}$-norm, given as

$$\|F\|_{L^{\infty,\mu}} = \|F\|^*_{L^{1,\mu}} = \text{ess} \sup_{x\in\Omega, y\in\partial\Omega} \{|b(x)|, |t(y)|\} \tag{4.15}$$

as anticipated.

It is well known that if F is equilibrated, i.e., $F \in \pi_0^*(G)$ for some $G \in \left(L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R}\right)^*$, then,

$$\int_\Omega b \, dV + \int_{\partial\Omega} t \, dA = 0, \quad \text{and} \quad \int_\Omega x \times b \, dV + \int_{\partial\Omega} x \times t \, dA = 0. \tag{4.16}$$

4.5. LD-forces represented by body forces and surface forces. Using the trace operator γ, for each $w \in LD(\Omega)$ we may define

$$\delta(w): \overline{\Omega} \to \mathbb{R}^3 \tag{4.17}$$

by $\delta(w)(x) = w(x)$ for $x \in \Omega$ and $\delta(w)(y) = \gamma(w)(y)$ for $y \in \partial\Omega$. The trace theorem 3.3 and the original definition in 3.1 of the norm on $LD(\Omega)$ imply that we defined a linear and continuous mapping

$$\delta: LD(\Omega) \longrightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3). \tag{4.18}$$

By the linearity of the trace mapping and using $\gamma(r) = r$ for $r \in \mathscr{R}$, we set

$$\gamma/\mathscr{R} \colon LD(\Omega)/\mathscr{R} \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R} \tag{4.19}$$

by $\gamma/\mathscr{R}([w]) = [\gamma(w)]$. Similarly, we set

$$\delta/\mathscr{R} \colon LD(\Omega)/\mathscr{R} \longrightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R} \tag{4.20}$$

by $\delta/\mathscr{R}([w]) = [\delta(w)]$. We note that the quotient mappings $\gamma/\mathscr{R}$ and $\delta/\mathscr{R}$ are bounded. For example, for any $r \in \mathscr{R}$

$$\|\delta(w) + r\|_{L^{1,\mu}} = \|\delta(w + r)\|_{L^{1,\mu}} \leqslant \|\delta\| \, \|w + r\|_{LD} \,, \tag{4.21}$$

so

$$\|[\delta(w)]\|_{L^{1,\mu}} = \inf_{r \in \mathscr{R}} \|\delta(w) + r\|_{L^{1,\mu}} \leqslant \|\delta\| \inf_{r \in \mathscr{R}} \|w + r\|_{LD} = \|\delta\| \, \|[w]\|_{LD} \,, \tag{4.22}$$

and the analogous argument applies to γ.

Thus we have the following commutative diagram

$$
\begin{array}{ccc}
L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3) & \xleftarrow{\quad\delta\quad} & LD(\Omega) \\[4pt]
{\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\[4pt]
L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R} & \xleftarrow{\;\delta/\mathscr{R}\;} & LD(\Omega)/\mathscr{R}.
\end{array}
\tag{4.23}
$$

The dual commutative diagram is

$$
\begin{array}{ccc}
L^{\infty,\mu}(\overline{\Omega}, \mathbb{R}^3) & \xrightarrow{\quad\delta^*\quad} & LD(\Omega)^* \\[4pt]
{\scriptstyle\pi^*}\Big\uparrow & & \Big\uparrow{\scriptstyle\pi^*} \\[4pt]
(L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R})^* & \xrightarrow{\;(\delta/\mathscr{R})^*\;} & (LD(\Omega)/\mathscr{R})^*.
\end{array}
\tag{4.24}
$$

In particular, the image under δ^* of an equilibrated force $F \in L^{\infty,\mu}(\overline{\Omega}, \mathbb{R}^3)$ is an equilibrated LD-force.

