LOAD CAPACITY RATIOS FOR STRUCTURES

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ABSTRACT. For a given structure made of a perfectly plastic material with a yield stress s_Y, we consider the load capacity ratio of the structure: the largest positive number C, depending only on the geometry of the structure, which satisfies the following property. For any loading distribution f on the structure whose maximum is $f_{\max}$, the structure will not undergo plastic collapse as long as $f_{\max} \leqslant s_Y C$, independently of the distribution of the load. The paper presents the mathematical aspects, related mechanical notions, algorithms and examples corresponding to load capacity ratios of structures.

KEYWORDS: Stress analysis; structural analysis; load capacity ratio; limit analysis; optimization; multi-parameter loading.

1. INTRODUCTION

1.1. **Background.** While the basic problem of limit analysis of rigid-perfectly plastic structures is to determine, for a given load distribution f, the largest positive multiplier λ_f such that the structure does not collapse under the loading μf for all $\mu < \lambda_f$ (see *e.g.*, [1, 2, 3, 4, 5]), the importance of multi-parameter load analysis has been recognized since the pioneering works on plastic structural analysis (*e.g.*, [6, 7, 8, 9, 1, 2, 10, 11, 12, 13] and the comprehensive bibliographical survey in [14]). For multi-parameter loading, given the load vectors $f_1, \dots, f_D$, one considers the load space consisting of all load vectors of the form

$$f = \sum_{d=1}^{D} \mu_d f_d, \qquad \mu_d \in \mathbb{R}. \tag{1.1}$$

Then, one naturally seeks the *yield point plastic interaction surface* (the term used in [15, 14, 16]) or *collapse surface* (as used in [17, 18]), containing all limit forces, *i.e.*, the collection of all loads of the form $\lambda_f f$ for some load vector f. The relevance of the collapse surface to strength analysis is quite clear intuitively. It is a surface that separates the loads that the structure can sustain without collapse from those it cannot support. A historical overview of the earlier work on the subject may be found in [17, 18]. Some general properties related to the analogy between the yield surface in stress space and the collapse surface in load space were studied by Hodge [19] and Shoemaker [20].

In spite of importance attributed to the determination of the collapse surface, the corresponding results on the subject are quite limited. On the one hand, the

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determination of the collapse surface is computationally involved (see [15, 17, 21, 18]). On the other hand, even if the collapse surface had been determined, additional attention should be given to its useful representation and use in cases with large numbers of parameters.

The approach to the analysis of collapse surfaces proposed in the present work may be described as follows. We seek the radius M of the largest sphere in load space contained within the collapse surface. Thus, one is guaranteed that the structure will not collapse under any load f as long as the magnitude $\|f\|$ of f is smaller than M. Since by its definition, M is the largest such upper bound, for any $M' > M$, the structure will collapse for some force f' with $\|f'\| = M'$. Various norms for the load vectors may be used in order to evaluate the magnitude $\|f\|$ of forces, however, in this work $\|f\|$ will indicate the maximum $f_{\max}$ of the applied load over the various points in the structure. Thus, once M is computed, one is guaranteed that the structure will not collapse as long as the load distribution on the structure is such that

$$f_{\max} < M, \tag{1.2}$$

independently of the distribution of f.

Noting that unlike earlier work on the subject, our load space contains all conceivable loading vectors pertaining to the given model of the structure, we present a method for computing a worst case loading distribution f_C whose maximum is M such that the structure will collapse plastically under the loading λf_C for all $\lambda > 1$.

It is shown that for the homogeneous isotropic case, the value M is the product of s_Y, the yield stress of the material that makes up the structure, and a number C which is a purely geometric property of the structure. (For a non-homogeneous, anisotropic situation, a straightforward modification should be implemented.) Thus, the structure will not collapse under a loading f if

$$f_{\max} < s_Y C. \tag{1.3}$$

The number C will be referred to as the *load capacity ratio* of the structure. This should not be confused with the general use of "load carrying capacity" in the plasticity literature, *e.g.*, [1].

It is noted that a design of a structure that is based on all conceivable loadings will be inherently inefficient. It seems to us that stress analysis based on all possible loads should be used after an initial design is made in order to find weaknesses in the structure to unexpected loadings. In Section 7 we give examples for the computations of "worst load cases" on structures. In addition, such an analysis may be used in order to compare various designs of the same structure. For cases where such a comprehensive analysis is not needed, we will also present the results for the situation where loads are applied to only a specific part of the body. For example, consider the plain strain problem shown in Figure 7.13 (note that the support is assumed to be damaged at the bottom left) and consider only forces that act on the left slope. Then, the structure will not collapse for any loading applied in that region as long as the maximum satisfies $f_{\max} \leqslant 0.023 s_Y$.

A worst case loading is shown in Figure 7.14(a). (Note that these values are not dimensionless.)

Limit analysis, as well as other notions within the field of plasticity, are usually formulated using terminology and results of convex analysis. Here however, since we study the norms of force vectors, we use the analytic analogs. Thus for example, the yield function is regarded here as a norm, or a semi-norm, on the space of stress matrices and the yield surface is a sphere in the stress space whose radius is the yield stress s_Y.

For the sake of completeness, we present below the basic results of limit analysis in the context of norms. It turns out that for the formulation of the basic results, the only constitutive information needed is a norm reflecting some failure criterion for the material. The following paragraphs describe intuitively the mechanical interpretation of the various analytical objects involved and the resulting expression for C.

1.2. **Stress Optimization.** The methods presented above follow from a theoretical analysis of optimal stress fields for a loaded structure. As is evident from vast literature on optimal plastic design (*e.g.*, [22] and works cited therein) and the intuitive observation that optimal structures are fully stressed when loaded, the notion of an optimal stress field gives a fresh point of view on this issue.

By a *structure* we will mean a model of a body having a finite number of degrees of freedom. A finite element model of a continuous body will serve as a standard example of a structure. In the following paragraphs, a structure will be defined only by its geometry, and unless we state otherwise, no reference will be made to its material properties.

Consider a structure Ω on which an external loading f is given. Assuming that the structure is statically indeterminate, there is a collection Φ_f of internal force vectors, or stress fields, such that each $\sigma \in \Phi_f$ is in equilibrium with f. Within the collection Φ_f, we wish to find the optimal stress field σ^{opt}. Throughout this work, the cost function for the consideration of optimality of stress fields will be defined in terms of a norm $\|\cdot\|$ on the space of stress fields. Thus, for an optimal stress field (or synonymously local force vector) $\sigma_f^{\text{opt}} \in \Phi_f$ and using the notation $s_f^{\text{opt}} = \|\sigma_f^{\text{opt}}\|$, one has

$$s_f^{\text{opt}} \leqslant \|\sigma\|, \quad \text{for all} \quad \sigma \in \Phi_f. \tag{1.4}$$

As an example for such a norm, consider a finite element model of a body consisting of uniform stress elements. Let σ_e be the constant matrix representing the stress within the element e and let $|\sigma_e|$ be the magnitude of the stress.[1] We set,

$$\|\sigma\| = \max_l |\sigma_l| = \max_{x\in\Omega} |\sigma(x)|. \tag{1.5}$$

[1]The magnitude of a stress matrix will be induced typically by some failure criterion or a yield function.

In other words, our cost function is the maximal equivalent stress and in our optimization we wish to find an equilibrating stress field for which the maximum of the corresponding equivalent stress is the least. An expression of the optimum s_f^{opt} will be given in Section 3.1.

It is noted that Truesdell and Toupin [23] present the results obtained by Signorini and extended by Grioli concerning bounds on the maximum of the a stress field in equilibrium with a given force (see the references cited in [23]). Later, Day [24] obtained bounds on the L^p-norms of equilibrating stress fields. These results, that are analogous to the treatment here in the sense that no constitutive assumption is made, give only lower bounds on the maximum of the stress components. Thus, as the bounds are not exact, they do not provide the optimal values for the maximal stresses.

As noted in Remark 3.6, with a different choice of norms, the stress optimization problem as defined above is transformed into the structural optimization problem as in [9, 22].

1.3. **The Stress Sensitivity.** If we normalize the optimal stress by the maximum of the applied force, we obtain the number

$$K_f = \frac{s_f^{\mathrm{opt}}}{f_{\max}} \tag{1.6}$$

which is reminiscent of the stress concentration factor of engineering stress analysis when one replaces the maximum of the nominal stress by the maximum of the applied load. Then, letting the applied force vary, we consider the *stress sensitivity* of the structure, a purely geometrical property, defined as

$$K = \max_f \left\{K_f\right\} = \max_f \frac{s_f^{\mathrm{opt}}}{f_{\max}}. \tag{1.7}$$

It will be shown, that the stress sensitivity is closely related to the load capacity ratio described above.

1.4. **The Mathematical Setting.** From the mathematical point of view, the problem of statically indeterminate structures is a solution of an under-determined system of linear equations of the form

$$B\sigma = f, \tag{1.8}$$

where the matrix B is of maximal rank. Thus, assuming we have a norm on the space of vectors σ, we are looking for the solution $\sigma^{\mathrm{opt}} \in \Phi_f$ of minimal norm. A particularly interesting norm one could use for the vectors σ is

$$\|\sigma\| = \max_{n=1,\dots,N} \{|\sigma_n|\} = \|\sigma\|^{\infty}, \tag{1.9}$$

where N is the dimension of the space of stress field entities σ (which is assumed to be finite for a structure).

