OPTIMIZATION FOR THE BALANCE EQUATIONS

REUVEN SEGEV

ABSTRACT. We present a framework for the analysis of optimization problems associated with balance equations on a given region Ω. As balance equations do not have unique solutions, we look for solutions of minimum L^p-norms, in particular, minimum L^∞-norm. Letting the data vary, we also look for the largest ratio K between the norm of the optimal solution and the norm of the data fields. This largest ratio is a purely geometric property of the region Ω. Among other examples, we show that for an elastic perfectly plastic body there is a maximal positive number C, the *load capacity ratio,* such that the body will not collapse under any pair (t, b), containing an external traction field t and a body force field b, if they are bounded by CY_0, where Y_0 is the yield stress. We also give expressions for K and C in terms of quantities that are analogous to the norm of the trace mapping for Sobolev spaces.

1. INTRODUCTION

The balance equations of various quantities in continuum physics may be written as

$$\nabla \cdot \sigma + b = 0 \quad \text{in } \Omega, \quad \sigma(\nu) = t, \quad \text{on } \partial\Omega, \tag{1.1}$$

where b is given in a region Ω, ν is the unit normal to the boundary, and the boundary condition t is given on $\partial\Omega$. The tensor field σ is the unknown and the system is under-determined. Thus, for the given data (t, b), there is a family $\Sigma_{(t,b)}$ of solutions to the problem. The weak version of balance equations may be written in the form

$$\int_\Omega \sigma(D(w)) = \int_{\partial\Omega} t \cdot w + \int_\Omega b \cdot w \tag{1.2}$$

for all vector fields w on Ω belonging to a certain function space, where D is some differential operator. Writing the integral operators as linear functionals on the space of fields w and keeping the same notation for

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the linear functionals (with an abuse of notation), the balance equation above may be rewritten as $\sigma(D(w)) = (t,b)(\delta(w))$, for all w, where $\delta(w) = (w|_{\partial\Omega}, w)$. This may be rewritten using the definition of dual mappings as $D^*(\sigma) = \delta^*(t,b)$. The last form of the balance equation for σ emphasizes that the solutions are given in terms of a generalized inverse D^{*+} of the differential operator D^*.

Since the solution is not unique, one may consider an optimization problem where, for given data (t,b), we look among all $\sigma \in \Sigma_{(t,b)}$ for the solution that minimizes a certain physically meaningful norm $\|\cdot\|$. Thus, we set

$$\sigma^{\text{opt}}_{(t,b)} = \inf_{D^*(\sigma)=\delta^*(t,b)} \|\sigma\|. \tag{1.3}$$

Usually, generalized inverses are constructed so that they minimize the L^2-norm. For many applications in engineering, the L^∞-norm, interpreted as the maximum of the field σ, is more relevant and we consider it in the examples below.

Next, we consider the ratio

$$K_{(t,b)} = \frac{\sigma^{\text{opt}}_{(t,b)}}{\|(t,b)\|} \tag{1.4}$$

for some norm on the space of data $\{(t,b)\}$. We refer to $K_{(t,b)}$ as the *optimal stress concentration factor* (see [7, 8] for explanation of the terminology). Finally, we consider the *generalized stress concentration factor*

$$K = \sup_{(t,b)} K_{(t,b)} = \sup_{(t,b)} \frac{\sigma^{\text{opt}}_{(t,b)}}{\|(t,b)\|}. \tag{1.5}$$

It is noted that by its definition, K is a purely geometric property of Ω.

Following our previous work [7, 8, 5, 9], we present here a framework where the analysis of the quantities defined above may be carried out. We also exhibit the results for some particular cases. The following result that we derive in the last section seems to us to be of interest for stress analysis.

Theorem 1.1. *Let Ω be a region occupied by an elastic-perfectly plastic body whose yield stress is Y_0 and such that the yield criterion is given in terms of a norm $|\cdot|$ applied to the stress deviator. Assume that the body is supported on an open subset of its boundary. Then, there is a maximal positive number C to which we refer as the* load capacity ratio *of Ω, such that the body will not collapse under any pair of essentially bounded force fields (t,b) as long as the essential suprema of their magnitudes are less than or equal to CY_0. The number C satisfies*

$$\frac{1}{C} = \sup_{w\in LD(\Omega)_D} \frac{\int_{\partial\Omega}|\gamma(w)| + \int_\Omega |w|}{\int_\Omega |\varepsilon(w)|}. \tag{1.6}$$

Here, γ is the trace mapping, i.e., $\gamma(w)$ is the boundary value of the vector field w so for a continuous vector field u defined on $\overline{\Omega}$, $\gamma(u|_{\Omega}) = u|_{\partial\Omega}$; $LD(\Omega)_D$ is the collection of incompressible integrable vector fields w for which $\varepsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^{\mathsf{T}})$, the corresponding linear strain, is integrable, and $|\varepsilon(w)(x)|$ is evaluated using the dual norm to that used for the yield criterion.

2. Balance Equations

2.1. **Jets for trivial vector bundles.** It will be convenient in the sequel to regard fields as sections of trivial vector bundles and we adopt some of the notation scheme used by Palais [4]. Thus, we consider a trivial vector bundle $\xi : E = \Omega \times \mathbf{V} \to \Omega$ where the fiber $\mathbf{V}$ is n-dimensional and the base manifold Ω is a bounded open subset of $\mathbb{R}^m$ having a smooth boundary. Obviously, a section w of ξ is uniquely represented by a mapping $\Omega \to \mathbf{V}$. It is assumed that $\mathbf{V}$ has a Riemannian structure $\langle \cdot, \cdot \rangle$. In fact, in all the examples we present, $\mathbf{V}$ is taken as $\mathbb{R}^n$. The k-th jet bundle of ξ is denoted by $J^k(\xi)$. The k-jet bundle is a vector bundle over Ω, and for each C^k-section $w\colon \Omega \to E$, we have a section $j_k(w)$ of $J^k(\xi)$—the k-jet extension of w. In general, denoting by $C^k(\xi)$ the vector space of C^k-sections of the vector bundle ξ, we have $j_k\colon C^k(\xi) \to C^0(J^k(\xi))$, a linear mapping. The jet extension $j_k(w)$ is represented by the collection of tensor fields $(w, \nabla w, \cdots, \nabla^k w)$. It is assumed that a norm $|\cdot|$ is given on $J^k(\xi)_x$. The dual vector space $J^k(\xi)^*_x$ is the fiber of the dual jet bundle $J^k(\xi)^*$. The same notation, $|\cdot|$, is used for the dual norm on $J^k(\xi)^*_x$.

