STRESS OPTIMIZATION FOR SUPPORTED BODIES

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In memory of Avinoam Zemach, 1946 – 2005

ABSTRACT. For a surface traction t, acting on a region Γ_t of the boundary of a given body Ω supported on $\Gamma_0 \subset \partial\Omega$, we consider the infimum $\sigma_t^{\mathrm{opt}} = \inf_\sigma \{\operatorname{ess\,sup}_x |\sigma(x)|\}$ over all stress fields σ in equilibrium with t, i.e., the smallest essential bound on all conceivable stress fields. Using the space of LD-vector fields on Ω, we show that σ_t^{opt} is attainable by some stress field $\widehat{\sigma}$ and that

$$\sigma_t^{\mathrm{opt}} = \sup_{w \in C^\infty(\overline{\Omega}, \mathbb{R}^3)} \frac{\left| \int_{\Gamma_t} t \cdot w \, \mathrm{d}A \right|}{\int_\Omega |\varepsilon| \, \mathrm{d}V},$$

where ε is the linear strain associated with the vector field w. Varying the traction, it is shown that

$$K = \sup_t \frac{\sigma_t^{\mathrm{opt}}}{\|t\|_\infty}$$

is given by $K = \|\gamma_0\|$, where γ_0 is the trace mapping. The boundary value problems associated with the suprema in the expressions for σ_t^{opt} and $\|\gamma_0\|$ are derived.

1. INTRODUCTION

As a sequel to our previous work [2, 3, 1], we consider stress fields on bodies whose maxima are the least. Let Ω represent the region occupied by the body in space so the body is supported on a part Γ_0 of its boundary and let t be the external surface traction acting on the part Γ_t of its boundary. Body forces may be included in the analysis using the same methods as in [3] but for the sake of simplicity we omit them here. The mechanical properties of the body are not specified, and so, there is a class of stress fields that satisfy the equilibrium conditions with the external loading. Clearly, distinct distributions of the mechanical properties within the body will result distinct equilibrating stress distributions. Each equilibrating stress field in this class has its own maximal value, and we denote by σ_t^{opt} the least maximum. The value of σ_t^{opt} seems to us to be meaningful in light of traditional engineering practices of stress analysis and attempts to generate optimal stress

1991 *Mathematics Subject Classification.* 35Q72; 46E35.

Key words and phrases. Continuum mechanics, stress analysis, optimization, mixed boundary conditions, trace.

November 3, 2005.

distributions in bodies by introducing various forms of heterogeneity. The main result we present here is an expression for σ_t^{opt}.

Specifically, the magnitude of the stress field at a point is evaluated using a norm on the space of matrices. While in most of the analysis, no specific norm is specified, in other parts we use the 2-norm $|\sigma(x)|_2 = \sqrt{\sigma_{ij}(x)\sigma_{ij}(x)}$. It is noted that failure criteria in stress analysis usually use semi-norms on the space of stresses rather than norms. By the maximum of a stress field we mean the essential supremum over the body of its magnitude. Thus, we ignore excessive values on regions of zero volume. The traction fields that we admit are essentially bounded also. The set Ω is assumed to be open, bounded and its boundary is assumed to be smooth. Furthermore, it is assumed that Γ_t and Γ_0 are disjoint open subsets of the boundary whose closures cover the boundary, and that their closures intersect on a smooth curve.

Subject to these assumptions (see further details in Section 2) our basic result is

Theorem 1.1. (*i*) ***The Existence of stresses***. *Given an essentially bounded traction field t on Γ_t, there is a collection Σ_t of essentially bounded symmetric tensor fields, interpreted physically as stress fields, that represent t in the form*

$$\int_{\Gamma_t} t\cdot w\,\mathrm{d}A = \int_{\Omega} \sigma_{ij}\varepsilon_{ij}(w)\,\mathrm{d}V, \quad \text{for all} \quad \sigma\in\Sigma_t,\ w\in C^\infty(\overline{\Omega},\mathbb{R}^3), \tag{1.1}$$

where, $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^T)$.

(*ii*) ***The Existence of optimal stress fields***. *There is a stress field $\widehat{\sigma}\in\Sigma_t$ such that*

$$\sigma_t^{\text{opt}} = \inf_{\sigma\in\Sigma_t}\left\{\operatorname*{ess\,sup}_{x\in\Omega}|\sigma(x)|\right\} = \operatorname*{ess\,sup}_{x\in\Omega}|\widehat{\sigma}(x)|\,. \tag{1.2}$$

(*iii*) ***The expression for*** σ_t^{opt}. *The optimum satisfies*

$$\sigma_t^{\text{opt}} = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\left|\int_{\Gamma_t} t\cdot w\,\mathrm{d}A\right|}{\int_\Omega |\varepsilon(w)|\ \mathrm{d}V}\,, \tag{1.3}$$

where the magnitude of $\varepsilon(w)(x)$ is evaluated using the norm dual to the one used for the values of stresses.

Item (*i*) above is of theoretical interest. It is a representation theorem for the virtual work performed by the traction field using tensor fields that we naturally interpret as stresses. It should be noted that the existence of stress is not assumed here a-priori. The expression for the representation by stresses turns out to be the principle of virtual work (1.1). Thus, the equilibrium conditions are derived mathematically on the basis of quite general assumptions. Item (*i*) also ensures us that the representing stress fields are also essentially bounded. Item (*ii*) states that the optimal value is actually attainable for some stress field and not just as a limit process. The expression in Item (*iii*) is notably simpler than estimating σ_t^{opt} on the basis of its definition. (One would have to generate all stress fields in Σ_t somehow, and evaluate the essential supremum for each.) It is conceivable that using Equation (1.3) one would be able to approximate σ_t^{opt} numerically using finite dimensional subspaces of $C^\infty(\overline{\Omega}, \mathbb{R}^3)$.