Our results use duality theory. We regard both f and σ as linear functionals operating on the spaces of virtual velocities, external and internal, respectively, and give the value $s_f^{\text{opt}} = \|\sigma_f^{\text{opt}}\|$ in terms of the mapping B and the action of f. Denoting a generic virtual velocity field by w, the expressions we obtain may be written generally in the forms

$$s_f^{\text{opt}} = \max_w \left\{ \frac{f^{\mathsf{T}} w}{\|B^{\mathsf{T}}(w)\|} \right\}, \qquad \frac{1}{C} = \max_w \left\{ \frac{\|w\|}{\|B^{\mathsf{T}}(w)\|} \right\}, \tag{1.10}$$

where the norms used for $\|B^{\mathsf{T}}(w)\|$ are dual to those implied by the failure criterion.

1.5. **Attaining the Optimal Stress: Plasticity and Limit Loads.** A natural question arising in the context of optimal stresses is whether the optimal stress field for a given loading can be realized. One conceivable way to attain the optimal stress field would be to introduce pre-stress or a residual stress field in the structure. Attempts for achieving such optimal stress fields are done in the processes of autofrettage and shot peening. Residual stresses are also common in biological structures.

Moreover, it turns out that perfectly plastic materials are naturally optimal in the following sense. For the case of a perfectly plastic structure for which

$$s_Y = s_f^{\text{opt}}, \tag{1.11}$$

and for which the plastic yield criterion is used for the evaluation of equivalent stresses, the mathematical theory of plasticity implies that an optimal stress field is distributed in the structure. In fact, using the terminology of plasticity theory, such a stress field in the body is a *limit stress distribution* where the structure is on the verge of plastic collapse.

Since the equilibrium equations are linear, it is natural to expect, and indeed proved in Section 3.1, that the optimal stress will depend on the loading in a homogeneous manner, *i.e.*,

$$s_{\lambda f}^{\text{opt}} = |\lambda|\, s_f^{\text{opt}}, \quad \text{for all} \quad \lambda \in \mathbb{R}. \tag{1.12}$$

Thus, the transformation

$$\pi : f \mapsto \pi(f) = \frac{s_Y}{s_f^{\text{opt}}} f \tag{1.13}$$

maps each loading f into a limit force bringing the structure to the verge of collapse (as follows immediately from the observation that $s_{\pi(f)}^{\text{opt}} = s_Y$). In other words, the transformation π is a projection of the space of all loads onto the collapse surface—the collection of loads that cause the structure to be in plastic limit states. Thus, for any loading f' such that

$$\|f'\| < \min_f \|\pi(f)\| = \min_f \left\{ \left\| \frac{s_Y}{s_f^{\text{opt}}} f \right\| \right\} = s_Y \cdot C, \tag{1.14}$$

where,

$$\frac{1}{C} = K = \max_{f} \left\{ \frac{s_f^{\mathrm{opt}}}{\|f\|} \right\},$$

collapse will not occur. As noted, K and C are purely geometric properties of the body.

1.6. **The Organization of the Paper.** The following sections present the details of the theoretical analysis as well as some steps needed in order to apply it to practical stress analysis. Section 2 sets up the mathematical notation and Section 3 contains the basic mathematical analysis of the notions presented above and the resulting expressions for their computations. The implementation to plasticity theory and the notion of load capacity ratio are presented in Section 4. Section 5 describes the steps necessary for the applications of the method to some particular mechanical systems. In particular, we consider trusses, frames, plane stress and plane strain systems models. Section 6 describes some of the numerical and algorithmical aspects involved with the implementations and finally, Section 7 presents the examples of the systems considered above.

The mathematical framework presented above is the finite dimensional analog of earlier work pertaining to continuous bodies. For the sake of completeness, we review in Appendix B the results obtained in [25] for continuous bodies loaded on their boundaries. See also [26, 27, 28] for some additional aspects of the theory for continuous bodies. We do not study here the way the finite dimensional structural model approximates a continuous body and consider it as a mechanical system for its own sake. In addition to the simplification it affords, the finite dimensional setting makes it possible to get some further concrete results. The paper is meant to be self-contained, and for this reason, it includes quite a lot of detail.

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2. Notation and Preliminaries

The infinitesimal kinematics of a structure Ω having D degrees of freedom is characterized by the vector space $\mathscr{W}$ containing the *external* or *global velocity fields* or *virtual displacements*. A velocity field $w \in \mathscr{W}$ is represented as a linear combination of D base functions ψ_d in the form

$$w = w^d \psi_d \tag{2.1}$$

where the summation convention is used.

The structure is assumed to be composed of L structural elements $\Omega_l \subset \Omega$, $l = 1, \ldots, L$, each having N_l degrees of freedom. The vector space of local virtual displacements of the element Ω_l will be denoted by $\mathscr{S}_l$. A member $\chi_l \in \mathscr{S}_l$ is interpreted kinematically as a *virtual strain field* within the element Ω_l. The Cartesian product space

$$\mathscr{S} = \prod_{l=1}^{L} \mathscr{S}_l \tag{2.2}$$

contains collections of strain fields over the various elements. Evidently, one cannot expect that the various components $\chi_1 \in \mathscr{S}_1, \dots, \chi_L \in \mathscr{S}_L$, of a generic $\chi \in \mathscr{S}$ be compatible.

Since displacement fields in the structure induce (by restriction, differentiation, etc.) displacement fields and strain fields in each of the elements, one has a linear mapping

$$A \colon \mathscr{W} \longrightarrow \mathscr{S}. \tag{2.3}$$

It is assumed that the structure is supported such that a rigid body motion is prevented. It follows from the Liouville theorem (*e.g.*, [29]) that the strain field in the structure vanishes if and only if the displacement $w = 0$. Accordingly, it is assumed throughout this work that the linear mapping A is injective. On the other hand, one cannot expect A to be surjective because of the existence of non-compatible strain fields.

Forces and stresses are introduced as elements of the respective dual spaces. Thus, an *external* or a *global force vector* f is an element of $\mathscr{W}^*$, a linear mapping $f \colon \mathscr{W} \to \mathbb{R}$. The evaluation $f(w) \in \mathbb{R}$ is interpreted mechanically as the virtual power expanded by the force distribution specified by f for the velocity field w so f may be regarded as a generalized force. Similarly, an element σ of $\mathscr{S}^*$ is interpreted as a *generalized local force field* or a *stress field* so that the evaluation $\sigma(\chi)$ is interpreted as the virtual power expended by the local force or stress σ for the local virtual displacement χ. The next elementary example illustrates the notation.

Example 2.1. Figure 2.1 shows a very simple truss structure having two degrees of freedom. The illustration on the left show the variables pertaining to $\mathscr{W}$. Any element of $\mathscr{W}$ is of the form $(w_1, w_2)^{\mathsf{T}}$. The variables corresponding to the to the space $\mathscr{S}$ are illustrated on the right. The elements of $\mathscr{S}$ may be chosen to be of the form $\chi = (V_1\varepsilon_1, V_2\varepsilon_2, V_3\varepsilon_3)^{\mathsf{T}}$, where ε_i is the virtual axial strain in the i-th bar, and V_i is its volume. Thus, an element of $\mathscr{S}^*$ will be of the form $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and the virtual work of the local forces is of the form $\sum_i \sigma_i \chi_i$. The matrix of the mapping A gives the linear approximation for χ in terms of w and is easily found to be of the form

$$\begin{pmatrix} V_1 \cos\theta_1 & -V_1 \sin\theta_1 \\ V_2 \cos\theta_2 & -V_2 \sin\theta_2 \\ V_3 \cos\theta_3 & -V_3 \sin\theta_3 \end{pmatrix}.$$

The equilibrium condition may be written using the principle of virtual work as

$$f(w) = \sigma(A(w)), \quad \text{for all} \quad w \in \mathscr{W}. \tag{2.4}$$

Using the dual (adjoint) mapping $A^* : \mathscr{S}^* \to \mathscr{W}^*$, defined by $A^*(\sigma)(w) = \sigma(A(w))$, the equilibrium condition assumes the form

$$A^*(\sigma) = f. \tag{2.5}$$

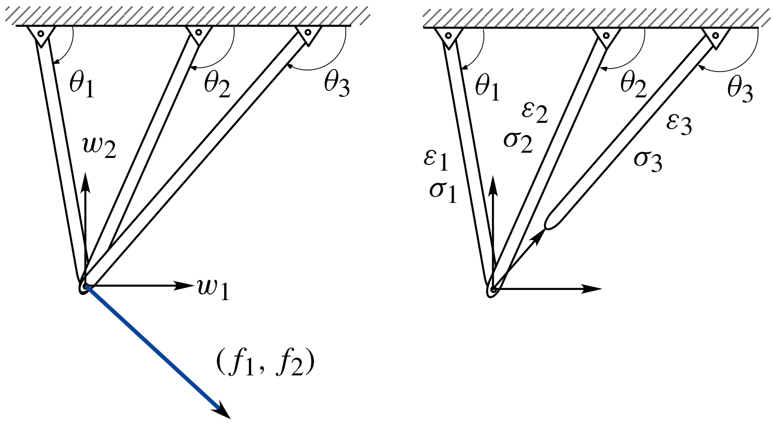

FIGURE 2.1. Example of global and local virtual displacements spaces

Thus, setting $B = A^*$ we retrieve Equation (1.8).