For $1 < p < \infty$, we set as usual $p' = p/(p-1)$ with the extension $p' = \infty$ for $p = 1$. A measurable section χ of $J^k(\xi)$ is L^p if

$$\|\chi\|_p = \left(\int_{\Omega} |\chi(x)|^p \right)^{1/p} < \infty. \tag{2.1}$$

For L^p-sections, $\|\cdot\|$ is indeed a norm. The space of L^p-sections of $J^k(\xi)$ is denoted as $L^p(J^k(\xi))$ and it is a Banach space with the norm above. We have the duality relation $L^p(J^k(\xi))^* = L^{p'}(J^k(\xi)^*)$. An $L^{p'}$-section $\widehat{\sigma}$ of $J^k(\xi)^*$ represents the continuous linear functional $\sigma \in L^p(J^k(\xi))^*$ in the form $\sigma(\chi) = \int_{\Omega} \widehat{\sigma}(x)(\chi(x))$ and in the sequel we will identify σ and $\widehat{\sigma}$ and omit the "$\widehat{\ }$".

The Sobolev space $W^p_k(\xi)$ contains the sections w of ξ such that $j_k(w) \in L^p(J^k(\xi))$. For any $w \in W^p_k(\xi)$, $\|w\| = \|j_k(w)\|_p$ is a norm for which $W^p_k(\xi)$ is a Banach space. Clearly, this choice of norm makes the jet mapping

$$j_k\colon W^p_k(\xi) \longrightarrow L^p(J^k(\xi)) \tag{2.2}$$

an isometric injection.

2.2. **Forces and stresses.** We will refer to an element F of $W^p_k(\xi)^*$ as a W^p_k-force and to an element σ of $L^{p'}(J^k(\xi)^*)$ as a (p,k)-stress. The standard representation of elements of the duals to the Sobolev spaces (e.g., [3, pp. 25–26]) is interpreted here physically as a representation theorem of forces by stresses. For each $F \in W^p_k(\xi)^*$, $\sigma_0 = F \circ j_k^{-1} \colon \mathrm{Image}\, j_k \to \mathbb{R}$ is a bounded linear functional with $\sigma_0 \circ j_k = F$. As the jet mapping is isometric, $\|\sigma_0\| = \|F\|$. Thus, using the Hahn-Banach theorem, one may extend σ_0 from $\mathrm{Image}\, j_k$ to $L^p(J^k(\xi))$ giving an element $\sigma \in L^{p'}(J^k(\xi)^*)$, with $\sigma \circ j_k = F$, or equivalently, $F = j_k^*(\sigma)$.

The last equation, representing the condition that a stress σ represents the force F, is a generalization of the force balance equation in mechanics. In fact, it may be written as $F(w) = \sigma(j_k(w))$, a generalization of the principle of virtual work to k-th order continuum mechanics.

2.3. **Stress optimization.** The stress representation procedure described above implies that if, $\|\sigma\| = \|\sigma_0\| = \|F\|$ which is possible by the Hahn-Banach theorem, then,

$$\|\sigma\| = \|F\| = \inf\left\{\|\sigma'\| \mid F = j_k^*(\sigma'), \sigma' \in L^{p'}(J^k(\xi)^*)\right\}. \tag{2.3}$$

It follows that $\|F\|$ is the least norm that a stress that represents the force F may have.

2.4. **Traces and loading distributions.** The trace mapping, $\gamma \colon W^p_k(\xi) \to W^p_{k_\partial}(\xi|_{\partial\Omega})$, where $k_\partial = k - 1/p$ is the reduced differentiability, assigns the boundary values to Sobolev sections. It is a well defined bounded linear mapping (see Palais [4, p. 27]) and for any $u \in C^\infty(\overline{\Omega})$ it satisfies $\gamma(u|_\Omega) = u|_{\partial\Omega}$. (It is noted that as ξ is a trivial bundle over $\Omega \subset \mathbb{R}^m$, $\xi|_{\partial\Omega}$ is well defined.)

Since we have the embedding $\iota_{k_\partial} \colon W^p_k(\xi) \hookrightarrow W^p_{k_\partial}(\xi)$, we set

$$\delta = (\gamma, \iota_{k_\partial}) \colon W^p_k(\xi) \longrightarrow W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi). \tag{2.4}$$

Clearly, δ is a bounded linear mapping where for $(u,v) \in W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi)$, $\|(u,v)\| = \|u\| + \|v\|$. We will refer to an element $t \in W^p_{k_\partial}(\xi|_{\partial\Omega})^*$ as a surface force, to an element $b \in W^p_{k_\partial}(\xi)^*$ as a body force, and to a pair $(t,b) \in \left(W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi)\right)^*$ as a loading pair.

We conclude that for a loading pair (t,b), we have a representation by a force $F = \delta^*(t,b)$, where $\delta^* \colon \left(W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi)\right)^* \to W^p_k(\xi)^*$ is the dual mapping.