Section 2 introduces the notation, assumptions and some background material. In particular, the space $LD(\Omega)$ of vector fields of integrable stretchings (or linear strains) (see [6, 7, 5]) is described. Following some preliminary material concerning the boundary conditions in Section 3, the proof of the theorem is given in Section 4, with some additional details in Appendix A. In Section 5 we derive formally the boundary value problem corresponding to the supremum in (1.3).

Next, we consider *generalized stress concentration factors* for the given body. For a given external loading, traditional stress concentration factors are used by engineers to specify the ratio between the maximal stress in the body and the maximum nominal stress obtained using simplified formulae where various geometric irregularities are not taken into account. Regarding these nominal stresses as boundary traction fields, we formulate the notion of a stress concentration factor for a stress field σ in equilibrium with the traction t mathematically as the ratio between the maximal stress and the maximum traction. Specifically, we set

$$K_{t,\sigma} = \frac{\operatorname{ess\,sup}_{x\in\Omega} |\sigma(x)|}{\operatorname{ess\,sup}_{y\in\Gamma_t} |t(y)|}\,. \tag{1.4}$$

In particular, the optimal stress concentration factor for the given traction t, is

$$K_t = \inf_{\sigma\in\Sigma_t} \{K_{t,\sigma}\} = \frac{\sigma_t^{\mathrm{opt}}}{\operatorname{ess\,sup}_{y\in\Gamma_t} |t(y)|}\,. \tag{1.5}$$

Finally, realizing that engineers may be uncertain as to the nature of the external loading, we let the external loading vary and define the *generalized stress concentration factor*, a purely geometric property of the body Ω, as

$$K = \sup_t \{K_t\}\,, \tag{1.6}$$

where t varies over all essentially bounded traction fields. In other words, K is the worst possible optimal stress concentration factor. Using the result on optimal stresses, we prove in Section 6 straightforwardly the following

Theorem 1.2. *The generalized stress concentration factor satisfies*

$$K = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\int_{\Gamma_t} |w|\;\mathrm{d}A}{\int_\Omega |\varepsilon(w)|\;\mathrm{d}V} = \|\gamma_0\|, \tag{1.7}$$

where γ_0 is the trace mapping for vector fields satisfying the boundary conditions on Γ_0.[1]

Later in Section 6, we derive the boundary value problem for the supremum of Equation (1.7). It turns out that the partial differential equation is the same as the one obtained for for σ_t^{opt} but the boundary conditions are different.

To prove the theorems we use standard tools of analysis and the theory of *LD*-spaces given by [6, 7, 5]. Results analogous to Theorems (1.1) and (1.2) were presented in our earlier work cited above. In [2], a weaker form of equilibrium is assumed, and in all earlier work we did not consider boundary conditions for the displacements on Γ_0. Also, the previous work does not contain the associated boundary value problems that we present here in Section 5 and Subsection 6.2.

2. Notation and Preliminaries

2.1. **Basic variables.** We consider a body under a given configuration in space. The space is modelled simply by $\mathbb{R}^3$ and the image of the body under the given configuration is the subset $\Omega \subset \mathbb{R}^3$. It is assumed that Ω is open and bounded and that it has a C^1-boundary $\partial\Omega$. Furthermore, there are two open subsets $\Gamma_0 \subset \partial\Omega$, and $\Gamma_t \subset \partial\Omega$ such that Γ_0 is the region where the body is supported and Γ_t is the region where the body is not supported so that a surface traction field t may be exerted on the body on Γ_t. Thus, it is natural to assume that Γ_0 and Γ_t are nonempty and disjoint, $\overline{\Gamma}_0 \cup \overline{\Gamma}_t = \partial\Omega$, and $\Lambda = \partial\Gamma_0 = \partial\Gamma_t$ is a smooth 1-dimensional submanifold of $\partial\Omega$. (The regularity assumptions may be generalized without affecting the validity of the constructions below.)

Basic objects in the construction are spaces of generalized velocity fields. A generic *generalized velocity field* (alternatively, *virtual velocity* or *virtual displacement*) will be denoted by w. In the sequel we consider a number of Banach spaces containing generalized velocities and a generic space of generalized velocities will be a denoted by $\mathbf{W}$. Generalized forces will be elements of the dual space $\mathbf{W}^*$. Thus, a *generalized force F* is a bounded linear functional $F\colon \mathbf{W} \longrightarrow \mathbb{R}$, such that $F(w)$ is interpreted as the virtual power (virtual work) performed by the force for the generalized velocity w. We recall that the dual norm of a linear functional F is defined as

$$\|F\| = \sup_{w\in\mathbf{W}} \frac{|F(w)|}{\|w\|} \,. \tag{2.1}$$

2.2. **Virtual stretchings (linear strains) and stresses.** As an example for the preceding paragraph, consider the case where $\mathbf{W}$ is the space $L^1(\Omega,\mathbb{R}^6)$ of L^1-symmetric tensor fields on Ω. A typical element $\varepsilon \in L^1(\Omega,\mathbb{R}^6)$ is interpreted as a *virtual stretching field* or a *linear strain field*. We will use $|\varepsilon(x)|$

[1] Further details on γ_0 are described in Section 3.