Since we are considering finite dimensional vector spaces, we have natural isomorphisms $\mathscr{W}^{**} \cong \mathscr{W}$, and $\mathscr{S}^{**} \cong \mathscr{S}$. We will assume that a failure criterion is given in terms of a norm $\|\sigma\|$ on the space $\mathscr{S}^*$ containing the stress-like objects. Examples of such failure criteria will be given in Section 5. Such a norm on $\mathscr{S}^*$ induces naturally a dual norm[2] on $\mathscr{S}$ that is given by

$$\|\chi\| = \max\left\{\frac{|\sigma(\chi)|}{\|\sigma\|}; \sigma \in \mathscr{S}^*, \sigma \neq 0\right\}. \tag{2.6}$$

This dual norm satisfies

$$\|\sigma\| = \max\left\{\frac{|\sigma(\chi)|}{\|\chi\|}; \chi \in \mathscr{S}, \chi \neq 0\right\}. \tag{2.7}$$

Similarly, we will assume that a pair of dual norms is given on $\mathscr{W}$ and $\mathscr{W}^*$ respectively satisfying duality relations analogous to Equations (2.6) and (2.7).

Example 2.2. Let $\tau \in L_{\mathrm{Symm}}(\mathbb{R}^3, \mathbb{R}^3)$ be a symmetric stress matrix and denote by $|\tau| \in \mathbb{R}^+$ the equivalent stress as based on some chosen failure criterion. It will be assumed that $|\tau|$ is a (semi) norm on the space of symmetric matrices. If σ is a stress distribution in the structure Ω, it is customary in traditional stress analysis to ensure that

$$\max_{x \in \Omega} |\sigma(x)| \tag{2.8}$$

remains bounded by a certain permitted value. From the mathematical point of view it will be advantageous to omit large values of $|\sigma(x)|$ if they occur in regions of zero volume and so we will consider

$$\|\sigma\|^\infty = \operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)|. \tag{2.9}$$

[2]The definition of the dissipation function in limit analysis (*e.g.*, [4, p. 116] is the analog of Equation (2.6).

Depending on whether $|\cdot|$ is a norm or a semi-norm on the space of symmetric matrices, $\|\cdot\|^\infty$ is a norm or a semi-norm on the space of stress fields. As mentioned above, failure will occur if $\|\sigma\|^\infty$ exceeds a certain value. Thus, traditional stress analysis provides one with mechanically meaningful (semi) norms on the space of stress fields.

In the particular case where the stress distribution σ is assumed to be piecewise uniform, *i.e.*, a constant matrix σ_l within each structural element Ω_l,

$$\|\sigma\|^\infty = \max_{l=1,\dots,L} |\sigma_l|. \tag{2.10}$$

Example 2.3. It is customary in finite element analysis to specify the external load in terms of force vectors $f = (f_1, \dots, f_G)$ applied at the grid points $g = 1, \dots, G$. Let $|f_g|$ be the magnitude (evaluated using some norm on $\mathbb{R}^3$) of the force at the grid point g, then,

$$\|f\|^\infty = \max_{g=1,\dots,G} |f_g|, \tag{2.11}$$

is a norm on $\mathscr{W}^*$ indicating the maximum of the applied load.

3. Basic Results for Structures

The methods we employ may be regarded as approximations of the expressions we presented for the case of continuous bodies. Thus, one may think of the expressions we will use as approximating searches in finite dimensional subspaces of the corresponding infinite dimensional vector spaces appearing in Equations (B.4), (B.6) and (B.8). However, for the sake of completeness and in order to obtain some additional results, we will present the theorems pertaining to structures and their proofs in detail in the following paragraphs.

3.1. Optimal Stresses.

We start with the results concerning optimal stress distributions that equilibrate a given external loading $f \neq 0$. Thus, for a structure given in terms of a linear injective mapping $A : \mathscr{W} \to \mathscr{S}$ and a given norm on $\mathscr{S}^*$, we consider the affine subspace $\Phi_f \subset \mathscr{S}^*$ of solutions to the equilibrium equation $A^*(\sigma) = f$. We set

$$s_f^{\text{opt}} = \min_{\sigma \in \Phi_f} \{\|\sigma\|\} \tag{3.1}$$

and refer to s_f^{opt} as the *optimal stress.* If there is some σ_f^{opt} with $\|\sigma_f^{\text{opt}}\| = s_f^{\text{opt}}$, we will refer to σ_f^{opt} as an *optimal stress field.*

Proposition 3.1. *The optimal stress can be evaluated by*

$$s_f^{\text{opt}} = \max \left\{ \frac{|f(w)|}{\|A(w)\|}, w \in \mathscr{W}, w \neq 0 \right\}. \tag{3.2}$$

In addition, the optimum is attained for some optimal stress field σ_f^{opt}.

Proof. Firstly, we note that since the kinematic mapping A is injective, its inverse

$$A^{-1} : \operatorname{Image} A \longrightarrow \mathscr{W}$$

exists and is a linear bijection. Given a load $f \in \mathscr{W}^*$, one can use A^{-1} to define a linear mapping

$$\widehat{\sigma} : \operatorname{Image} A \longrightarrow \mathbb{R} \quad \text{by} \quad \widehat{\sigma}(\chi_0) = f(A^{-1}(\chi_0)). \tag{3.3}$$

Clearly, $\widehat{\sigma}(A(w)) = f(w)$, for all $w \in \mathscr{W}$.

Let $\sigma' \in \Phi_f \subset \mathscr{S}^*$ be any solution of the equilibrium equation so

$$\sigma'(A(w)) = f(w), \quad \text{for all} \quad w \in \mathscr{W}. \tag{3.4}$$

Hence,

$$\sigma'(\chi_0) = f(A^{-1}(\chi_0)) = \widehat{\sigma}(\chi_0), \quad \text{for all} \quad \chi_0 \in \operatorname{Image} A. \tag{3.5}$$

It follows that the restriction of any solution $\sigma' : \mathscr{S} \to \mathbb{R}$ from $\mathscr{S}$ to $\operatorname{Image} A \subset \mathscr{S}$ is $\widehat{\sigma}$ defined above. In other words, any solution $\sigma' \in \Phi_f$ is an extension of $\widehat{\sigma}$ from $\operatorname{Image} A$ to the entire $\mathscr{S}$. It is also noted that as a member of $(\operatorname{Image} A)^*$ the norm of $\widehat{\sigma}$ is given by

$$\|\widehat{\sigma}\| = \max_{\chi_0 \in \operatorname{Image} A} \frac{|\widehat{\sigma}(\chi_0)|}{\|\chi_0\|} = \max_{w \in \mathscr{W}} \frac{|f(w)|}{\|A(w)\|}. \tag{3.6}$$

We recall that the Hahn-Banach theorem for linear functionals (*e.g.*, [30]) asserts that $\widehat{\sigma}$ has an extension

$$\sigma^{\mathrm{hb}} : \mathscr{S} \longrightarrow \mathbb{R} \tag{3.7}$$

whose norm is identical to the norm of $\widehat{\sigma}$. Hence, there is some $\sigma^{\mathrm{hb}} \in \mathscr{S}^*$ with

$$\sigma^{\mathrm{hb}}(A(w)) = \widehat{\sigma}(A(w)) = f(w), \quad \text{for all} \quad w \in \mathscr{W} \tag{3.8}$$

and

$$\|\sigma^{\mathrm{hb}}\| = \max_{\chi \in \mathscr{S}} \frac{|\sigma^{\mathrm{hb}}(\chi)|}{\|\chi\|} = \|\widehat{\sigma}\| = \max_{\chi_0 \in \operatorname{Image} A} \frac{|\widehat{\sigma}(\chi_0)|}{\|\chi_0\|} = \max_{w \in \mathscr{W}} \frac{|f(w)|}{\|A(w)\|}. \tag{3.9}$$

Evidently, for any solution $\sigma' \in \Phi_f$,

$$\|\sigma'\| = \max_{\chi \in \mathscr{S}} \frac{|\sigma'(\chi)|}{\|\chi\|} \geqslant \|\sigma^{\mathrm{hb}}\| = \|\widehat{\sigma}\| = \max_{\chi_0 \in \operatorname{Image} A} \frac{|\widehat{\sigma}(\chi_0)|}{\|\chi_0\|}. \tag{3.10}$$

Hence, σ^{hb} is an optimal stress field and the optimal stress $s_f^{\mathrm{opt}} = \|\sigma^{\mathrm{hb}}\|$ satisfies Equation (3.2). □

It is noted that since A is linear and injective, the norm on $\mathscr{S}$ defines a norm on $\mathscr{W}$ by

$$\|w\|_A = \|A(w)\|, \quad \text{for all} \quad w \in \mathscr{W}. \tag{3.11}$$

Thus, the expression for the optimal stress may be rewritten as

$$s_f^{\mathrm{opt}} = \max_{w \in \mathscr{W}} \frac{|f(w)|}{\|w\|_A}, \tag{3.12}$$

so using the definition of the dual norm we have

Corollary 3.2. *The optimal stress is given by*

$$s_f^{\mathrm{opt}} = \|f\|_A. \tag{3.13}$$

It is noted that in general the mechanical interpretation of the norm $\|f\|_A$ is completely different than that of the norm defined in Example 2.3.