2.5. **Loading distributions and stresses.** The framework we presented above can be described by the following diagram

$$\begin{array}{ccccc} W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi) & \xleftarrow{\delta} & W^p_k(\xi) & \xrightarrow{j_k} & L^p(J^k(\xi)) \\ \\ \left(W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi)\right)^* & \xrightarrow{\delta^*} & W^p_k(\xi)^* & \xleftarrow{j^*_k} & L^{p'}(J^k(\xi)^*). \end{array} \tag{2.5}$$

From the analysis above we conclude the following

Theorem 2.1. *Let* $(t,b) \in \left(W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi)\right)^*$ *be a loading pair, then, the following assertions hold.* *(i)* Existence of stresses. *There is a subspace* $\Sigma_{(t,b)} \subset L^{p'}(J^k(\xi)^*)$ *of stresses that represent the loading pair such that for* $\sigma \in \Sigma_{(t,b)}$, $\delta^*(t,b) = j^*_k(\sigma)$. *(ii)* Optimal stresses. *Let* $\sigma^{\text{opt}}_{(t,b)} = \inf_{\sigma\in\Sigma_{(t,b)}}\{\|\sigma\|\}$. *Then, there is a stress* $\widehat{\sigma} \in \Sigma_{(t,b)}$ *such that* $\sigma^{\text{opt}}_{(t,b)} = \|\widehat{\sigma}\|$. *(iii)* The expression for $\sigma^{\text{opt}}_{(t,b)}$. *The optimum satisfies*

$$\sigma^{\text{opt}}_{(t,b)} = \|\delta^*(t,b)\| = \sup_{w\in W^p_k(\xi)} \frac{|t(\gamma(w)) + b(w)|}{\|j_k(w)\|_p}. \tag{2.6}$$

(iv) The generalized stress concentration factor. *Let the generalized stress concentration factor be defined by*

$$K = \sup_{(t,b)} \frac{\sigma^{\text{opt}}_{(t,b)}}{\|(t,b)\|}, \quad (t,b) \in \left(W^p_{k_\partial}(\xi|_{\partial\Omega}) \times W^p_{k_\partial}(\xi)\right)^*. \tag{2.7}$$

Then, $K = \|\delta\|$.

Proof. There remains to prove (*iv*). Using Equation (2.6), we have $K = \|\delta^*\| = \|\delta\|$. □

2.6. **The junction problem for fluxes.** In this simplest example we consider the case where ξ is the trivial line bundle $\Omega \times \mathbb{R}$ and the Sobolev space is $W^1_1(\Omega)$. The various spaces are shown in the diagram below. Sections of $J^1(\Omega\times\mathbb{R})$ are identified with mappings $\Omega \to \mathbb{R}^{m+1}$ regarded as pairs $(\varphi, v) \in L^1(\Omega, \mathbb{R}\times\mathbb{R}^m)$, and $j_1(\varphi)$ may be identified with $(\varphi, \nabla\varphi)$.

$$\begin{array}{ccccc} L^1(\partial\Omega)\times L^1(\Omega) & \xleftarrow{\delta} & W^1_1(\Omega) & \xrightarrow{j_1} & L^1(\Omega,\mathbb{R}^{m+1}) \\ \\ \left(L^1(\partial\Omega)\times L^1(\Omega)\right)^* & \xrightarrow{\delta^*} & W^1_1(\Omega)^* & \xleftarrow{j^*_1} & L^1(\Omega,\mathbb{R}^{m+1})^* \\ \Vert & & & & \Vert \\ L^\infty(\partial\Omega)\times L^\infty(\Omega) & & & & L^\infty(\Omega,\mathbb{R}^{m+1}). \end{array} \tag{2.8}$$

One may think of this framework as a continuous logistics problem in the region Ω. For some extensive property, e.g., the mass of a certain material or the thermal energy, we regard the loading pair $(t, b) \in L^\infty(\partial\Omega) \times L^\infty(\Omega)$ as prescribed boundary flux distribution and rate of change of density. These are balanced by a stress object, a pair $(\sigma_0, \sigma_1) \in L^\infty(\Omega) \times L^\infty(\Omega, \mathbb{R}^m)$, where σ_1 is interpreted as a flow field and σ_0 is interpreted as the production rate distribution of the property in Ω.

Clearly, the condition $\delta^*(t, b) = j_1^*(\sigma_0, \sigma_1)$ is the weak formulation of the boundary value problem

$$\sigma_{1i,i} + b = \sigma_0 \quad \text{in}\, \Omega, \quad \text{and} \quad \sigma_{1i}\nu_i = t \quad \text{on}\, \partial\Omega \tag{2.9}$$

for differentiable fields σ_1.

We conclude that Theorem 2.1 implies the following. (i) *Existence of flow objects*. Given a pair $(t, b) \in L^\infty(\partial\Omega) \times L^\infty(\Omega)$ consisting of a boundary flux and a density rate distributions, there is a subspace $\Sigma_{(t,b)} \subset L^\infty(\Omega) \times L^\infty(\Omega, \mathbb{R}^m)$ containing pairs of essentially bounded production rates and flow vector fields for which the weak version of the balance equations (2.9) holds. It is noticed that we did not use any form of Cauchy's flux or stress existence theorem other than Theorem (2.1,i). (ii) *Optimal stresses*. Let

$$\sigma^{\text{opt}}_{(t,b)} = \inf_{\sigma \in \Sigma_{(t,b)}} \left\{ \operatorname*{ess\,sup}_{x \in \Omega} \{ |\sigma_0(x)|, |\sigma_1(x)| \} \right\}, \tag{2.10}$$

where $|\sigma_1(x)|$ is any particular norm on $\mathbb{R}^m$ used for the values of the flow vector field. Then, there is a pair $(\widehat{\sigma}_0, \widehat{\sigma}_1) \in \Sigma_{(t,b)}$ such that

$$\sigma^{\text{opt}}_{(t,b)} = \operatorname*{ess\,sup}_{x \in \Omega} \{ |\widehat{\sigma}_0(x)|, |\widehat{\sigma}_1(x)| \}. \tag{2.11}$$

(iii) *The expression for* $\sigma^{\text{opt}}_{(t,b)}$. The optimum satisfies

$$\sigma^{\text{opt}}_{(t,b)} = \sup_{\varphi} \frac{\left| \int_{\partial\Omega} t\varphi + \int_\Omega b\varphi \right|}{\int_\Omega (|\varphi| + |\nabla\varphi|)}, \quad \varphi \in C^\infty(\overline{\Omega}),$$

where for computing $|\nabla\varphi|$ we use the norm on $\mathbb{R}^m$ that is dual to the one used for the flow vector $\sigma_1(x)$. (iv) *The generalized stress concentration factor*. The generalized stress concentration factor, or rather flow amplification factor for our interpretation, is defined as

$$K = \sup_{(t,b)} \frac{\sigma^{\text{opt}}_{(t,b)}}{\operatorname{ess\,sup}_{x,y} \{ |t(y)|, |b(x)| \}}, \tag{2.12}$$

$(t, b) \in L^\infty(\partial\Omega) \times L^\infty(\Omega)$. Then, as $C^\infty(\overline{\Omega})$ is dense in the Sobolev spaces,

$$K = \|\delta\| = \sup_{\varphi \in C^\infty(\overline{\Omega})} \frac{\int_\Omega |\varphi| + \int_{\partial\Omega} |\varphi|}{\int_\Omega (|\varphi| + |\nabla\varphi|)}. \tag{2.13}$$

3. Other Differential Operators

3.1. **Other differential operators.** In our principal example, the differential operator considered is not the jet mapping, and as a consequence, the space we consider is not a Sobolev space. Thus we consider the following situation.