to denote the norm of the matrix $\varepsilon(x)$. Various such norms are described in [1]. Thus,

$$\|\varepsilon\|_1 = \int_{\Omega} |\varepsilon(x)| \, \mathrm{d}V. \tag{2.2}$$

The dual space $L^1(\Omega, \mathbb{R}^6)^* = L^\infty(\Omega, \mathbb{R}^6)$ contains symmetric essentially bounded tensor fields σ that act on the stretching fields by

$$\sigma(\varepsilon) = \int_{\Omega} \sigma(x)(\varepsilon(x)) \, \mathrm{d}V. \tag{2.3}$$

Here, we use the same notation for the functional σ and the essentially bounded tensor field representing it and we regard the matrix $\sigma(x)$ as a linear form on the space of matrices so $\sigma(x)(\varepsilon(x)) = \sigma(x)_{ij}\varepsilon(x)_{ji}$. Naturally, an element $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$ is interpreted as a *stress field*. The dual norm of a stress field is given as

$$\|\sigma\| = \|\sigma\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)| \,. \tag{2.4}$$

Here, $|\sigma(x)|$ is calculated using the norm on the space of matrices which is dual to that used for the evaluation of $|\varepsilon(x)|$ (see [1] for details). Thus, the choice of the space $L^1(\Omega, \mathbb{R}^6)$ for stretchings is natural when one is looking for the maximum of the stress tensor.

2.3. **The space of boundary velocity fields and boundary tractions.** As another example to be used later, consider the space $L^1(\Gamma_t, \mathbb{R}^3)$ of integrable vector fields on the "free" part of the boundary. Its dual space is

$$L^1(\Gamma_t, \mathbb{R}^3)^* = L^\infty(\Gamma_t, \mathbb{R}^3), \tag{2.5}$$

so that a generalized force in this case will be represented by an essentially bounded vector field t on Γ_t. Using the same notation for the functional and the vector field representing it, we have

$$t(u) = \int_{\Gamma_t} t(y) \cdot u(y) \, \mathrm{d}A \tag{2.6}$$

so t may be interpreted as a traction field on Γ_t as expected. The dual norm of the traction field t is

$$\|t\| = \|t\|_\infty = \operatorname*{ess\,sup}_{y \in \Gamma_t} |t(y)| \,, \tag{2.7}$$

again, the relevant maximum.

2.4. **The space $LD(\Omega)$ and its elementary properties.** A central role in the subsequent analysis is played by the space $LD(\Omega)$ containing vector fields of integrable stretchings (see [5, 6, 7]). We summarize below its definition and basic relevant properties (see [7] for proofs and details).

2.4.1. *Definition.* For an integrable vector field $w \in L^1(\Omega, \mathbb{R}^3)$, let ∇w denote its distributional gradient and consider the corresponding stretching (a tensor distribution)

$$\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^T). \tag{2.8}$$

The vector field w has an integrable stretching if the distribution $\varepsilon(w)$ is an integrable symmetric tensor field, i.e., it belongs to $L^1(\Omega, \mathbb{R}^6)$. For the sake of simplifying the notation, we use ε for both the stretching mapping here and its value in the example above. The space $LD(\Omega)$ is defined by

$$LD(\Omega) = \left\{ w \colon \Omega \to \mathbb{R}^3;\ w \in L^1(\Omega, \mathbb{R}^3),\ \varepsilon(w) \in L^1(\Omega, \mathbb{R}^6) \right\}. \tag{2.9}$$

A natural norm is provided by

$$\|w\| = \|w\|_{LD} = \|w\|_1 + \|\varepsilon(w)\|_1 \tag{2.10}$$

and it induces on $LD(\Omega)$ a Banach space structure. Clearly, the stretching mapping

$$\varepsilon \colon LD(\Omega) \longrightarrow L^1(\Omega, \mathbb{R}^6) \tag{2.11}$$

is linear and continuous.

2.4.2. *Approximations.* The space of restrictions to Ω of smooth mappings in $C^\infty(\overline{\Omega}, \mathbb{R}^3)$, is dense in $LD(\Omega)$, so any LD-vector field may be approximated by restrictions of smooth vector fields defined on the closure $\overline{\Omega}$.

2.4.3. *Trace mapping.* There is a unique continuous and linear trace mapping

$$\gamma \colon LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3) \tag{2.12}$$

satisfying the consistency condition

$$\gamma(u|_\Omega) = u|_{\partial\Omega} \tag{2.13}$$

for any continuous mapping $u \in C^0(\overline{\Omega}, \mathbb{R}^3)$. Furthermore, the trace mapping is surjective. Thus, although LD-mappings are defined on the open set Ω, they have meaningful L^1 boundary values.

2.4.4. *Equivalent norm.* Let Γ be an open subset of $\partial\Omega$ and for $w \in LD(\Omega)$ let

$$\|w\|_\Gamma = \int_\Gamma |\gamma(w)|\ \mathrm{d}A + \|\varepsilon(w)\|_1, \tag{2.14}$$

then, $\|w\|_\Gamma$ is a norm on $LD(\Omega)$ which is equivalent to the original norm defined in Equation (2.10).