In conclusion we emphasize that the optimal stress field need not be unique and that the procedure outlined above does not provide one with any optimal stress distribution.

3.2. **Stress Concentration Factors and Stress Sensitivity.** We recall the notion of *stress concentration factor*, used in practical stress analysis, indicating the ratio between the maximal stress in a loaded body as obtained by computations or experiments, and the nominal maximal stress for the body as predicted by elementary formulas for simplified geometries. We generalize the notion of stress concentration factor in the following way. Let a structure Ω be loaded by a load vector f and let σ be a stress field in equilibrium with f. We define the stress concentration $K_{f,\sigma}$ for the pair f and σ by

$$K_{f,\sigma} = \frac{\|\sigma\|}{\|f\|}. \tag{3.14}$$

We already saw in Example 2.2 that $\|\sigma\|^\infty$ represents the maximal equivalent stress. Similarly, as in Example 2.3 one can use

$$\|f\| = \|f\|^\infty \tag{3.15}$$

so $\|f\|$ indicates the maximum of the magnitude of the applied load. Thus, in the generalization of the notion of stress concentration, we are able to consider arbitrary bodies and structures and arbitrary loadings by replacing the nominal maximal stress by the norm of the applied load.

In particular, one may consider the optimal stress concentration factor, *i.e.*, $K_f = K_{f,\sigma_f^{\mathrm{opt}}}$, the stress concentration factor corresponding to the optimal stress. Thus,

$$K_f = \frac{\|\sigma_f^{\mathrm{opt}}\|}{\|f\|} = \frac{s_f^{\mathrm{opt}}}{\|f\|} = \frac{1}{\|f\|}\min_{\sigma\in\Phi_f}\|\sigma\| = \min_{\sigma\in\Phi_f} K_{f,\sigma}. \tag{3.16}$$

Finally, we let the applied load vary and define

$$K = \max_{f\in\mathscr{W}^*} K_f = \max_{f\in\mathscr{W}^*} \frac{s_f^{\mathrm{opt}}}{\|f\|}. \tag{3.17}$$

We will refer to K as the *stress sensitivity* of the structure. It is noted that once the norms for the stress and load vectors are chosen, the stress sensitivity depends only on the geometry of the structure. Thus, similarly to the stress concentration factor of traditional engineering, the stress sensitivity is a purely geometric quantity. Next, we give an expression for the computation of the stress sensitivity.

Proposition 3.3. *The stress sensitivity can be computed by*

$$K = \max_{w\in\mathscr{W}} \frac{\|w\|}{\|A(w)\|}. \tag{3.18}$$

Proof. Using the expression (3.2) for the optimal stress in the definition (3.17) of the stress sensitivity, we have

$$\begin{aligned} K &= \max_{f\in\mathscr{W}^*} \frac{s_f^{\text{opt}}}{\|f\|}, \\ &= \max_{f\in\mathscr{W}^*}\left\{\frac{1}{\|f\|}\max_{w\in\mathscr{W}}\left\{\frac{|f(w)|}{\|A(w)\|}\right\}\right\}, \\ &= \max_{w\in\mathscr{W}}\left\{\frac{1}{\|A(w)\|}\max_{f\in\mathscr{W}^*}\left\{\frac{|f(w)|}{\|f\|}\right\}\right\}, \\ &= \max_{w\in\mathscr{W}} \frac{\|w\|}{\|A(w)\|}, \end{aligned} \tag{3.19}$$

where in arriving at the last line we used an analog of Equation (2.7). □

Corollary 3.4. *Let*

$$A^{-1}\colon \operatorname{Image} A \longrightarrow \mathscr{W} \tag{3.20}$$

be the inverse of the A. Then,

$$K = \|A^{-1}\|. \tag{3.21}$$

Proof. Note that the existence of A^{-1} is guaranteed by the assumption that A is injective. In addition, A^{-1} is an isomorphism with inverse A, hence,

$$\begin{aligned} \|A^{-1}\| &= \max\left\{\frac{\|A^{-1}(\chi_0)\|}{\|\chi_0\|};\ \chi_0 \neq 0,\ \chi_0 \in \operatorname{Image} A\right\}, \\ &= \max\left\{\frac{\|w\|}{\|A(w)\|};\ w\neq 0,\ w\in\mathscr{W}\right\}. \end{aligned} \tag{3.22}$$

□

Remark 3.5. It is noted that in this subsection, the norm used for load vectors was not specified and the only requirement is that the norm used for external velocity fields in $\mathscr{W}$ will be its dual norm. While it might be natural to use the $\|f\|^\infty$ for forces, as we do throughout this paper, this is not required by the analysis. The situation is different for continuous bodies. In the infinite dimensional case, the choice of norms on the global velocity fields will determine whether the mapping analogous to A will be continuous or not and the whole analysis is based on a proper choice of norms.

Remark 3.6. While in the problem of least maximal stresses it is natural to use the ∞-norm for stress distributions, the results above, in which the norm was not specified, apply also to structural optimization. If the specific cost function

$\psi(\sigma(x))$ for the structural optimization problem (*e.g.*, [31, 22]) is a norm $|\sigma(x)|$ on the space of stress matrices, then, the total cost

$$\Psi(\sigma) = \int_\Omega \psi(\sigma(x))\mathrm{d}x \tag{3.23}$$

is just the 1−norm of the stress. In the context of structural optimization, the expression for the optimal stress is related to the duality results in [31, 22] and one can offer an immediate interpretation for the stress sensitivity.

3.3. **Restricted Stress Sensitivity.** So far, in the definition of the stress sensitivity, we considered the case where no a-priori information is given on the force vector and we looked for the maximum of the optimal stresses over all possible load vectors $f \in \mathscr{W}^*$. In this subsection we consider the case where some simple constraints are given on the applied loads.

Specifically, we consider load vectors f for which the components $f_{i_1}, f_{i_2}, \ldots, f_{i_z}$ vanish. That is, we consider the case where no forces are associated with the degrees of freedom in the set $Z = \{i_1, \ldots, i_z\} \subset \{1, 2, \ldots, D\}$ while the components of the forces for the complementary degrees of freedom $\overline{Z} = \{j_1, \ldots, j_{D-z}\}$ are arbitrary. Clearly, we consider a decomposition of $\mathscr{W}$ into the $(D-z)$-dimensional subspace $\mathscr{M}$ spanned by the $\{j_1, \ldots, j_{D-z}\}$ degrees of freedom and a subspace $\overline{\mathscr{M}}$ of $\mathscr{W}$ spanned by the $\{i_1, \ldots, i_z\}$ degrees of freedom (containing elements for with the components in $\overline{Z}$ vanish). Thus we may write

$$\mathscr{W} = \mathscr{M} \oplus \overline{\mathscr{M}} \tag{3.24}$$

and we have two natural projections:

$$\pi\colon \mathscr{W} \longrightarrow \mathscr{M}, \quad \overline{\pi}\colon \mathscr{W} \longrightarrow \overline{\mathscr{M}}, \tag{3.25}$$

obtained by setting the appropriate components to zero, and two natural inclusions

$$\pi^*\colon \mathscr{M}^* \longrightarrow \mathscr{W}^*, \quad \overline{\pi}^*\colon \overline{\mathscr{M}}^* \longrightarrow \mathscr{W}^*. \tag{3.26}$$

Conversely, there are two inclusions

$$\iota\colon \mathscr{M} \longrightarrow \mathscr{W}, \quad \overline{\iota}\colon \overline{\mathscr{M}} \longrightarrow \mathscr{W} \tag{3.27}$$

and two projections

$$\iota^*\colon \mathscr{W}^* \longrightarrow \mathscr{M}^*, \quad \overline{\iota}^*\colon \mathscr{W}^* \longrightarrow \overline{\mathscr{M}}^*. \tag{3.28}$$

These induce an isomorphism

$$\mathscr{W}^* = \mathscr{M}^* \oplus \overline{\mathscr{M}}^* \tag{3.29}$$

with

$$f(w) = \iota^*(f)(\pi(w)) + \overline{\iota}^*(f)(\overline{\pi}(w)). \tag{3.30}$$