Let $\mathscr{D}'(\xi)$ denote the space of Schwartz-distribution sections (e.g., [6, Chapter 3]) of ξ. Let $D\colon \mathscr{D}'(\xi) \to \mathscr{D}'(J^k(\xi))$ be a k-th order linear differential operator where all derivatives are taken in the distributional sense. Consider the space $\mathbf{W} = D^{-1}\left\{L^p(J^k(\xi))\right\}$. Assume that $\|w\| = \|D(w)\|_p$, $w \in \mathbf{W}$, is a norm on $\mathbf{W}$. Then, $\mathbf{W}$ is endowed with this norm and so $D\colon \mathbf{W} \to L^p(J^k(\xi))$ is a linear isometric injection (as injectivity is implied by isometry).

3.2. **Example: The space $LD(\Omega)$.** Let Ω be a subset of $\mathbb{R}^3$, and let $\mathscr{R}$ be the 6-dimensional space of rigid vector fields $\Omega \to \mathbb{R}^3$, i.e., the collection of vector fields of the form $w(x) = a + b \times x$, $a, b \in \mathbb{R}^3$. Regarding vector fields as sections of the trivial bundle $\Omega \times \mathbb{R}^3$, clearly, $\mathscr{R} \subset L^1(\xi)$.

Let $\pi_{\mathscr{R}}\colon \mathscr{D}'(\xi) \longrightarrow \mathscr{R}$ be a linear projection. (For examples of such projections, see [14, 8].) Thus, identifying $J^1(\xi)$ with $\Omega \times \mathbb{R}^3 \times \mathbb{R}^9$, we use the linear strain mapping $\varepsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^{\mathsf{T}})$ for any distribution-section w of ξ, and consider the linear differential operator $(\pi_{\mathscr{R}}, \varepsilon)\colon \mathscr{D}'(\xi) \to \mathscr{R} \times L^1(\Omega, \mathbb{R}^6)$. Here we identify the space of symmetric 3×3 matrices with $\mathbb{R}^6$ and regard it as a subspace of all 3×3 matrices which is identified with $\mathbb{R}^9$. Temam [14] showed that $\|w\| = \|\pi_{\mathscr{R}}(w)\|_1 + \|\varepsilon(w)\|_1$ is a norm and the space $(\pi_{\mathscr{R}}, \varepsilon)^{-1}\left\{L^1(J^1(\xi))\right\}$ of integrable stretchings (linear strains) is denoted by $LD(\Omega)$. Thus,

$$(\pi_{\mathscr{R}}, \varepsilon)\colon LD(\Omega) \longrightarrow \mathscr{R} \times L^1(\Omega, \mathbb{R}^6) \subset L^1(J^1(\xi)) \tag{3.1}$$

is a norm-preserving linear injection. One of the basic properties (loc. cit.) of the space $LD(\Omega)$ is that the trace mapping $\gamma\colon LD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^3)$ is a well defined bounded operator. As a result, we can use $\delta\colon LD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^3) \times L^1(\Omega, \mathbb{R}^3)$ in the setting described above.

4. Quotient Spaces

In case D is not injective, $\|w\| = \|D(w)\|$, $w \in \mathbf{W} = D^{-1}\left\{L^p(J^k(\xi))\right\}$ is not a norm on $\mathbf{W}$. However, we may form the quotient space $\mathbf{W}/\operatorname{Kernel} D$ and consider $E = D/\operatorname{Kernel} D\colon \mathbf{W}/\operatorname{Kernel} D \to L^p(J^k(\xi))$. Thus, E is an injection so $\|w\| = \|E(w)\|$, is a norm on $\mathbf{W}/\operatorname{Kernel} D$.

Thus, we may extend the definition of a force to be an element F of $(\mathbf{W}/\operatorname{Kernel} D)^*$. We conclude that the representation of force by stresses as in Subsection 2.2 and the optimization result in Subsection 2.3 apply here too. Hence, any given force $F \in (\mathbf{W}/\operatorname{Kernel} D)^*$ is represented by some stress $\sigma \in L^{p'}(J^k(\xi)^*)$ in the form $F = E^*(\sigma)$, and $\|F\| = \inf\{\|\sigma\|\}$ over all $\sigma \in L^{p'}(J^k(\xi)^*)$ such that $F = E^*(\sigma)$.

4.1. **Example: The space of *LD*-distortions.** Instead of the differential operator $(\pi_{\mathscr{R}}, \varepsilon)$, we consider $(0, \varepsilon)\colon LD(\Omega) \to L^1(J^1(\xi))$ which we will often write simply as ε. Clearly, ε is not injective. In fact, it follows from a theorem by Liouville (see [14, pp. 18–19]) that $\operatorname{Kernel} \varepsilon = \mathscr{R}$—the space of rigid fields. We refer to an element $u \in LD(\Omega)/\mathscr{R}$ as a *distortion*. Thus, we have the following diagram

$$\begin{array}{ccc} LD(\Omega) & \xrightarrow{\varepsilon} & L^1(\Omega, \mathbb{R}^6) \\ \downarrow \pi & & \| \\ LD(\Omega)/\mathscr{R} & \xrightarrow{\varepsilon/\mathscr{R}} & L^1(\Omega, \mathbb{R}^6), \end{array} \tag{4.1}$$

where π is the natural projection on the quotient and with some abuse of notation we restricted the co-domain of ε and $\varepsilon/\mathscr{R}$ to the symmetric tensor fields. Clearly, $\varepsilon/\mathscr{R}$ is an isometric injection.