As the norm of a mapping and its dual are equal, we have[1]

$$
\begin{aligned}
\|\delta/\mathscr{R}\| = \|(\delta/\mathscr{R})^*\| &= \sup_{G \in (L^{1,\mu}(\overline{\Omega},\mathbb{R}^3)/\mathscr{R})^*} \frac{\|(\delta/\mathscr{R})^*(G)\|}{\|G\|} \\[6pt]
&= \sup_{G \in (L^{1,\mu}(\overline{\Omega},\mathbb{R}^3)/\mathscr{R})^*} \frac{\inf_{(\delta/\mathscr{R})^*(G) = (\varepsilon/\mathscr{R})^*(\sigma)} \|\sigma\|}{\|G\|} \,.
\end{aligned}
\tag{4.26}
$$

Using the fact that the two mappings π^* are isometric injections onto the respective subspaces of equilibrated forces, we may replace G above by an equilibrated force $F \in L^{\infty,\mu}(\overline{\Omega}, \mathbb{R}^3)$, and $(\delta/\mathscr{R})^*(G) = (\varepsilon/\mathscr{R})^*(\sigma)$ is replaced by $\delta^*(F) = \varepsilon^*(\sigma)$. Thus, we obtain

$$\|\delta/\mathscr{R}\| = \sup_F \frac{\inf_{\delta^*(F) = \varepsilon^*(\sigma)} \left\{ \operatorname{ess\,sup}_{i,m,x} \{|\sigma_{im}(x)|\} \right\}}{\operatorname{ess\,sup}_{i,x,y} \{|b_i(x)|, |t_i(y)|\}} \,, \tag{4.27}$$

---

[1]Note that we cannot use

$$\|\delta\| = \|\delta^*\| = \sup_{F \in L^{\infty,\mu}(\overline{\Omega},\mathbb{R}^3)} \frac{\|\delta^*(F)\|}{\|F\|^{L^{\infty,\mu}}} = \sup_F \frac{\inf_{\delta^*(F) = \varepsilon^*(\sigma)} \{\|\sigma\|\}}{\|F\|^{L^{\infty,\mu}}} \tag{4.25}$$

because ε^* is not surjective so there might be no σ satisfying the condition $\delta^*(F) = \varepsilon^*(\sigma)$.

over all equilibrated forces in $L^{\infty,\mu}(\overline{\Omega}, \mathbb{R}^3)$. Explicitly, the condition $\delta^*(F) = \varepsilon^*(\sigma)$ is the principle of virtual work

$$\int_{\Omega} b \cdot w \, \mathrm{d}V + \int_{\partial B} t \cdot w \, \mathrm{d}A = \int_{\Omega} \sigma \cdot \varepsilon(w) \, \mathrm{d}V \tag{4.28}$$

as anticipated and we conclude that

$$K = \|\delta/\mathscr{R}\| . \tag{4.29}$$

Remark 4.1. If we want to regard $\delta/\mathscr{R}$ as a mapping between function spaces we should use the decompositions of the respective spaces into Whitney sums. We already noted that $LD(\Omega)/\mathscr{R}$ is isometrically isomorphic to $LD(\Omega)_0$—the space of LD-vector fields having zero rigid components. Now $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)_0$ is bijective to $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R}$ but as a subspace of $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ it has a different norm (see Paragraph 2.4). Since we are interested in the quotient norm in order to use the essential supremum for the dual norm, we will endow $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)_0$ with the quotient norm $\|w_0\| = \inf_{r \in \mathscr{R}} \|w_0 - r\|_{L^{1,\mu}}$— which brings us back to the problem of best approximation by rigid velocity as described in the end of Paragraph 2.3. Thus, $\delta/\mathscr{R}$ becomes identical to the restriction

$$\delta_0 = \delta\big|_{LD(\Omega)_0} : LD(\Omega)_0 \longrightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)_0 \tag{4.30}$$

of δ to vector fields having zero rigid components. Its norm is given by

$$\|\delta/\mathscr{R}\| = \|\delta_0\| = \sup_{w_0 \in LD(\Omega)_0} \frac{\inf_{r \in \mathscr{R}} \left\{ \int_{\Omega} \sum_i |w_{0i} - r_i| \, \mathrm{d}V + \int_{\partial \Omega} \sum_i |w_{0i} - r_i| \, \mathrm{d}A \right\}}{\frac{1}{2} \int_{\Omega} \sum_{i,m} |w_{0i,m} + w_{0m,i}| \, \mathrm{d}V} .$$

$$\tag{4.31}$$

Again, one may use smooth vector fields to evaluate the supremum as these are dense in $LD(\Omega)$.

5. CONCLUDING REMARKS

In this section we emphasize some immediate consequences of the analysis presented above.