3. Constructions Associated with the Boundary Conditions

3.1. **The space $L^1(\partial\Omega,\mathbb{R}^3)_0$.** Let $L^1(\partial\Omega,\mathbb{R}^3)_0 \subset L^1(\partial\Omega,\mathbb{R}^3)$ be the vector space of vector fields on $\partial\Omega$ such that for each $u \in L^1(\partial\Omega,\mathbb{R}^3)_0$, $u(y) = 0$ for almost all $y \in \Gamma_0$. It is noted that the restriction mapping

$$\rho_0\colon L^1(\partial\Omega,\mathbb{R}^3) \longrightarrow L^1(\Gamma_0,\mathbb{R}^3), \quad \rho_0(u) = u|_{\Gamma_0} \tag{3.1}$$

is linear and continuous. Thus, since

$$L^1(\partial\Omega,\mathbb{R}^3)_0 = \rho_0^{-1}\{0\}, \tag{3.2}$$

$L^1(\partial\Omega,\mathbb{R}^3)_0$ is a closed subspace of $L^1(\partial\Omega,\mathbb{R}^3)$.

The restriction mapping

$$\rho_t\colon L^1(\partial\Omega,\mathbb{R}^3)_0 \longrightarrow L^1(\Gamma_t,\mathbb{R}^3), \quad \rho_t(u) = u|_{\Gamma_t} \tag{3.3}$$

is also linear and continuous. In addition, as $\partial\Gamma_0 = \partial\Gamma_t = \Lambda$ have zero area measure,

$$\int_{\partial\Omega} |u| \,\mathrm{d}A = \int_{\Gamma_t} |\rho_t(u)| \,\mathrm{d}A, \quad u \in L^1(\partial\Omega,\mathbb{R}^3)_0, \tag{3.4}$$

so ρ_t is a norm-preserving injection.

Consider the zero extension mapping

$$e_0\colon L^1(\Gamma_t,\mathbb{R}^3) \longrightarrow L^1(\partial\Omega,\mathbb{R}^3)_0, \tag{3.5}$$

defined by

$$e_0(u)(y) = \begin{cases} u(y) & \text{for } y \in \Gamma_t, \\ 0 & \text{for } y \notin \Gamma_t. \end{cases} \tag{3.6}$$

Clearly, $\rho_t \circ e_0$ is the identity on the space $L^1(\Gamma_t,\mathbb{R}^3)$. Moreover, for any $u \in L^1(\partial\Omega,\mathbb{R}^3)_0$, $e_0(\rho_t(u))(y) = u(y)$ almost everywhere (except for $y \in \Lambda$), so $e_0 \circ \rho_t$ is the identity on $L^1(\partial\Omega,\mathbb{R}^3)_0$. We conclude,

Lemma 3.1. *The mappings ρ_t and e_0 induce an isometric isomorphism of the spaces $L^1(\partial\Omega,\mathbb{R}^3)_0$ and $L^1(\Gamma_t,\mathbb{R}^3)$. The dual mappings e_0^* and ρ_t^* induce an isometric isomorphism of the spaces $L^1(\Gamma_t,\mathbb{R}^3)^*$ and $L^1(\partial\Omega,\mathbb{R}^3)_0^*$. Every element $t_0 \in L^1(\partial\Omega,\mathbb{R}^3)_0^*$ is represented uniquely by an essentially bounded $t \in L^\infty(\Gamma_t,\mathbb{R}^3)$ in the form*

$$t_0(u) = \int_{\Gamma_t} t \cdot u \,\mathrm{d}A. \tag{3.7}$$

3.2. **The space $LD(\Omega)_0$ of velocity fields satisfying the boundary conditions.** Recalling the definition of the equivalent norm on $LD(\Omega)$ in Equation (2.14), we set $\Gamma = \Gamma_0$ in that equation. Henceforth, we will use on $LD(\Omega)$ the equivalent norm

$$\|w\| = \|w\|_{\Gamma_0} = \int_{\Gamma_0} |\gamma(w)| \,\mathrm{d}A + \|\varepsilon(w)\|_1. \tag{3.8}$$

Consider the vector subspace $LD(\Omega)_0$ defined by

$$LD(\Omega)_0 = \gamma^{-1}\left\{L^1(\partial\Omega,\mathbb{R}^3)_0\right\} \subset LD(\Omega). \tag{3.9}$$

Thus, $LD(\Omega)_0$ is the subspace containing vector fields on Ω whose boundary values vanish on Γ_0 almost everywhere. Since γ is continuous and $L^1(\partial\Omega,\mathbb{R}^3)_0$ is a closed subspace of $L^1(\partial\Omega,\mathbb{R}^3)$, $LD(\Omega)_0$ is a closed subspace of $LD(\Omega)$. Combining this with Lemma (3.1) we obtain immediately

Lemma 3.2. *The mapping*

$$\gamma_0 = \rho_t \circ \gamma|_{LD(\Omega)_0} \colon LD(\Omega)_0 \longrightarrow L^1(\Gamma_t,\mathbb{R}^3) \tag{3.10}$$

is a linear and continuous surjection. Dually,

$$\gamma_0^* = \left(\gamma|_{LD(\Omega)_0}\right)^* \circ \rho_t^* \colon L^\infty(\Gamma_t,\mathbb{R}^3) \longrightarrow LD(\Omega)_0^* \tag{3.11}$$

is a continuous injection.

Observing Equation (3.8), for each $w \in LD(\Omega)_0$,

$$\|w\| = \|\varepsilon(w)\|_1. \tag{3.12}$$

Lemma 3.3. *The mapping*

$$\varepsilon_0 = \varepsilon|_{LD(\Omega)_0} \colon LD(\Omega)_0 \to L^1(\Omega,\mathbb{R}^6) \tag{3.13}$$

is an isometric injection.