4.2. **Equilibrated forces.** While for a vector space $\mathbf{W}$ of sections of ξ, the space $LD(\Omega)$ in the current example, the elements of $\mathbf{W}^*$ are interpreted as forces, the elements of $(\mathbf{W}/\mathscr{R})^*$, $(LD(\Omega)/\mathscr{R})^*$ in the present example, are interpreted as equilibrated forces. In fact we have the sequences

$$\begin{array}{ccccc} \mathscr{R} & \xrightarrow{\iota} & \mathbf{W} & \xrightarrow{\pi} & \mathbf{W}/\mathscr{R} \\ \mathscr{R}^* & \xleftarrow{\iota^*} & \mathbf{W}^* & \xleftarrow{\pi^*} & (\mathbf{W}/\mathscr{R})^*. \end{array} \tag{4.2}$$

where ι is the inclusion mapping. We note that $\operatorname{Image} \iota = \operatorname{Kernel} \pi$ and similarly, $\operatorname{Image} \pi^* = \operatorname{Kernel} \iota^*$. For a force $F \in \mathbf{W}^*$, $\iota^*(F)$ is interpreted the the total (or resultant) force. Thus, elements of $\operatorname{Image} \pi^* = \operatorname{Kernel} \iota^*$ are interpreted as equilibrated forces. Recalling that π^* is a norm preserving injection relative to the quotient norm (see Taylor [10, p. 227]), we may identify the equilibrated forces with elements of $(\mathbf{W}/\mathscr{R})^*$.

4.3. **Stresses for unsupported bodies under equilibrated loadings.** The dual counterpart of the diagram (4.1) is

$$\begin{array}{ccc} LD(\Omega)^* & \xleftarrow{\varepsilon^*} & L^\infty(\Omega, \mathbb{R}^6) \\ \uparrow \pi^* & & \| \\ (LD(\Omega)/\mathscr{R})^* & \xleftarrow{(\varepsilon/\mathscr{R})^*} & L^\infty(\Omega, \mathbb{R}^6), \end{array} \tag{4.3}$$

where $(LD(\Omega)/\mathscr{R})^*$ represents the collection of equilibrated forces.

On the other hand, as the trace mapping is well defined for $LD(\Omega)$, we have

$$\begin{array}{ccc} L^1(\partial\Omega,\mathbb{R}^3)\times L^1(\Omega,\mathbb{R}^3) & \xleftarrow{\ \delta\ } & LD(\Omega) \\ \pi\big\downarrow & & \big\downarrow\pi \\ (L^1(\partial\Omega,\mathbb{R}^3)\times L^1(\Omega,\mathbb{R}^3))/\mathscr{R} & \xleftarrow{\ \delta/\mathscr{R}\ } & LD(\Omega)/\mathscr{R} \end{array} \tag{4.4}$$

where $\delta/\mathscr{R}$ makes the diagram commutative. The dual diagram is

$$\begin{array}{ccc} L^\infty(\partial\Omega,\mathbb{R}^3)\times L^\infty(\Omega,\mathbb{R}^3) & \xrightarrow{\ \delta^*\ } & LD(\Omega)^* \\ \pi^*\big\uparrow & & \big\uparrow\pi^* \\ ((L^1(\partial\Omega,\mathbb{R}^3)\times L^1(\Omega,\mathbb{R}^3))/\mathscr{R})^* & \xrightarrow{\ (\delta/\mathscr{R})^*\ } & (LD(\Omega)/\mathscr{R})^*. \end{array} \tag{4.5}$$

Regarding elements of $((L^1(\partial\Omega,\mathbb{R}^3)\times L^1(\Omega,\mathbb{R}^3))/\mathscr{R})^*$ as equilibrated loading pairs, we may apply the basic result to this situation as follows. (*i*) *Existence of stresses*. Given an equilibrated loading pair $(t,b)\in L^\infty(\partial\Omega,\mathbb{R}^3)\times L^\infty(\Omega,\mathbb{R}^3)$, there is a subspace $\Sigma_{(t,b)}\subset L^\infty(\Omega,\mathbb{R}^6)$ containing essentially bounded symmetric stress tensor fields σ that represent the loading pair by $\delta^*(t,b)=\pi^*\circ(\varepsilon/\mathscr{R})^*(\sigma)$. The condition that σ represents the loading pair (t,b) is the principle of virtual work

$$\int_{\partial\Omega} t\cdot\gamma(w)+\int_{\Omega} b\cdot w=\int_{\Omega}\sigma_{ij}\varepsilon(w)_{ij}. \tag{4.6}$$

Again, it is noticed that we did not use any form of Cauchy's stress existence theorem. (*ii*) *Optimal stresses*. Let

$$\sigma^{\text{opt}}_{(t,b)}=\inf_{\sigma\in\Sigma_{(t,b)}}\left\{\operatorname*{ess\,sup}_{x\in\Omega}\{|\sigma(x)|\}\right\}, \tag{4.7}$$

where $|\sigma(x)|$ is any particular norm on $\mathbb{R}^6$ used for the values of the stress field. Then, there is a stress field $\widehat{\sigma}\in\Sigma_{(t,b)}$ such that $\sigma^{\text{opt}}_{(t,b)}=\operatorname{ess\,sup}_{x\in\Omega}\{|\widehat{\sigma}(x)|\}$. The optimum satisfies

$$\sigma^{\text{opt}}_{(t,b)}=\sup_{w}\frac{\left|\int_{\partial\Omega}t\cdot w+\int_{\Omega}b\cdot w\right|}{\int_{\Omega}|\varepsilon(w)|},\quad w\in C^\infty(\overline{\Omega},\mathbb{R}^3),$$

where for computing $|\varepsilon(w)|$ we use the norm on $\mathbb{R}^9$ that is dual to the one used for the stress matrix $\sigma(x)$. (*iii*) *The generalized stress concentration factor*. Let the generalized stress concentration factor be defined as

$$K=\sup_{(t,b)}\frac{\sigma^{\text{opt}}_{(t,b)}}{\operatorname{ess\,sup}_{x,y}\{|t(y)|,|b(x)|\}}, \tag{4.8}$$

where the loading pair $(t,b) \in L^\infty(\partial\Omega,\mathbb{R}^3)\times L^\infty(\Omega,\mathbb{R}^3)$ is equilibrated. Then, $K = \|\delta/\mathscr{R}\|$.