5.1. **Generalized stress concentration factors for surface forces.** The forgoing analysis may be simplified naturally to the case where only surface forces are applied to the body by making the following modifications.

- The body force is omitted,
- there is no need to use the measure μ and $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ is replaced by $L^1(\partial \Omega, \mathbb{R}^3)$,
- the extension δ is replaced by the trace mapping $\gamma \colon LD(\Omega) \to L^1(\partial \Omega, \mathbb{R}^3)$ (particularly in Subsection 4.5).

Thus, the generalized stress concentration factor is defined now as

$$K = \sup_{t \in L^{\infty}(\partial \Omega, \mathbb{R}^3)} \inf_{\sigma \in \Sigma_t} \frac{\operatorname{ess\,sup}_x \{|\sigma(x)|\}}{\operatorname{ess\,sup}_y \{|t(y)|\}}, \tag{5.1}$$

and the corresponding result is

$$K = \|\gamma/\mathscr{R}\| . \tag{5.2}$$

5.2. **Representation of Forces by** LD**-Functionals.** We note that δ is not surjective. However, we can state the following

Proposition 5.1. Image δ is dense in $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$.

Proof. We first show that $C^\infty(\overline{\Omega})$ is dense in $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$. Let $u \in L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ be an arbitrary field and $\varepsilon > 0$ an arbitrary positive number. The restriction $u|_{\partial\Omega}$ is in $L^1(\partial\Omega, \mathbb{R}^3)$ and may be approximated by a smooth mapping $u_\partial \colon \partial\Omega \to \mathbb{R}^3$ such that

$$\left\| u|_{\partial\Omega} - u_\partial \right\|_{L^1} < \frac{\varepsilon}{3}.$$

Now, u_∂ may be extended to a smooth field $\hat{u}_\partial$ that vanishes outside an arbitrarily chosen open neighborhood U of $\partial\Omega$ in $\overline{\Omega}$ such that

$$\int_\Omega |\hat{u}_\partial| \, \mathrm{d}V < \frac{\varepsilon}{3}. \tag{5.3}$$

We may also approximate $u|_\Omega$ by a smooth function u_0 having a compact support D in Ω such that $\left\| u|_\Omega - u_0 \right\|_{L^1} < \varepsilon/3$. Denoting the zero-extension of u_0 to $\overline{\Omega}$ by $\hat{u}_0$, set

$$u_\varepsilon = \hat{u}_\partial + \hat{u}_0.$$

Thus,

$$\begin{aligned}
\int_\Omega |u - u_\varepsilon| \, \mathrm{d}V &= \int_\Omega |u - \hat{u}_0 - \hat{u}_\partial| \, \mathrm{d}V \\
&\leqslant \int_\Omega |u - \hat{u}_0| \, \mathrm{d}V + \int_\Omega |\hat{u}_\partial| \, \mathrm{d}V \\
&< \frac{2\varepsilon}{3},
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
\int_{\partial\Omega} |u - u_\varepsilon| \, \mathrm{d}A &= \int_{\partial\Omega} |u - \hat{u}_0 - \hat{u}_\partial| \, \mathrm{d}A \\
&\leqslant \int_{\partial\Omega} |u - \hat{u}_\partial| \, \mathrm{d}A \\
&< \frac{\varepsilon}{3}.
\end{aligned} \tag{5.5}$$

Thus,

$$\|u - u_\varepsilon\|_{L^{1,\mu}} = \int_\Omega |u - u_\varepsilon| \, \mathrm{d}V + \int_{\partial\Omega} |u - u_\varepsilon| \, \mathrm{d}A < \varepsilon. \tag{5.6}$$

Now, the restrictions of smooth mappings on $\overline{\Omega}$ to Ω are dense in $LD(\Omega)$ and for each $u \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$,

$$\delta\left(u|_\Omega\right) = u.$$

Thus, the dense subset $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ is contained in Image δ which implies the assertion $\qquad\square$

Since the natural quotient projection $\pi \colon L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3) \to L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R}$ is surjective and continuous, and since $\delta/\mathscr{R} \circ \pi = \pi \circ \delta$, we have the following.

Corollary 5.2. The image of the mapping $\delta/\mathscr{R}$ is dense in $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R}$.