For an element $g \in \mathscr{M}^*, \pi^*(g)$ is a force for which the components associated with the collection Z of degrees of freedom vanish. This follows immediately from

$$\pi^*(g)(w) = g(\pi(w)) \tag{3.31}$$

so no power is expended on the degrees of freedom in Z. Moreover, we may look for the optimal stress, $s^{\text{opt}}_{\pi^*(g)} \in \mathscr{W}^*$. Finally, one may consider the stress sensitivity pertaining to the external force vectors in $\mathscr{M}^*$, *i.e.*,

$$K^{\mathscr{M}^*} = \max_{g \in \mathscr{M}^*} \frac{s^{\text{opt}}_{\pi^*(g)}}{\|\pi^*(g)\|}. \tag{3.32}$$

For the computation of the restricted stress sensitivity $K^{\mathscr{M}^*}$, it is assumed henceforth that,

$$\|\pi^*(g)\| = \|g\|, \quad g \in \mathscr{M}^*. \tag{3.33}$$

This assumption applies to all cases considered here and to all traditional finite element models (see Appendix A for motivation). Using the result for optimal stress we have

$$\begin{aligned} K^{\mathscr{M}^*} &= \max_{g \in \mathscr{M}^*} \left\{ \frac{1}{\|\pi^*(g)\|} \max_{w \in \mathscr{W}} \left\{ \frac{\pi^*(g)(w)}{\|A(w)\|} \right\} \right\}, \\ &= \max_{w \in \mathscr{W}} \left\{ \frac{1}{\|A(w)\|} \max_{g \in \mathscr{M}^*} \left\{ \frac{\pi^*(g)(w)}{\|\pi^*(g)\|} \right\} \right\}, \\ &= \max_{w \in \mathscr{W}} \left\{ \frac{1}{\|A(w)\|} \max_{g \in \mathscr{M}^*} \left\{ \frac{g(\pi(w))}{\|g\|} \right\} \right\}. \end{aligned} \tag{3.34}$$

We conclude that

$$K^{\mathscr{M}^*} = \max_{w \in \mathscr{W}} \left\{ \frac{\|\pi(w)\|}{\|A(w)\|} \right\}. \tag{3.35}$$

The last expression allows some reduced computational effort in the calculation of $K^{\mathscr{M}^*}$ in comparison with those pertaining to K.

4. The Implementation to Plasticity

In this section we show how the foregoing analysis may be applied to the mathematical theory of perfectly plastic bodies (see [32, 33, 34, 35, 4, 29]) undergoing small deformations. It is assumed henceforth that the yield function is given as a norm $|\cdot|$ on the space of symmetric 3×3 matrices or a vector subspace of that space. In particular, traditional yield functions are norms on the space of traceless matrices, the stress deviatoric components. The yield stress is denoted by s_Y so the structure can support any stress field σ such that

$$\|\sigma\|^{\infty} = \operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)| \leqslant s_Y. \tag{4.1}$$

For the structural models considered here and using traditional finite base functions, the structure can support the stress field σ, if

$$\|\sigma\|^{\infty} = \max_{l} \left\{ \max_{x \in \Omega_l} |\sigma(x)| \right\} \leqslant s_Y. \tag{4.2}$$

It is noted that $\max_{x\in\Omega_l}|\sigma(x)|$ is a norm on the finite dimensional space of stress fields (or trace-less stress fields) on the finite element Ω_e with a particularly simple form for the case of uniform stress elements.

4.1. **Optimal Stresses and Limit Analysis.** Let f be a load vector acting on the structure. Let $\mu_f \in \mathbb{R}^+$, be the number such that

$$s^{\text{opt}}_{\mu_f f} = s_Y. \tag{4.3}$$

Observing expression (3.2), it is clear that the optimal stress satisfies the homogeneity condition

$$s^{\text{opt}}_{\mu f} = |\mu|\, s^{\text{opt}}_f, \quad \text{for all} \quad \mu \in \mathbb{R}. \tag{4.4}$$

It follows that

$$\mu_f = \frac{s_Y}{s^{\text{opt}}_f}. \tag{4.5}$$

It is recalled (*e.g.*, [32, 33, 34, 35, 4]) that for a load vector f, the *limit design factor* or *factor of safety* λ_f of the theory of plasticity is defined to be the largest positive number such that there is a stress field σ in equilibrium with $\lambda_f f$, with $\|\sigma\|^\infty \leqslant s_Y$, *i.e.*,

$$\lambda_f = \sup\{\lambda \in \mathbb{R}^+;\ \exists\sigma,\ \|\sigma\|^\infty \leqslant s_Y,\ A^*(\sigma) = \lambda f\}. \tag{4.6}$$

It follows that

$$\begin{aligned}
\lambda_f &= \sup\left\{\lambda \in \mathbb{R}^+;\ \exists\sigma,\ \left\|\sigma/\lambda\right\|^\infty \leqslant \frac{s_Y}{\lambda},\ A^*\left(\sigma/\lambda\right) = f\right\},\\
&= \sup\left\{\lambda \in \mathbb{R}^+;\ \exists\sigma',\ \|\sigma'\|^\infty \leqslant \frac{s_Y}{\lambda},\ A^*(\sigma') = f\right\},\\
&= \sup\left\{\lambda \in \mathbb{R}^+;\ \exists\sigma',\ \lambda \leqslant \frac{s_Y}{\|\sigma'\|^\infty},\ A^*(\sigma') = f\right\},\\
&= \sup\left\{\frac{s_Y}{\|\sigma'\|^\infty};\ A^*(\sigma') = f\right\},\\
&= \frac{s_Y}{\inf_{A^*(\sigma')=f}\|\sigma'\|},\\
&= \frac{s_Y}{s^{\text{opt}}_f}.
\end{aligned} \tag{4.7}$$

Thus,

$$\lambda_f = \mu_f = \frac{s_Y}{s^{\text{opt}}_f}, \tag{4.8}$$

and by the existence of an optimal stress field, we conclude that the limit multiplier is attained for the stress distribution $\sigma^{\text{opt}}_{\mu_f f}$.

The foregoing analysis implies that within the framework of the mathematical theory of plasticity, optimal stress fields can be realized automatically if one chooses a material for which

$$s_Y = s^{\text{opt}}_f. \tag{4.9}$$

In such a situation the structure is on the threshold of plastic collapse. In case $s_f^{\text{opt}} > s_Y$, the structure cannot support the load and plastic collapse occurs at a force of a smaller magnitude.

4.2. **Load Capacity Ratio of Structures.** As mentioned in the introduction, the load capacity ratio of a structure is a purely geometric property of the structure that, together with the yield stress, provides a maximal bound on the magnitude of the external load that the structure can support without plastic collapse, regardless of the way the load is distributed. The bound is maximal in the sense that for any higher bound, there will be some loads that will cause the collapse of the structure.

Recalling the definition of the stress sensitivity in (3.17) and using a yield criterion for the evaluation of the magnitude of a stress matrix, we have

Proposition 4.1. *Let s_Y be the yield stress of a homogeneous isotropic perfectly plastic structure. Using the yield function as a norm on the space of stress deviator matrices, let the load capacity ratio of the structure be defined by*

$$C = \frac{1}{K}. \tag{4.10}$$

Then,

(*i*) *the structure can support any load f' with*

$$\|f'\| \leqslant C s_Y \tag{4.11}$$

without collapsing;

(*ii*) *for any $C' > C$, the structure will collapse for some load g with $\|g\| = C' s_Y$.*

Proof. We first recall that the structure can support the load f without collapsing if and only if $s_f^{\text{opt}} \leqslant s_Y$. Next, we note that it follows from the definition of K that

$$C = \inf_f \frac{\|f\|}{s_f^{\text{opt}}}. \tag{4.12}$$

It follows that for all loads f,

$$\frac{s_Y \|f\|}{s_f^{\text{opt}}} \geqslant C s_Y. \tag{4.13}$$

Thus, if $\|f'\| \leqslant C s_Y$,

$$\frac{s_Y \|f'\|}{s_{f'}^{\text{opt}}} \geqslant \|f'\| \tag{4.14}$$

and $s_Y \geqslant s_{f'}^{\text{opt}}$, proving (*i*).

To prove (*ii*), we first note that it follows from Equation (4.12) that for any $C' > C$, there is some f' with

$$\frac{s_Y \|f'\|}{s_{f'}^{\text{opt}}} < C' s_Y. \tag{4.15}$$

Considering

$$g = \frac{f'}{\|f'\|} C' s_Y,$$

we note that $\|g\| = C' s_Y$. Recalling that for any $a > 0$ and any loading f, Equation (4.4) implies that $s^{\text{opt}}_{af} = a s^{\text{opt}}_{f}$, we have

$$\frac{\|g\|}{s^{\text{opt}}_{g}} s_Y = \frac{\|f'\|}{s^{\text{opt}}_{f'}} s_Y < C' s_Y = \|g\| \tag{4.16}$$

and (*ii*) follows. □

Remark 4.2. The construction may be illustrated further as follows. Consider the equivalence relation on $\mathscr{S}^*$ defined by $f \sim f'$ if $f = af'$ for some $a > 0$. Each such equivalence class $[f] \in \mathscr{S}^*/\sim$ in the quotient space is a ray in $\mathscr{S}^*$ and it follows from the definition in Equation (3.16) that K_f depends only on the equivalence class. Each such ray contains a single limit force f_Y such that $s^{\text{opt}}_{f_Y} = s_Y$. The collapse manifold

$$\Psi_Y = \left\{ f_Y : s^{\text{opt}}_{f_Y} = s_Y \right\} \tag{4.17}$$

is the collection of all such loads. The mapping

$$\pi : \mathscr{S}^* \longrightarrow \Psi_Y, \quad \pi(f) = \frac{s_Y}{s^{\text{opt}}_{f}} f, \tag{4.18}$$

cf. Equation (1.11), is a projection of each ray on the corresponding limit load. The structure cannot support loads that are outside the limit manifold. Taking $s_Y = 1$ for convenience, the load capacity ratio is the radius of the largest ball in $\mathscr{S}^*$ contained in the collapse manifold. The ball of radius C does not contain in its interior any point belonging to the collapse manifold. Any ball with a larger radius will necessarily contain loads outside the collapse manifold.