5. Subspaces

We now return to the case where $D\colon \mathbf{W}\to L^p(J^k(\xi))$ is injective. Let $\mathbf{W}_0$ be a subspace of $\mathbf{W}$. Then, $D_0 = D|_{\mathbf{W}_0}\colon \mathbf{W}_0 \to L^p(J^k(\xi))$ is again an isometric injection and the representation of elements of $\mathbf{W}_0^*$ by stresses as well as the rule about optimal stresses still hold. The next example uses this fact where the subspace $\mathbf{W}_0$ contains fields that satisfy homogeneous boundary conditions.

5.1. **Supported bodies and the space $LD(\Omega)_0$.** Consider the situation where the body Ω is supported on an open subset ${}_0 \subset \partial\Omega$ and let $LD(\Omega)_0$ be the subspace of $LD(\Omega)$ containing fields that satisfy the boundary conditions, i.e., $LD(\Omega)_0 = \left\{w\in LD(\Omega)\mid \gamma(w)|_{{}_0} = 0\right\}$. Since both γ and the restriction operator are continuous and linear, $LD(\Omega)_0$ is a closed subspace of $LD(\Omega)$. Thus, we have the isometric injection

$$(\pi_{\mathscr{R}},\varepsilon)_0 = (\pi_{\mathscr{R}},\varepsilon)|_{LD(\Omega)_0}\colon LD(\Omega)_0 \longrightarrow \mathscr{R}\times L^1(\Omega,\mathbb{R}^6). \tag{5.1}$$

Let $\pi^0_{\mathscr{R}}\colon L^1({}_0,\mathbb{R}^3)\to\mathscr{R}$ be a projection on the space of rigid motions for fields defined on ${}_0$. Define the projection $\pi_{\mathscr{R}}\colon L^1(\Omega,\mathbb{R}^3)\to\mathscr{R}$ by $\pi_{\mathscr{R}}(w) = \pi^0_{\mathscr{R}}(\gamma(w)|_{{}_0})$. Since all operators on the right are continuous in w, so is $\pi_{\mathscr{R}}$. Thus, one may use this projection mapping in Equation (3.1). However, it is clear that $LD(\Omega)_0 = \operatorname{Kernel}\pi_{\mathscr{R}}$.

We conclude that with this choice of projection on the space of rigid motions, $\varepsilon_0 = \varepsilon|_{LD(\Omega)_0}\colon LD(\Omega)_0 \to L^1(\Omega,\mathbb{R}^6)$ is an isometric injection.

5.2. **Stress analysis for supported bodies.** As the body is supported on ${}_0$, traction may be applied on a disjoint part ${}_t\subset\partial\Omega$. Thus, it is natural to assume that ${}_0$ and ${}_t$ are nonempty and disjoint open sets, ${}_0\cup{}_t = \partial\Omega$, and $=\partial_0 = \partial_t$ is a differentiable 1-dimensional submanifold of $\partial\Omega$.

The boundary tractions are naturally elements of $L^\infty({}_t,\mathbb{R}^3) = L^1({}_t,\mathbb{R}^3)^*$. Let $L^1(\partial\Omega,\mathbb{R}^3)_0 = \left\{u\in L^1(\partial\Omega,\mathbb{R}^3)\mid u(y)=0 \text{ a.e. on } {}_0\right\}$, then, there are natural isometric isomorphisms $L^1({}_t,\mathbb{R}^3)\cong L^1(\partial\Omega,\mathbb{R}^3)_0$, and $L^\infty({}_t,\mathbb{R}^3)\cong L^1(\partial\Omega,\mathbb{R}^3)^*_0$ (see [9, Sec. 3] for details).

The situation may be described by the following commutative diagrams where ι denote a generic inclusion of a subspace.

$$\begin{array}{ccccc}
L^1(\partial\Omega,\mathbb{R}^3)\times L^1(\Omega,\mathbb{R}^3) & \xleftarrow{\ \delta\ } & LD(\Omega) & \xrightarrow{(\pi_{\mathscr{R}},\varepsilon)} & \mathscr{R}\times L^1(\Omega,\mathbb{R}^6) \\
{\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} \\
L^1({}_t,\mathbb{R}^3)\times L^1(\Omega,\mathbb{R}^3) & \xleftarrow{\ \delta_0\ } & LD(\Omega)_0 & \xrightarrow{\ \varepsilon_0\ } & L^1(\Omega,\mathbb{R}^6).
\end{array} \tag{5.2}$$

Here, δ_0 is the restriction of δ to $LD(\Omega)_0$. The dual diagram is

$$\begin{array}{ccccc} L^\infty(\partial\Omega,\mathbb{R}^3)\times L^\infty(\Omega,\mathbb{R}^3) & \xrightarrow{\delta^*} & LD(\Omega)^* & \xleftarrow{(\pi_{\mathscr{R}},\varepsilon)^*} & \mathscr{R}^*\times L^\infty(\Omega,\mathbb{R}^6) \\ {\scriptstyle\iota^*}\big\downarrow & & \big\downarrow{\scriptstyle\iota^*} & & \big\downarrow{\scriptstyle\iota^*} \\ L^\infty({}_t,\mathbb{R}^3)\times L^\infty(\Omega,\mathbb{R}^3) & \xrightarrow{\delta_0^*} & LD(\Omega)_0^* & \xleftarrow{\varepsilon_0^*} & L^\infty(\Omega,\mathbb{R}^6). \end{array} \tag{5.3}$$

We conclude that the following can be stated for the stress analysis of supported bodies. (i) *Existence of stresses*. Given a loading pair $(t,b)\in L^\infty({}_t,\mathbb{R}^3)\times L^\infty(\Omega,\mathbb{R}^3)$, there is a subspace $\Sigma_{(t,b)}\subset L^\infty(\Omega,\mathbb{R}^6)$ such that for each $\sigma\in\Sigma_{(t,b)}$, $\delta_0^*(t,b)=\varepsilon_0^*(\sigma)$. ($ii$) *Optimal stresses*. The optimum $\sigma^{\text{opt}}_{(t,b)}$ satisfies

$$\sigma^{\text{opt}}_{(t,b)}=\sup_w\frac{\left|\int_{\partial\Omega}t\cdot w+\int_\Omega b\cdot w\right|}{\int_\Omega|\varepsilon_0(w)|},\quad w\in LD(\Omega)_0.$$

The optimum is attainable by some stress field $\hat\sigma\in\Sigma_{(t,b)}$. ($ii$) *The generalized stress concentration factor*. The generalized stress concentration factor is given by $K=\|\delta_0\|$.