Proof. Equation (3.12) implies immediately that $\|w\| = \|\varepsilon(w)\|_1$ for all $w \in LD(\Omega)_0$. Being a linear isometry, the zero element is the only element that is mapped to zero, so ε_0 is injective. In addition to relying on the technical property (2.4.4) of $LD(\Omega)$ to show that ε_0 is injective, it should be mentioned that this follows from the fact that for any vector field w on Ω, $\varepsilon(w) = 0$ only if w is a rigid vector field, i.e., if w is of the form $w(x) = a + b \times x$, $a, b \in \mathbb{R}^3$. Now, the only rigid vector field that vanishes on the open set Γ_0 is the zero vector field. □

4. Stresses and Their Maxima

Let $t \in L^\infty(\Gamma_t,\mathbb{R}^3)$ be a traction field on the free part of the boundary. Then, $\gamma_0^*(t)$ is an element of $LD(\Omega)_0^*$ representing t. The basic properties of elements of $LD(\Omega)_0^*$ are as follows.

Lemma 4.1. *Each $S \in LD(\Omega)_0^*$ may be represented by some non-unique tensor field $\sigma \in L^\infty(\Omega,\mathbb{R}^6)$ in the form*

$$S = \varepsilon_0^*(\sigma) \quad \text{or} \quad S(w) = \int_\Omega \sigma(x)(\varepsilon_0(w)(x))\,\mathrm{d}V. \tag{4.1}$$

The dual norm of S satisfies

$$\|S\| = \inf_\sigma \|\sigma\|_\infty = \inf_\sigma \left\{ \operatorname*{ess\,sup}_{x\in\Omega} |\sigma(x)| \right\}, \tag{4.2}$$

where the infimum is taken is taken over all tensor fields σ, satisfying $S = \varepsilon_0^*(\sigma)$, i.e, tensor fields representing S. There is a $\widehat{\sigma} \in L^\infty(\Omega, \mathbb{R}^6)$ for which the infimum is attained.

Proof. The assertion follows from the fact that ε_0 is a linear and isometric injection as in Lemma (3.3) and using the Hahn-Banach theorem. See Appendix A for the details of the technical lemma used and its proof. □

Applying this lemma to $S = \gamma_0^*(t)$ one may draw the following conclusions.

Conclusion 4.2. Forces on the body given by essentially bounded surface tractions are represented by tensor fields on the body. These tensor fields are naturally interpreted as stress fields. The condition that a stress tensor field σ represents the surface traction t is

$$\gamma_0^*(t) = \varepsilon_0^*(\sigma), \tag{4.3}$$

and explicitly,

$$\int_{\Gamma_t} t \cdot \gamma_0(w)\, \mathrm{d}A = \int_{\Omega} \sigma(\varepsilon_0(w))\, \mathrm{d}V, \tag{4.4}$$

for each vector field $w \in LD(\Omega)_0$, i.e., a vector field of integrable stretching satisfying the boundary condition on Γ_0. This condition is just the principle of virtual work which is a weak form of the equation of equilibrium and the corresponding boundary conditions. Thus, we have derived both the existence of stresses and the equilibrium conditions analytically under mild assumptions.

It is noted that the subscript 0, only indicating the restriction of the various operations to fields satisfying the boundary conditions, may be omitted above. Also, as the restrictions of smooth vector fields on $\overline{\Omega}$ are dense in $LD(\Omega)$, it is sufficient to verify that the condition holds for smooth fields on $\overline{\Omega}$. For such fields, the integrand on the left may be replaced simply by $t \cdot w$.

Conclusion 4.3. There is an optimal stress field $\widehat{\sigma}$ representing t and

$$\|\gamma_0^*(t)\| = \|\widehat{\sigma}\|_\infty = \inf_{\sigma} \left\{ \operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)| \right\}, \tag{4.5}$$

where the infimum is taken over all stress fields σ satisfying $\gamma_0^*(t) = \varepsilon_0^*(\sigma)$, i.e., all stress fields in equilibrium with t. Thus, the infimum on the right is the optimal maximal stress σ_t^{opt}. In addition, by the definition of the dual norm we have

$$\|\gamma_0^*(t)\| = \sup_{w \in LD(\Omega)_0} \frac{\left|\gamma_0^*(t)(w)\right|}{\|w\|} \tag{4.6}$$

$$= \sup_{w \in LD(\Omega)_0} \frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1}, \tag{4.7}$$

where in the last line we used Equation (3.12). We conclude that

$$\sigma_t^{\mathrm{opt}} = \sup_{w \in LD(\Omega)_0} \frac{\left| \int_{\Gamma_t} t \cdot \gamma_0(w) \, \mathrm{d}A \right|}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} . \tag{4.8}$$

Recalling that the restrictions of smooth mappings on $\overline{\Omega}$ are dense in $LD(\Omega)$ and that for such mappings the trace mapping is just the restriction, the optimal stress may be evaluated as

$$\sigma_t^{\mathrm{opt}} = \sup_{w} \frac{\left| \int_{\Gamma_t} t \cdot w \, \mathrm{d}A \right|}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} , \tag{4.9}$$

where the supremum is taken over all smooth mappings in $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ that vanish on Γ_0.

It is noted that the value of σ_t^{opt} depends on the norm used for strain matrices.