Remark 4.3. For an inhomogeneous structure, the yield stress varies from one material point to another so we have a function

$$s_Y \colon \Omega \longrightarrow \mathbb{R}^+, \tag{4.19}$$

where $s_Y(x)$ is the yield stress of the material at point $x \in \Omega$. Assuming naturally that the function $s_Y(x)$ is sufficiently regular and bounded above and below by some positive numbers, the structure will not collapse if there is an equilibrating stress field σ such that

$$|\sigma(x)| \leqslant s_Y(x), \quad \text{for all} \quad x \in \Omega, \tag{4.20}$$

or equivalently, if

$$\|\sigma\|^{\infty}_{H} = \operatorname*{ess\,sup}_{x \in \Omega} \left\{ \frac{|\sigma(x)|}{s_Y(x)} \right\} \leqslant 1. \tag{4.21}$$

Using $\|\cdot\|_H^\infty$, given a load vector f, we define the optimal stress for the heterogeneous structure by

$$s_f^{\mathrm{opt}} = \inf_{\sigma\in\Phi_f} \|\sigma\|_H^\infty.$$

Having defined a norm for $\mathscr{S}^*$, the appropriate norm on $\mathscr{S}$ is obtained using Equation (2.6) as

$$\|\chi\|_H^1 = \int_\Omega s_Y(x)\left|\chi(x)\right| \mathrm{d}x. \tag{4.22}$$

Using the weighted norms, the basic results, *e.g.*, Equation (4.11), hold when one uses $s_Y = 1$.

5. Selected Models

In order to demonstrate the applications of the foregoing analysis, we consider a number of typical structural models. For each of these models, one has to construct the vector spaces $\mathscr{W}$ and $\mathscr{S}$, the structural mapping A and the appropriate norm as induced by the failure criterion. In order to simplify the examples, we consider mainly structural elements with uniform stress distributions. Specifically, we describe the applications to trusses, frames, plane stress and plain strain structural models. The actual examples are given in Section 7.

5.1. **Trusses.** For the truss model, the space $\mathscr{W}$ of external virtual displacements contains naturally vectors of nodes' velocities and the space $\mathscr{S}$ of internal virtual displacements contains the distribution of axial virtual strains along the struts multiplied by the volumes of the corresponding struts. For a generic element $\chi \in \mathscr{S}$, the virtual strains in the various bars need not be compatible. Thus, $\mathscr{W}^*$ contains vectors describing the distribution of nodal forces and $\mathscr{S}^*$ contains the vector of struts tensile stresses. An element $\chi \in \mathscr{S}$ is of the form $\chi = \left(V_1\varepsilon_1,\dots,V_L\varepsilon_L\right)^\mathsf{T}$, where ε_l is the virtual strain in the l-th element and V_l is the volume of the element. An element, $\sigma \in \mathscr{S}^*$ is of the form $\sigma = (\sigma_1,\dots,\sigma_L)$ with $\sigma(\chi) = \sum_l V_l\varepsilon_l\sigma_l$. The matrix A is standard. The norm of a stress vector should reflect the maximal stress in the truss and hence

$$\|\sigma\| = \|\sigma\|^\infty = \max_{l=1,\dots,L} \left|\sigma_l\right|. \tag{5.1}$$

Using Equation (2.6), the norm of the local displacements, *i.e.*, tensile strain vectors, should be taken as

$$\|\chi\| = \|\chi\|^1 = \sum_{l=1,\dots,L} V_l\left|\varepsilon_l\right|. \tag{5.2}$$

The resulting expression for the optimal stress corresponding to a load vector f is

$$s_f^{\mathrm{opt}} = \max_w \frac{\sum_d f_d w_d}{\sum_l \left|\sum_d A_{ld} w_d\right|}. \tag{5.3}$$

The expression for the stress sensitivity will assume the form

$$K = \max_w \frac{\|w\|}{\sum_l \left|\sum_d A_{ld} w_d\right|}. \tag{5.4}$$

As mentioned in Remark 3.5, it is not required but is natural to use

$$\|f\| = \|f\|^{\infty} = \max_i |f_i|, \tag{5.5}$$

where $|f_i|$ is the magnitude of the load acting at the i-th node. The resulting norm for velocity fields is

$$\|w\| = \|w\|^1 = \sum_i |w_i|, \tag{5.6}$$

where $|w_i|$ is the magnitude of the velocity of the i-th node.

Alternatively, one could choose

$$\|f\| = \max_d |f_d|, \quad \|w\| = \sum_d |w_d|, \tag{5.7}$$

where now f_d and w_d are the components of force vector and velocity corresponding to the d-th degree of freedom. For this choice of norms, the expression for the stress sensitivity takes the relatively simple form

$$K = \max_w \frac{\sum_d |w_d|}{\sum_l |\sum_d A_{ld} w_d|}. \tag{5.8}$$

5.2. **Plane Frames.** The analysis of a frame structure differs from the rest of the structural models presented in this work as the frame members are modeled using linearly, rather than uniformly, distributed strains in accordance with the traditional Euler-Bernoulli hypothesis.

For a plane frame model having a symmetric cross section, each node is assumed to have two translational degrees of freedom and one rotational degree of freedom. The collection of the virtual displacements corresponding to the various degrees of freedom makes up the space $\mathscr{W}$. Accordingly, a generalized force vector $f \in \mathscr{W}^*$ will consist of force vectors and couples at the various nodes.

In the analysis of frames we neglect the shear stresses relative to the normal stresses, also denoted as σ. The interpolated virtual displacement field in a typical beam that makes up the frame is taken a linear for the axial component and as a polynomial of degree 3 for the transversal component. This way, the interpolated displacement field of the center-line agrees with the displacements at the end points and the derivatives along the beam of the transverse deflection agree with the rotations at the end points. The resulting axial strain field varies linearly along the beam and within the various cross sections. We therefore subdivide each beam in the structure to smaller beam element segments where the curvature is approximately uniform.

The norm used for the normal stress fields σ is

$$\|\sigma\| = \|\sigma\|^{\infty} = \max_{x \in \Omega} |\sigma(x)| = \max_{l=1,\dots,L} \left\{ \max_{x \in \Omega_l} |\sigma(x)| \right\}, \tag{5.9}$$

where $\{\Omega_l\}$ is the partition of the structure into finite elements. Thus,

$$\|\chi\| = \sum_{l=1}^{L} \int_{\Omega_l} |\varepsilon_l(x)| \, \mathrm{d}V. \tag{5.10}$$

Noting that the strain distribution within the section of a beam element is linear rather than uniform, as the last equation indicates, $\|A(w)\|$ will contain geometric properties of the cross section. However, Equations (5.3) and (5.4) still apply where the matrix A is replaced by a matrix $\tilde{A}$ whose elements depend on the geometric properties of the cross-section.

5.3. **Plane Stress Finite Element Model.** Letting $\mathscr{W}$ consist of vectors of virtual displacements in the form $w = \{w_1, \dots, w_P\}^{\mathsf{T}}$, where w_p is the virtual displacement vector at the p-th node, we use linear interpolation functions to obtain uniform strain triangular finite elements. Thus, any local virtual displacement will be of the form $\chi = \{V_1\varepsilon_1, \dots, V_L\varepsilon_L\}^{\mathsf{T}}$, where V_l is the volume (or area) of the l-th element and ε_l is the 2-dimensional strain matrix in that element. The mapping A associates with a nodal displacement field the corresponding element-wise uniform virtual strain field (multiplied by the corresponding volume) using the linear interpolation functions.

A local force will be of the form $\sigma = \{\sigma_1, \dots, \sigma_L\}$, where σ_l is the 2×2 uniform stress matrix in the l-th element. The virtual power is calculated by

$$\sigma(\chi) = \sum_{l=1}^{L} \sigma_l \cdot \varepsilon_l V_l, \tag{5.11}$$

where the dot indicates the usual inner product of two square matrices.