6. Product Structures

6.1. **Product structures on subbundles of $J^k(\xi)$.** Consider the case where there is a subbundle $\eta\subset J^k(\xi)$ such that $D\colon\mathbf{W}\to\operatorname{Image}D\subset L^p(\eta)$ and η has the direct sum structure $\eta=\eta_1\oplus\eta_2$ for complementary subbundles η_1 and η_2. Thus, $L^p(\eta)=L^p(\eta_1)\oplus L^p(\eta_2)$, and denoting the natural projections as $(\pi_1,\pi_2)\colon\eta\to\eta_1\oplus\eta_2$ we have for $\alpha=1,2$, $D=(D_1,D_2)$, where, $D_\alpha\colon\mathbf{W}\to L^p(\eta_\alpha)$, is defined by $D_\alpha(w)(x)=\pi_\alpha(D(w)(x))$. Equivalently, setting $\pi_\alpha^\circ\colon L^p(\eta)\to L^p(\eta_\alpha)$, by $\pi_\alpha^\circ(\chi)=\pi_\alpha\circ\chi$, we have $D_\alpha=\pi_\alpha^\circ\circ D$.

Consider the seminorm $|\cdot|_Y$ on η given by $|\chi|_Y=|\pi_1(\chi)|$, and the induced seminorm $\|\cdot\|_Y$ on $\mathbf{W}$ given by $\|w\|_Y=\|D_1(w)\|$. Then, $\|\cdot\|_Y$ is a norm on $\mathbf{W}_Y=D^{-1}\{L^p(\eta_1)\}$ and $D_Y=D|_{\mathbf{W}_Y}\colon\mathbf{W}_Y\to L^p(\eta_1)$ is an isometric linear injection.

Again, a force $F\in\mathbf{W}_Y^*$ is represented by some stress $\sigma\in L^{p'}(\eta_1^*)$ in the form $F=D_Y^*(\sigma)$ and

$$\|F\|=\inf\left\{\|\sigma\|\mid F=D_Y^*(\sigma),\ \sigma\in L^{p'}(\eta_1^*)\right\}. \tag{6.1}$$

The situation is illustrated in the following diagrams.

$$\begin{array}{ccccccc} \mathbf{W} & \xrightarrow{D} & L^p(\eta) & \qquad & \mathbf{W}^* & \xleftarrow{D^*} & L^{p'}(\eta^*) \\ \uparrow \iota & & \iota \uparrow\downarrow \pi_1^\circ & & \downarrow \iota^* & & \iota^* \downarrow\uparrow \pi_1^{\circ *} \\ \mathbf{W}_Y & \xrightarrow{D_1} & L^p(\eta_1), & & \mathbf{W}^*_Y & \xleftarrow{D_1^*} & L^{p'}(\eta_1^*). \end{array} \tag{6.2}$$

6.2. **Stress analysis for elastic-plastic bodies.** As an example for the foregoing discussion we consider stress analysis for elastic-perfectly plastic bodies. Using the notation of Subsections 5.1–5.2, the differential operator we consider is $\varepsilon_0 \colon LD(\Omega)_0 \to L^1(\Omega, \mathbb{R}^6)$. The analysis below is motivated by the fact that yield criteria in the theory of plasticity are usually semi-norms on the space of stress matrices rather than norms. Specifically, the yield criteria may be expressed usually as the application of a norm to the deviatoric component of the stress matrix. Here, the direct sum decomposition is of the form $\mathbb{R}^6 = D \oplus P$ where D is the space of trace-less or deviatoric matrices and $P = \{aI \mid a \in \mathbb{R}\}$. We denote by π_P the usual projection of the space of matrices on the subspace P, i.e., $\pi_P(m) = \frac{1}{3} m_{ii} I$, and by π_D, the projection on D so $\pi_D(m) = m_D = m - \pi_P(m)$. Thus, the pair (π_D, π_P) induces an isomorphism of the space of symmetric matrices with $D \oplus P$. Again, we use the same notation $|\cdot|$ for both the norm on $\mathbb{R}^6$, whose elements are interpreted as strain values, and the dual norm on $\mathbb{R}^{6*}$, whose elements are interpreted as stress values. Thus, the yield function is of the form $|m|_Y = |\pi_D(m)|$. For example, if we take $|\cdot|$ to be the Frobenius norm on $\mathbb{R}^{6*}$ we get the von-Mises yield criterion.

For the space of stress fields we will therefore have the seminorm $\|\cdot\|_Y$ defined by $\|\sigma\|_Y = \|\pi_D \circ \sigma\|_\infty$ and no yielding will occur as long as $\|\sigma\|_Y \leqslant Y_0$, where Y_0 is the yield stress. The situation is depicted in the following diagrams

$$\begin{array}{ccccc} L^1({}_t, \mathbb{R}^3) \times L^1(\Omega, \mathbb{R}^3) & \xleftarrow{\delta} & LD(\Omega)_0 & \xrightarrow{\varepsilon_0} & L^1(\Omega, \mathbb{R}^6) \\ \| & & \uparrow \iota & & \iota \uparrow\downarrow \pi_D^\circ \\ L^1({}_t, \mathbb{R}^3) \times L^1(\Omega, \mathbb{R}^3) & \xleftarrow{\delta_D} & LD(\Omega)_D & \xrightarrow{\varepsilon_D} & L^1(\Omega, D), \end{array} \tag{6.3}$$

$$\begin{array}{ccccc} L^\infty({}_t, \mathbb{R}^3) \times L^\infty(\Omega, \mathbb{R}^3) & \xrightarrow{\delta^*} & LD(\Omega)_0^* & \xleftarrow{\varepsilon_0^*} & L^\infty(\Omega, \mathbb{R}^6) \\ \| & & \downarrow \iota^* & & \iota^* \downarrow\uparrow \pi_D^{\circ *} \\ L^\infty({}_t, \mathbb{R}^3) \times L^\infty(\Omega, \mathbb{R}^3) & \xrightarrow{\delta_D^*} & LD(\Omega)_D^* & \xleftarrow{\varepsilon_D^*} & L^\infty(\Omega, D). \end{array} \tag{6.4}$$