Proposition 5.3. The mappings δ^* and $(\delta/\mathscr{R})^*$ are injective.

This is implied immediately as the images of the corresponding maps are dense in the respective Banach spaces (see [10, p. 226]).

We conclude that δ^* and $(\delta/\mathscr{R})^*$ are embeddings of the spaces of forces represented by bounded body forces and surface forces into the corresponding spaces of bounded functionals on $LD(\Omega)$ and $LD(\Omega)/\mathscr{R}$.

Combining this with the representation of LD-functionals by stresses as in Subsection 4.3 we obtain the representation of equilibrated forces in $(L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R})^*$ by stresses, i.e., for any equilibrated force $F \in (L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R})^*$, there exists some stress field $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$ such that

$$\delta^*(F) = \varepsilon^*(\sigma). \tag{5.7}$$

5.3. **Optimal stresses for given loadings.** Let $F = (b, t) \in L^{\infty,\mu}(B, \mathbb{R}^3)$ be an equilibrated force so that there is an $F_0 \in (L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)/\mathscr{R})^*$ such that $\pi^*(F_0) = F$. Then,

$$\pi^* \circ (\delta/\mathscr{R})^*(F_0) = \delta^* \circ \pi^*(F_0), \tag{5.8}$$

so $\pi^*((\delta/\mathscr{R})^*(F_0)) = \delta^*(F)$, and as π^* is norm preserving, we get $\|(\delta/\mathscr{R})^*(F_0)\| = \|\delta^*(F)\|$.

Using the representation of LD-functionals by stresses an in Subsection 4.3 and in [8], we conclude that

$$\|\delta^*(F)\| = \inf_{F = (\varepsilon/\mathscr{R})^*(\sigma)} \||\sigma|\|_{L^\infty}. \tag{5.9}$$

In addition, the infimum is attained by some optimal stress field $\hat{\sigma} \in L^\infty(\Omega, \mathbb{R}^6)$ so

$$\|\delta^*(F)\| = \||\hat{\sigma}|\|_{L^\infty}. \tag{5.10}$$

In other words, for the optimal stress field $\hat{\sigma}$

$$\operatorname*{ess\,sup}_{x \in \Omega} \{|\hat{\sigma}|\} = \|\delta^*(F)\| \tag{5.11}$$

Using

$$\delta^*(F)(w) = F(\delta(w)) = \int_\Omega b \cdot w \, \mathrm{d}V + \int_{\partial\Omega} t \cdot \bar{w} \, \mathrm{d}A, \tag{5.12}$$

where for simplicity we use $\bar{w} = \gamma(w)$, we obtain

$$\operatorname*{ess\,sup}_{x \in \Omega} \{|\hat{\sigma}|\} = \sup_{w \in LD(\Omega)} \frac{\left| \int_\Omega b \cdot w \, \mathrm{d}V + \int_{\partial\Omega} t \cdot \bar{w} \, \mathrm{d}A \right|}{\|w\|_{LD}}. \tag{5.13}$$

Note that we may calculate the supremum using smooth fields w due to the fact that they are dense in $LD(\Omega)$. Since the trace mapping is just the restriction

for such smooth mappings, we may replace the expression for the maximum of the optimal stress by

$$\operatorname*{ess\,sup}_{x\in\Omega}\{|\hat{\sigma}|\} = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\left|\int\limits_\Omega b\cdot w\,\mathrm{d}V + \int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A\right|}{\|\pi_{\mathscr{R}}(w)\| + \int\limits_\Omega |\varepsilon(w)|\,\mathrm{d}V}. \tag{5.14}$$

Equilibrium of the external forces implies that the numerator is invariant under the addition of a rigid velocity field and the same holds for $\varepsilon(w)$. Thus, the supremum is attained for a velocity field that satisfies $\pi_{\mathscr{R}}(w) = 0$ and finally

$$\operatorname*{ess\,sup}_{x\in\Omega}\{|\hat{\sigma}|\} = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\left|\int\limits_\Omega b\cdot w\,\mathrm{d}V + \int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A\right|}{\int\limits_\Omega |\varepsilon(w)|\,\mathrm{d}V}. \tag{5.15}$$

APPENDIX A. ELEMENTARY PROPERTIES OF QUOTIENT SPACES

We describe below some elementary properties of quotient spaces of normed spaces (e.g., [10, p. 227]).