For the case of plane stress, the von-Mises yield criterion provides a norm (rather than a semi-norm) on the space of symmetric 2×2 matrices by

$$|\tau| = \left(\tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2\right)^{\frac{1}{2}}. \tag{5.12}$$

Naturally, the norm to be used for stress fields will be

$$\|\sigma\| = \|\sigma\|^{\infty} = \max_{l=1,\dots,L} |\sigma_l|. \tag{5.13}$$

Thus, the norm to be used for the strain fields is of the form

$$\|\chi\| = \|\chi\|^{1} = \sum_{l} V_l |\varepsilon_l|, \tag{5.14}$$

where $|\varepsilon|$ is the dual norm computed using Equation (2.6). Using straightforward differentiation one obtains

$$|\varepsilon| = \max_{\tau \in \mathbb{R}^{2\times 2}_{\text{symm}}} \frac{\tau \cdot \varepsilon}{|\tau|} = \frac{2}{\sqrt{3}} \left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2\right)^{\frac{1}{2}}. \tag{5.15}$$

Finally, $\|A(w)\|$ assumes the form

$$\|A(w)\| = \sum_{l=1}^{L} \frac{2V_l}{\sqrt{3}} \left[\left(A^l(w^l)\right)^{\mathsf{T}} \begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^l(w^l) \right]^{\frac{1}{2}}, \tag{5.16}$$

where $w^l = w|_{\Omega_l}$, and A^l is the natural restriction of the interpolation mapping to the degrees of freedom associated with the l-th element.

5.4. **Plane Strain Finite Element Model.** For plane strain models, the interpolations of node velocities, stress and strain fields are similar to those corresponding to plane stress models. However, the von-Mises yield criterion for a stress matrix τ is now

$$|\tau| = \left[(\tau_{11} - \tau_{22})^2 + 4\tau_{12}^2\right]^{\frac{1}{2}} \tag{5.17}$$

which is not a norm as it vanishes on the subspace of all spherical matrices $\tau = aI$. Nevertheless, the von-Mises yield criterion is a norm on the space of deviatoric (trace-less) matrices. Thus, as customary in the theory of plasticity (see [4]), one considers only the deviatoric part of the strain fields in the construction of $\mathscr{S}$. In other words, $\mathscr{S}$ will contain only virtual strain fields associated with incompressible virtual displacement fields. In order that the mapping A be well defined and injective, one has to consider only incompressible vector fields w in the construction of $\mathscr{W}$.

The induced dual norm of an incompressible strain matrix will now assume the form

$$|\varepsilon| = \max\left\{\frac{\tau \cdot \varepsilon}{|\tau|};\ \tau_{11} + \tau_{22} = 0\right\} = \left[\frac{1}{4}(\varepsilon_{11} - \varepsilon_{22})^2 + \varepsilon_{12}^2\right]^{\frac{1}{2}}. \tag{5.18}$$

It is noted that the incompressibility constraint could be substituted to the equation above resulting

$$|\varepsilon| = \sqrt{\varepsilon_{11}^2 + \varepsilon_{12}^2}. \tag{5.19}$$

However, we do not follow this line in order to examine the incompressibility constraint from a more general point of view.

Finally, $\|A(w)\|$ assumes the form

$$\|A(w)\| = \sum_{l=1}^{L} \frac{V_l}{2} \left[(A^l(w^l))^{\mathsf{T}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^l(w^l)\right]^{\frac{1}{2}} = \sum_{l}^{L} \|D^l(w^l)\|_2. \tag{5.20}$$

Remark 5.1. As mentioned in Subsection 1.2, by limiting ourselves in this work to structures, we consider given finite dimensional models of mechanical systems. Thus, it is not our objective here to estimate how well do these models approximate continuous, infinite dimensional models. Nevertheless, the following remarks may be appropriate at this point. While it is well known by now that the displacements corresponding to the limit state may be discontinuous (*e.g.*, [33, 4, 29]), similarly to [34, p. 236], we use continuous displacement fields in the finite element function space. Since the space of continuous functions is dense in the space of possible solutions to the continuum limit analysis problem, the values of K and C for the continuum problem, as given in Appendix B, can be approximated in principle using continuous fields to any desired accuracy. Thus, while discontinuous finite element fields may result more accurate results for a given mesh size (see the discussion on the use of continuous fields in [36]), the use of continuous virtual displacement fields is in accordance with the approximation process.

6. Methods

In this section we review the optimization methods used in order to compute expressions such as (3.2) and (3.18).

6.1. **Optimal Stresses.**

We consider first the methods used for the computation of the optimal stresses using Equation (3.2) for the various examples.

6.1.1. *Trusses and Frames.* As discussed in Sections 5.1 and 5.2, in order to evaluate the optimal stress for a given load vector f, one has to use Equation (5.3) or equivalently,

$$\frac{1}{s_f^{\mathrm{opt}}} = \min_{f(w)=1} \left\{ \sum_{n=1}^{N} \left| \sum_{d=1}^{D} A_{nd} w_d \right| \right\}. \tag{6.1}$$

The last expression may be represented as a linear programming problem. Following a standard procedure in linear programming (*e.g.*, [37, 38, ?]), one introduces the N slack variables $\{t_1, \dots, t_N\}$, with $t_n > 0$ for all $n = 1, \dots, N$, and requires that

$$-t_n \leqslant \sum_{d=1}^{D} A_{nd} w_d \leqslant t_n. \tag{6.2}$$

With these definitions, the optimal stress problem may be set in the following linear programming standard form.

$$\frac{1}{s_f^{\mathrm{opt}}} = \min \left\{ \sum_{n=1}^{N} t_n \right\}, \tag{6.3}$$

subject to the linear equality and inequality constraints

$$\sum_{d=1}^{D} f_d w_d - 1 = 0, \quad t_n - \sum_{d=1}^{D} A_{nd} w_d \geqslant 0, \quad t_n + \sum_{d=1}^{D} A_{nd} w_d \geqslant 0. \tag{6.4}$$

In solving the linear programming problem we employed the primal-dual interior point method suggested in [38].

6.1.2. *Plane Stress Finite Element Models.* For the cases of plane stress and plane strain finite element models, the norm to be used on $\mathscr{S}$ is given by Equation (5.14) using (5.15) and (5.18), respectively. The norms for the strain matrices in these two equations imply that linear programming cannot be applied for the computations. Thus, we use the method of Second Order Cone Programming as in [39, 40].

We recall that a second order cone in $\mathbb{R}^m$ is a collection

$$Q = \left\{ (x, x_0);\, x \in \mathbb{R}^{m-1},\, x_0 \in \mathbb{R}^+,\, x_0 \geqslant |x| \right\},$$

where $|x|$ is the standard Euclidean norm in $\mathbb{R}^{m-1}$. For a linear function p, the standard second order cone programming problem seeks

$$\min_x \{p(x)\} \tag{6.5}$$

subject to

$$
\begin{aligned}
H_j(x) &= h_j, j = 1, \ldots, m, \\
G_i(x) &\leqslant g_i, i = 1, \ldots, n. \\
x &\in Q_1 \times \cdots \times Q_r,
\end{aligned}
\tag{6.6}
$$

where H_j and G_i are linear and the Q_p are r-second order cones.

It is noted that for the model of plane stress finite elements we may write Equation (5.16) in the form

$$
\|A(w)\| = \sum_{l=1}^{L} \|B^l(w^l)\|_2, \tag{6.7}
$$

where $\|\cdot\|_2$ denotes the Euclidean norm. In order to represent the expression for the optimal stress as a second order cone programming problem, we introduce the L positive slack variables t_l, and the L inequality constraints

$$
\|B^l(w^l)\|_2 \leqslant t_l \tag{6.8}
$$

that define the second order cones $Q_l = \left\{\left(B^l(w^l), t_l\right)\right\}$. Thus, the optimal stress assumes the form

$$
\frac{1}{s_f^{\text{opt}}} = \min\left\{\sum_{l}^{L} t_l\right\}, \tag{6.9}
$$

subject to

$$
\begin{aligned}
f(w) &= 1, \\
t_l &\geqslant 0, \; l = 1, \ldots, L, \\
\|B^l(w^l)\|_2 &\leqslant t_l, \; l = 1, \ldots, L,
\end{aligned}
\tag{6.10}
$$

where we have an optimization problem in $D+L$ variables, L second order inequality constraints, L linear inequality constraints and one linear equality constraint.

6.1.3. *Plane Strain Finite Element Models.* The procedure for the plane strain finite element models is similar to the one corresponding to plane stress models. The basic difference is that now one has to satisfy the incompressibility constraints. For simplicity, we chose to introduce the incompressibility conditions elementwise by requiring that

$$
h^l(w) = V^l(\varepsilon_{11}^l + \varepsilon_{22}^l) = A^l(w)_1 + A^l(w)_2 = 0. \tag{6.11}
$$

The stress optimization problem assumes the form

$$
\frac{1}{s_f^{\text{opt}}} = \min\left\{\sum_{l}^{L} t_l\right\}, \tag{6.12}
$$

subject to

$$\begin{aligned} f(w) &= 1, \\ h^l(w) &= 0,\ l = 1,\dots,L, \\ t_l &\geqslant 0,\ l = 1,\dots,L, \\ \|D^l(w^l)\|_2 &\leqslant t_l,\ l = 1,\dots,L, \end{aligned} \tag{6.13}$$

where $D^l(w^l)$ are defined by Equation 5.20.