5. The Associated Boundary Value Problem

In this section we derive formally the boundary value problem associated with the supremum on the right hand side of Equation (4.9), i.e., the Euler-Lagrange equation and boundary conditions. Thus, we will assume that the vector fields w are restrictions of smooth fields defined on $\overline{\Omega}$ so all differentiation and trace operations apply, and we will neglect questions like the existence of solutions of the resulting boundary value problem that possess these regularity properties. We also simplify the notation by writing

$$\int_{\partial\Omega} t \cdot w \, \mathrm{d}A \quad \text{for} \quad \int_{\Gamma_t} t \cdot \gamma_0(w) \, \mathrm{d}A, \quad \text{etc.} \tag{5.1}$$

Thus, let

$$Q(w) = \frac{\left| \int_{\Gamma_t} t \cdot w \, \mathrm{d}A \right|}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} , \tag{5.2}$$

then, we are looking for the extremum of $Q(w)$ as the solution w of

$$DQ(w) = 0,$$

where DQ denotes the Frechet derivative of Q. In other words

$$DQ(w)(u) = \frac{\mathrm{d}}{\mathrm{d}s} \left\{ \frac{\left| \int_{\Gamma_t} t \cdot (w + su) \, \mathrm{d}A \right|}{\int_\Omega |\varepsilon(w + su)| \, \mathrm{d}V} \right\}_{s=0} = 0 \tag{5.3}$$

for all vector fields u that vanish on Γ_0. We will use the 2-norm on the space of stretching matrices so

$$|\varepsilon(w)(x)| = |\varepsilon(w)(x)|_2 = \left(\varepsilon_{ij}(w)(x) \varepsilon_{ij}(w)(x) \right)^{\frac{1}{2}} \tag{5.4}$$

and in the following calculations we will omit the dependence on $x \in \Omega$ in the notation.

5.1. **Preliminary calculations.** Note that

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{f(s)}{g(s)} = 0$$

if

$$\frac{\mathrm{d}f}{\mathrm{d}s}g - f\frac{\mathrm{d}g}{\mathrm{d}s} = 0, \tag{5.5}$$

and that

$$\frac{\mathrm{d}\,|f|}{\mathrm{d}f} = \operatorname{sign} f = \frac{f}{|f|}. \tag{5.6}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left|\int_{\partial\Omega} t\cdot(w+su)\,\mathrm{d}A\right|_{s=0} = \operatorname{sign}\left[\int_{\partial\Omega} t\cdot w\,\mathrm{d}A\right]\int_{\partial\Omega} t\cdot u\,\mathrm{d}A. \tag{5.7}$$

Also,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_{\Omega}|\varepsilon(w+su)|_2\ \mathrm{d}V\right\}_{s=0} = \int_{\Omega}\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\varepsilon_{ij}(u)\,\mathrm{d}V \tag{5.8}$$

$$= \int_{\Omega}\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\,u_{i,j}\,\mathrm{d}V \quad \text{(using the symmetry of } \varepsilon\text{)} \tag{5.9}$$

$$= \int_{\Omega}\left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\,u_i\right)_{,j}\mathrm{d}V - \int_{\Omega}\left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\right)_{,j}u_i\,\mathrm{d}V \tag{5.10}$$

$$= \int_{\partial\Omega}\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\,u_i\nu_j\,\mathrm{d}A - \int_{\Omega}\left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\right)_{,j}u_i\,\mathrm{d}V. \tag{5.11}$$

Thus, using Equations (5.11), (5.7), (5.5), in (5.3), we have

$$0 = \operatorname{sign}\left(\int_{\partial\Omega} t\cdot w\,\mathrm{d}A\right)\int_{\partial\Omega} t\cdot u\,\mathrm{d}A\cdot\int_{\Omega}|\varepsilon(w)|\ \mathrm{d}V - \left|\int_{\partial\Omega} t\cdot w\,\mathrm{d}A\right|\left[\int_{\partial\Omega}\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\,u_i\nu_j\,\mathrm{d}A - \int_{\Omega}\left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\right)_{,j}u_i\,\mathrm{d}V\right]. \tag{5.12}$$

Multiplying the equation by sign $\left(\int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A\right)$ and using $(\operatorname{sign} z)^2 = 1$, and $z = \operatorname{sign}(z)\,|z|$, we rewrite the condition as

$$0 = \int\limits_{\partial\Omega} t\cdot u\,\mathrm{d}A \cdot \int\limits_{\Omega} |\varepsilon(w)|\,\mathrm{d}V - \int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A \left[\int\limits_{\partial\Omega} \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2} u_i \nu_j\,\mathrm{d}A - \int\limits_{\Omega} \left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\right)_{,j} u_i\,\mathrm{d}V\right] \quad (5.13)$$

for all fields u vanishing on Γ_0.

5.2. **The differential equation and boundary conditions.** In order to extract the differential equation and boundary condition from the previous equation we first consider all fields u that vanish on all of $\partial\Omega$. Thus, we obtain (clearly, $\int_{\partial\Omega} t\cdot w\,\mathrm{d}A = 0$ will give a minimum of Q)

$$0 = \int\limits_{\Omega} \left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\right)_{,j} u_i\,\mathrm{d}V, \quad (5.14)$$

implying the differential equation

$$\left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\right)_{,j} = 0, \quad \text{in } \Omega. \quad (5.15)$$

Using Equation (5.14) in (5.13), we obtain the boundary condition

$$0 = \int\limits_{\partial\Omega} t\cdot u\,\mathrm{d}A \cdot \int\limits_{\Omega} |\varepsilon(w)|\,\mathrm{d}V - \int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A \cdot \int\limits_{\partial\Omega} \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2} u_i\nu_j\,\mathrm{d}A$$

for all vector fields u on the boundary. This may be rewritten as

$$0 = \int\limits_{\partial\Omega} \left[\left(\int\limits_{\Omega} |\varepsilon(w)|\,\mathrm{d}V\right) t_i - \left(\int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A\right) \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\nu_j\right] u_i\,\mathrm{d}A$$

that finally gives the complicated boundary conditions as the integral equation

$$\left(\int\limits_{\Omega} |\varepsilon(w)|\,\mathrm{d}V\right) t_i - \left(\int\limits_{\partial\Omega} t\cdot w\,\mathrm{d}A\right) \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|_2}\nu_j = 0 \quad \text{on} \quad \Gamma_t. \quad (5.16)$$

(Note that $\varepsilon_{ij}(w)/\,|\varepsilon(w)|_2$ is a field on the boundary.)