For a given loading pair $(t, b) \in L^\infty(_t, \mathbb{R}^3) \times L^\infty(\Omega, \mathbb{R}^3)$, the expression defining the optimal stress is

$$\sigma^{\text{opt}}_{(t,b)} = \inf_{\iota^*(t,b)=\varepsilon^*_0(\sigma)} \|\sigma\|_Y, \tag{6.5}$$

and the expression for the generalized stress concentration factor is

$$K = \sup_{(t,b)} \frac{\sigma^{\text{opt}}_{(t,b)}}{\|t\|_\infty}. \tag{6.6}$$

In analogy with the results for the earlier examples one has

$$\sigma^{\text{opt}}_{(t,b)} = \sup_{w \in LD(\Omega)_D} \frac{\left|\int_{\partial\Omega} t \cdot w + \int_\Omega b \cdot w\right|}{\int_\Omega |\varepsilon_0(w)|} \tag{6.7}$$

and $K = \|\delta_D\|$.

We now present the mechanical interpretation of $\sigma^{\text{opt}}_{(t,b)}$ and K in the framework of plasticity theory. For an elastic-perfectly plastic body, the condition for unavoidable collapse under the loading pair (t, b) is $\sigma^{\text{opt}}_{(t,b)} > Y_0$. Let $\mathscr{B}$ be the collection of all loading pairs for which collapse does not occur, that is, those pairs for which $\sigma^{\text{opt}}_{(t,b)} \leqslant Y_0$. The boundary of $\mathscr{B}$ is the collapse manifold , i.e., $= \left\{(t, b) \mid \sigma^{\text{opt}}_{(t,b)} = Y_0\right\}$. One may write $\sigma = \sigma_1/\lambda$, $\|\sigma_1\|_Y = Y_0$ and noting that $\|\sigma/\|\sigma\|_Y\|_Y = 1$, the expression for the optimal stress becomes

$$\sigma^{\text{opt}}_{(t,b)} = \inf_{\substack{\varepsilon^*(\sigma_1/\lambda)=\delta^*(t,b),\\ \lambda\in\mathbb{R}^+,\, \sigma_1\in\partial B}} \|\sigma_1/\lambda\|_Y, \tag{6.8}$$

where B is the ball in $L^\infty(\Omega, D)$ of radius Y_0 . (In order to simplify the notation we omit the subscript D in δ_D in the sequel.) Thus,

$$\sigma^{\text{opt}}_{(t,b)} = \inf_{\substack{\varepsilon^*(\sigma_1/\lambda)=\delta^*(t,b),\\ \lambda\in\mathbb{R}^+,\, \sigma_1\in\partial B}} \frac{Y_0}{\lambda}, \tag{6.9}$$

$$\frac{Y_0}{\sigma^{\text{opt}}_{(t,b)}} = \sup\left\{\lambda \mid \exists \sigma_1 \in \partial B,\, \varepsilon^*(\sigma_1) = \delta^*(\lambda(t, b))\right\}. \tag{6.10}$$

Clearly, in the last equation ∂B may be replaced by $\overline{B}$ because if we consider σ with $\|\sigma\|_Y < 1$, then, $\sigma_1 = \sigma/\|\sigma\|_Y$ is on ∂B and the corresponding λ will be multiplied by $\|\sigma\|_Y < 1$. The unit ball B contains the stress fields that are essentially bounded by the yield stress and we are looking for the largest multiplication of the loading pair for which there is an equilibrating stress field that is essentially bounded by Y_0. Thus, we are looking for

$$\frac{Y_0}{\sigma^{\text{opt}}_{(t,b)}} = \lambda^* = \sup\{\lambda \mid \exists \sigma \in B,\ \varepsilon^*(\sigma) = \gamma^*(\lambda(t,b))\}, \tag{6.11}$$

which is the limit analysis factor (e.g., Christiansen [1, 2] and Teman & Strang [12]).

For the application to plasticity, we use the term *load capacity* for $C = 1/K$. Hence,

$$C = \frac{1}{\sup_{(t,b)}(\sigma^{\text{opt}}_{(t,b)}/\|(t,b)\|_\infty)} = \inf_{(t,b)} \frac{\|(t,b)\|_\infty}{\sigma^{\text{opt}}_{(t,b)}}. \tag{6.12}$$

For every loading pair (t,b) we set

$$(t,b) = \frac{(t,b)}{\sigma^{\text{opt}}_{(t,b)}/Y_0}, \tag{6.13}$$

so using $\sigma^{\text{opt}}_{\lambda(t,b)} = \|\delta^*(\lambda(t,b))\| = \lambda\sigma^{\text{opt}}_{(t,b)}$ for any $\lambda > 0$, one has

$$\sigma^{\text{opt}}_{(t,b)} = Y_0, \qquad \frac{\|(t,b)\|_\infty}{\sigma^{\text{opt}}_{(t,b)}} = \frac{\|(t,b)\sigma^{\text{opt}}_{(t,b)}/Y_0\|_\infty}{\sigma^{\text{opt}}_{(t,b)}} = \|(t,b)\|_\infty/Y_0. \tag{6.14}$$

It follows that for any loading pair (t,b), (t,b) belongs to the collapse manifold and the operation above is a projection onto the collapse manifold. Thus,

$$C = \inf_{(t,b)} \frac{\|(t,b)\|_\infty}{\sigma^{\text{opt}}_{(t,b)}} = \inf_{(t,b)\in} \|(t,b)\|_\infty/Y_0, \tag{6.15}$$

and indeed, $CY_0 = \inf_{(t,b)\in}\|(t,b)\|_\infty$ is the largest radius of a ball containing only loading pairs for which collapse does not occur.

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Current address: Department of Mechanical Engineering, Ben-Gurion University, Beer-Sheva, Israel

E-mail address: rsegev@bgu.ac.il

URL: http://www.bgu.ac.il/~rsegev