A.1. **The quotient norm.** Let $\mathbf{W}$ be a normed vector space with a norm $\|\cdot\|$ and $\mathscr{R}$ a closed subspace of $\mathbf{W}$ (e.g., a finite dimensional subspace). Then, the quotient norm $\|\cdot\|_0$ is defined on $\mathbf{W}/\mathscr{R}$ by

$$\|w_0\|_0 = \inf_{w\in w_0} \|w\|. \tag{A.1}$$

Denoting by $\pi\colon \mathbf{W} \longrightarrow \mathbf{W}/\mathscr{R}$ the natural linear projection $\pi(w) = [w]$, we clearly have

$$\|\pi(w)\|_0 = \|\pi(w+r)\|_0 = \inf_{r\in\mathscr{R}} \|w+r\|,$$

for any $r\in\mathscr{R}$. The quotient norm makes the projection mapping π continuous and the topology it generates on the quotient space is equivalent to quotient topology.

A.2. **Dual spaces.** We note that as the projection π is surjective, its dual mapping

$$\pi^*\colon (\mathbf{W}/\mathscr{R})^* \longrightarrow \mathbf{W}^* \tag{A.2}$$

is injective. Clearly, it is linear and continuous relative to the dual norms. If $\phi \in$ Image π^* so $\phi = \pi^*(\phi_0)$, $\phi_0 \in (\mathbf{W}/\mathscr{R})^*$, then, for each $r\in\mathscr{R}$,

$$\begin{aligned} \phi(r) &= \pi^*(\phi_0)(r) \\ &= \phi_0(\pi(r)) \\ &= \phi_0(0) \\ &= 0. \end{aligned} \tag{A.3}$$

On the other hand, if for $\phi \in \mathbf{W}^*$, $\phi(r) = 0$ for all $r\in\mathscr{R}$, then, we may define $\phi_0 \in (\mathbf{W}/\mathscr{R})^*$ by $\phi_0(w_0) = \phi(w)$, for some $w\in\mathbf{W}$ such that $\pi(w) = w_0$. The choice of $w \in w_0$ is immaterial because $\phi(w+r) = \phi(w) + \phi(r) = \phi(w)$, for any $r\in\mathscr{R}$. We conclude that

$$\text{Image } \pi^* = \mathscr{R}^\perp = \{\phi \in \mathbf{W}^* \mid \phi(r) = 0 \text{ for all } r\in\mathscr{R}\}.$$

Next we consider the dual norm of elements of the dual to the quotient space. For $\phi_0 \in (\mathbf{W}/\mathscr{R})^*$, we have

$$\|\phi_0\| = \sup_{w_0 \in \mathbf{W}/\mathscr{R}} \frac{|\phi_0(w_0)|}{\|w_0\|_0}. \tag{A.4}$$

Thus,

$$
\begin{aligned}
\|\phi_0\| &= \sup_{w_0 \in \mathbf{W}/\mathscr{R}} \left\{ \frac{|\phi_0(\pi(w))|}{\inf_{r \in \mathscr{R}} \|w + r\|} \,\middle|\, \text{for some } w \in w_0 \right\} \\
&= \sup_{w_0 \in \mathbf{W}/\mathscr{R}} \left\{ \sup_{r \in \mathscr{R}} \frac{|\pi^*(\phi_0)(w)|}{\|w + r\|} \,\middle|\, \text{for some } w \in w_0 \right\} \\
&= \sup_{w_0 \in \mathbf{W}/\mathscr{R}} \left\{ \sup_{r \in \mathscr{R}} \frac{|\pi^*(\phi_0)(w + r)|}{\|w + r\|} \,\middle|\, \text{for some } w \in w_0 \right\} \\
&= \sup_{w_0 \in \mathbf{W}/\mathscr{R}} \left\{ \sup_{w' \in w_0} \frac{|\pi^*(\phi_0)(w')|}{\|w'\|} \right\} \\
&= \sup_{w' \in \mathbf{W}} \frac{|\pi^*(\phi_0)(w')|}{\|w'\|} \\
&= \|\pi^*(\phi_0)\|.
\end{aligned}
\tag{A.5}
$$

We conclude that π^* is norm preserving.

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