5.3. **Simplification of the differential equation and boundary conditions.** The equation and boundary conditions of (5.15) and (5.16) may be further simplified by using a "hat" ($\widehat{\ }$) to indicate unit vectors (tensors) so

$$\widehat{\varepsilon}_{ij}(w) = \frac{\varepsilon(w)_{ij}}{|\varepsilon(w)|}. \tag{5.17}$$

Thus, Equation (5.16) may be rewritten as

$$t_i = \frac{\int_{\partial\Omega} t \cdot w \,\mathrm{d}A}{\int_{\Omega} |\varepsilon(w)| \,\mathrm{d}V} \widehat{\varepsilon}(w)_{ij}\nu_j. \tag{5.18}$$

For a maximizing w, the basic result implies that the quotient above is just $\pm\sigma_t^{\mathrm{opt}}$, so we finally have the following boundary value problem

$$\widehat{\varepsilon}(w)_{ij,j} = 0, \quad \text{in } \Omega, \qquad t_i = \pm\sigma_t^{\mathrm{opt}}\widehat{\varepsilon}(w)_{ij}\nu_j, \quad \text{in } \partial\Omega. \tag{5.19}$$

6. The Relation To Stress Concentration and the Norm of the Trace Mapping

We now turn to the simple proof of Theorem (1.2) and the derivation of the boundary value problem associated with the generalized stress concentration factor.

6.1. **Proof of Theorem (1.2).** We had

$$\sigma_t^{\mathrm{opt}} = \sup_{w \in LD(\Omega)_0} \frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1},$$

so

$$K = \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \frac{\sigma_t^{\mathrm{opt}}}{\|t\|} \tag{6.1}$$

$$= \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \left\{ \sup_{w \in LD(\Omega)_0} \frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1 \|t\|} \right\}, \tag{6.2}$$

$$= \sup_{w \in LD(\Omega)_0} \frac{1}{\|\varepsilon(w)\|_1} \left\{ \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \frac{|t(\gamma_0(w))|}{\|t\|} \right\}. \tag{6.3}$$

Let $i\colon \mathbf{W} \to \mathbf{W}^{**}$ be the natural injection given by $i(w)(f) = f(w)$ and let $w^{**} = i(w)$. Then, it is standard that $\|w^{**}\| = \|w\|$ (e.g., [4, pp., 191-192]). Thus,

$$\|w\| = \sup_{f \in \mathbf{W}^*} \frac{|w^{**}(f)|}{\|f\|} = \sup_{f \in \mathbf{W}^*} \frac{|f(w)|}{\|f\|}, \tag{6.4}$$

and we have

$$\sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \frac{|t(\gamma_0(w))|}{\|t\|} = \|\gamma_0(w)\|. \tag{6.5}$$

Using $\|w\| = \|\varepsilon(w)\|_1$ for $w \in LD(\Omega)_0$, we finally obtain

$$K = \sup_{w \in LD(\Omega)_0} \frac{\|\gamma_0(w)\|}{\|w\|} = \|\gamma_0\| \tag{6.6}$$

by the definition of the norm of a linear mapping. □

6.2. **The boundary value problem corresponding to the generalized stress concentration factor.** We wish to maximize formally the quotient

$$Q = \frac{\int_{\Gamma_t} |\gamma_0(w)| \, \mathrm{d}A}{\int_\Omega |\varepsilon(w)|_2 \, \mathrm{d}V} \tag{6.7}$$

so all the disclaimers and remarks regarding the notation made in the beginning of Section 5 still apply. Thus, using

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\{ \int_{\Gamma_t} |w + su| \, \mathrm{d}A \right\}_{s=0} = \int_{\Gamma_t} \frac{w_i}{|w|} u_i \, \mathrm{d}A, \tag{6.8}$$

the condition $DQ(w)(u) = 0$, for all vector fields u vanishing on Γ_0, becomes

$$0 = \int_{\Gamma_t} \frac{w_i}{|w|} u_i \, \mathrm{d}A \cdot \int_\Omega |\varepsilon(w)| \, \mathrm{d}V - \\ \int_{\Gamma_t} |w| \, \mathrm{d}A \left(\int_{\Gamma_t} \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|} \nu_j u_i \, \mathrm{d}A - \int_\Omega \left(\frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|} \right)_{,j} u_i \, \mathrm{d}V \right). \tag{6.9}$$

Again we apply this condition first to all fields that vanish on $\partial\Omega$ to obtain the condition (5.14) again so the same partial differential equation (5.15) holds. Then, using the condition (5.14) above, we obtain the boundary conditions

$$0 = \int_{\Gamma_t} \left[\left(\int_\Omega |\varepsilon(w)| \, \mathrm{d}V \right) \frac{w_i}{|w|} - \left(\int_{\Gamma_t} |w| \, \mathrm{d}A \right) \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|} \nu_j \right] u_i \, \mathrm{d}A. \tag{6.10}$$

This condition may be rewritten as

$$\frac{w_i}{|w|} = \frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} \frac{\varepsilon_{ij}(w)}{|\varepsilon(w)|} \nu_j, \quad \text{on} \quad \Gamma_t. \tag{6.11}$$

For the extremizing w we have

$$\frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} = \|\gamma_0\| = K, \tag{6.12}$$

and with the notation of Subsection (5.3), the boundary value problem assumes the form

$$\widehat{\varepsilon}(w)_{ij,j} = 0, \quad \text{in } \Omega, \qquad \widehat{w}_i = K \widehat{\varepsilon}(w)_{ij} \nu_j, \quad \text{in } \Gamma_t. \tag{6.13}$$

Acknowledgements. This work was partially supported by the Paul Ivanier Center for Robotics Research and Production Management at Ben-Gurion University.

APPENDIX A. THE TECHNICAL LEMMA

Lemma A.1. *Let* $\mathbf{W}$ *and* $\mathbf{V}$ *be two Banach spaces and* $\varphi\colon \mathbf{W} \longrightarrow \mathbf{V}$ *an isometric injection.*
(*i*) *For each* $S \in \mathbf{W}^*$ *there is some (non-unique)* $\sigma \in \mathbf{V}^*$*, such that*

$$S = \varphi^*(\sigma). \tag{A.1}$$

(*ii*) *The dual norm of* S *satisfies*

$$\|S\| = \inf_{\sigma} \|\sigma\|, \tag{A.2}$$

where the infimum is taken over all σ *representing* S*, i.e., those satisfying*

$$S = \varphi^*(\sigma). \tag{A.3}$$

(*iii*) *There is a* $\widehat{\sigma} \in \mathbf{V}^*$ *such that*

$$\|S\| = \inf_{\sigma} \|\sigma\| = \|\widehat{\sigma}\|. \tag{A.4}$$

Proof. Given $S \in \mathbf{W}^*$, we may use the fact that

$$\varphi^{-1}\colon \mathrm{Image}\varphi \subset \mathbf{V} \longrightarrow \mathbf{W} \tag{A.5}$$

is a well defined linear isometry to write

$$\left| S(\varphi^{-1}(v)) \right| \leqslant \|S\| \|\varphi^{-1}(v)\| = \|S\| \|v\|. \tag{A.6}$$

It follows that

$$S \circ \varphi^{-1}\colon \mathrm{Image}\varphi \longrightarrow \mathbb{R} \tag{A.7}$$

is a bounded linear functional on the subspace $\mathrm{Image}\varphi \subset \mathbf{V}$. We recall that the Hahn-Banach theorem asserts that if $\mathbf{U} \subset \mathbf{V}$ is a vector subspace and τ is a bounded linear functional on $\mathbf{U}$, then, τ may be extended to a bounded linear functional σ on $\mathbf{V}$ such that

$$\sigma(u) = \tau(u), \quad \text{for all} \quad u \in \mathbf{U}, \tag{A.8}$$

and

$$\|\sigma\| = \sup_{v\in\mathbf{V}} \frac{|\sigma(v)|}{\|v\|} = \sup_{u\in\mathbf{U}} \frac{|\tau(u)|}{\|u\|} = \|\tau\|. \tag{A.9}$$

Applying the Hahn-Banach theorem to the situation at hand, we conclude that the functional $S \circ \varphi^{-1}$ may be extended to a linear functional σ on $\mathbf{V}$ such that

$$\sigma(u) = S \circ \varphi^{-1}(u), \quad \text{for all} \quad u \in \mathrm{Image}\varphi, \tag{A.10}$$

or equivalently,

$$S(w) = \sigma(\varphi(w)). \tag{A.11}$$

By the definition of the dual mapping we conclude that $S = \varphi^*(\sigma)$.

In general, for any $\sigma \in \mathbf{V}^*$

$$\|\varphi^*(\sigma)\| = \sup_{w\in\mathbf{W}} \frac{|\varphi^*(\sigma)(w)|}{\|w\|} = \sup_{w\in\mathbf{W}} \frac{|\sigma(\varphi(w))|}{\|w\|} \leqslant \sup_{w\in\mathbf{W}} \frac{\|\sigma\|\|\varphi(w)\|}{\|\varphi(w)\|}, \quad \text{(A.12)}$$

so

$$\|\varphi^*(\sigma)\| \leqslant \|\sigma\|. \quad \text{(A.13)}$$

On the other hand, for any $\sigma \in \mathbf{V}^*$, such that $S = \varphi^*(\sigma)$

$$\sup_{w\in\mathbf{W}} \frac{|S(w)|}{\|w\|} = \sup_{v\in\text{Image}\varphi} \frac{\left|S\circ\varphi^{-1}(v)\right|}{\|v\|} = \sup_{v\in\text{Image}\varphi} \frac{|\sigma(v)|}{\|v\|}, \quad \text{(A.14)}$$

so by the Hahn-Banach theorem

$$\|S\| = \|\varphi^*(\sigma)\| = \|\widehat{\sigma}\|, \quad \text{(A.15)}$$

where $\widehat{\sigma}$ is the element of $\mathbf{V}^*$extending $S\circ\varphi^{-1}$and having the same norm as in Equation (A.9).

We conclude that

$$\|S\| = \inf_{\sigma} \|\sigma\|, \quad \text{(A.16)}$$

where infimum is taken over all σ satisfying $S = \varphi^*(\sigma)$. The infimum is attained for $\widehat{\sigma}$ as above. □

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