6.2. **On the Computation of the Stress Sensitivity and Load Capacity Ratio.** In order to evaluate the stress sensitivity and the load capacity ratio one can rewrite Equation (3.18) in the form

$$C = \frac{1}{K} = \min_{\|w\|=1} \|A(w)\|. \tag{6.14}$$

The function $\|A(w)\|$ is a convex function. The set $\{w;\ \|w\| = 1\}$ is not a convex set hence the difficulty in computing the minimum.

For the 1-norm, the set $\|w\|^1 = 1$ is the union of 2^D convex sets and using $\|A(w)\| = \|A(-w)\|$, the computation may be presented by the solution of 2^{D-1} linear programming problems. Similar observations hold for the other norms considered.

For a possibly more efficient approach, consider the optimization problem

$$K = \frac{1}{C} = \max_w \frac{\|w\|^1}{\|A(w)\|^1} = \max_w \frac{\sum_d |w_d|}{\sum_n \left|\sum_d A_{nd} w_d\right|} = \max_{\|A(w)\|^1 \leqslant 1} \sum_d |w_d|\,. \tag{6.15}$$

Thus, one is looking for the maximum of a convex function $\|w\|^1$ over the convex polytope $\|A(w)\|^1 \leqslant 1$ (see for example [41] as an indication of the difficulties involved). Since the maximum of the convex function will occur at the vertices of the convex polytope, one can use algorithms of the Double Description Method for calculating the vertices of a polytope defined by inequalities (see [42]). In practice, using algorithms of the double description method did not prove to be advantageous in large problems in comparison with other methods.

6.3. **The Computation of the Collapse Load.** In this subsection we consider the method used for evaluating a (non unique) collapse load, *i.e.*, a loading distribution f_C for which the maximum in the expression for K is attained. Thus,

$$K = \max_f \frac{s_f^{\text{opt}}}{\|f\|} = \frac{s_{f_C}^{\text{opt}}}{\|f_C\|}. \tag{6.16}$$

It follows that for the case of plasticity

$$\|f_C\| = s_Y C, \quad s_{f_C}^{\text{opt}} = s_Y \tag{6.17}$$

and the structure will collapse for $f = f_C(1+\varepsilon)$ for any $\varepsilon > 0$.

Recalling the results for the optimal stresses, we have,

$$
\begin{aligned}
K &= \max_f \left\{ \max_w \frac{|f(w)|}{\|f\| \|A(w)\|} \right\} = \frac{|f_C(w_C)|}{\|f_C\| \|A(w_C)\|}, \\
&= \frac{|f_C(w_C)|}{\|f_C\| \|w_C\|} \cdot \frac{\|w_C\|}{\|A(w_C)\|}, \qquad (6.18)
\end{aligned}
$$

for some maximizing pair f_C, w_C. In general,

$$
\frac{|f_C(w_C)|}{\|w_C\|} \leqslant \|f_C\|, \qquad \frac{\|w_C\|}{\|A(w_C)\|} \leqslant K, \qquad (6.19)
$$

so in order for last two inequalities to hold, we must have

$$
\frac{|f_C(w_C)|}{\|w_C\|} = \sup_w \frac{|f_C(w)|}{\|w\|} = \|f_C\|, \qquad \frac{\|w_C\|}{\|A(w_C)\|} = \sup_w \frac{\|w\|}{\|A(w)\|} = K.
$$

In other words, w_C is both the maximizing point (norming point) for f_C and for $\|w\|/\|A(w)\|$.

An analogous analysis may be carried out for the case of restricted stress sensitivity as in Section 3.3. We have,

$$
\begin{aligned}
K^{\mathcal{M}^*} &= \max_{g \in \mathcal{M}^*} \left\{ \max_{w \in \mathcal{W}} \frac{|g(\pi(w))|}{\|g\| \|A(w)\|} \right\} = \frac{|g_C(\pi(w_C))|}{\|g_C\| \|A(w_C)\|}, \\
&= \frac{|g_C(\pi(w_C))|}{\|g_C\| \|\pi(w_C)\|} \cdot \frac{\|\pi(w_C)\|}{\|A(w_C)\|}, \qquad (6.20)
\end{aligned}
$$

for some maximizing (not necessarily unique) pair g_C, w_C. As in general

$$
\frac{|g_C(\pi(w_C))|}{\|\pi(w_C)\|} \leqslant \|g_C\|, \qquad \frac{\|\pi(w_C)\|}{\|A(w_C)\|} \leqslant K, \qquad (6.21)
$$

the load g_C is a maximizer of

$$
K^{\mathcal{M}^*} = \max_{g \in \mathcal{M}^*} \frac{s^{\text{opt}}_{\pi^*(g)}}{\|\pi^*(g)\|}, \qquad (6.22)
$$

and w_C is a maximizer of both

$$
K^{\mathcal{M}^*} = \max_{w \in \mathcal{W}} \frac{\|\pi(w)\|}{\|A(w)\|} \quad \text{and} \quad \|g_C\| = \max_{w \in \mathcal{W}} \frac{|g_C(\pi(w))|}{\|g_C\|}. \qquad (6.23)
$$

These observations motivate the following iterative scheme that we used for the estimation of the stress sensitivity and load capacity ratio of a structure. The algorithm applies in the case of the 1-norm for displacement vectors in $\mathbb{R}^3$ and may be modified appropriately for the use of the Euclidean norm as in Subsections 7.2 and 7.4.2.

(1) Initialization:
- Select at random a load distribution f_1^d.

- Compute the optimal stress and the associated norming virtual displacement w_1^i.

(2) Update the load distribution so that $f^d_{m+1} = \operatorname{\mathbf{sign}} w^d_m$.
(3) The condition for stopping the iterations is $f^d_{m+1} = \operatorname{\mathbf{sign}} w^d_m$.
(4) Save the last values obtained for the optimal stress, virtual displacement, and load as s_m^{opt}, w_m, f_m, respectively.
(5) Repeat steps (1) – (4) for $m = 1, \dots, M$.
(6) End: $K = \max_m \left\{ s_m^{\text{opt}} \right\}$ and the corresponding virtual displacement and loading are w_C and f_C.

It is noted that the criterion for ending the iterations in (3) above is discrete. We stop the iterations when the components f^d_{m+1} have the same signs as the corresponding components of w^d_m. In the case of the Euclidean norm, step 2 above has been changed as follows. Let f^n_m be the force vector at the n-th node for the m-th iteration and let w^n_m be the corresponding virtual displacement vector at that node. Then, the updated load distribution is

$$f^n_{m+1} = \frac{w^n_m}{\left| w^n_m \right|}.$$

The criterion for stopping the iterations was replaced by

$$\sum_n f^n_{m+1} \cdot \frac{w^n_{m+1}}{\left| w^n_{m+1} \right|} > 0.99N,$$

where N is the total number of nodes in the model where forces may be applied. In other words, we check that the extremizing normalized virtual displacement for the $m+1$ iteration is parallel to the normalized extremizing load at that iteration. In the examples we describe below, the maximum numbers of iterations needed to satisfy the criteria above were of the same orders of magnitude as the numbers of degrees of freedom of the structures.

6.4. **Bounds on K Using $\|A^{-1}\|$.** For the 1-norm, the following analysis provides upper bounds on K using linear programming. In fact, the method outlined below provided in practice very accurate values for K and C although in principle it should provide only upper bounds for K.

Using Corollary 3.4, the expression for the stress sensitivity may be rewritten now as

$$K = \max_w \frac{\|w\|^1}{\|A(w)\|^1} = \max_{\chi \in \operatorname{Image} A} \frac{\|A^{-1}(\chi)\|^1}{\|\chi\|^1} = \|A^{-1}\|^1. \tag{6.24}$$

Thus, the stress sensitivity is the 1-norm of the inverse of the kinematic mapping. It would not be practical to calculate A^{-1} but we give an estimate for its norm as follows.

Consider the collection $\mathscr{A}$ of all left inverses of A defined on $\mathscr{S}$, *i.e.*,

$$\mathscr{A} = \left\{ A^+ : \mathscr{S} \longrightarrow \mathscr{W};\ A^+ \circ A = I_{\mathscr{W}} \right\}. \tag{6.25}$$

where $I_{\mathscr{W}}$ is the identity on $\mathscr{W}$. Clearly, every $A^+ \in \mathscr{A}$ is an extension of A^{-1} from Image A to the entire space $\mathscr{S}$. For any such extension A^+ we have,

$$\begin{aligned} \|A^+\|^1 &= \max_{\chi\in\mathscr{S}} \frac{\|A^+(\chi)\|^1}{\|\chi\|^1}, \\ &\geqslant \max_{\chi\in\text{Image}\,A} \frac{\|A^+(\chi)\|^1}{\|\chi\|_1}, \\ &= \max_{\chi\in\text{Image}\,A} \frac{\|A^{-1}(\chi)\|^1}{\|\chi\|^1}, \\ &= \|A^{-1}\|^1. \end{aligned} \tag{6.26}$$

Thus, the stress sensitivity may be bounded by

$$K = \|A^{-1}\|^1 \leqslant \min_{A^+\in\mathscr{A}} \|A^+\|^1. \tag{6.27}$$

Representing a mapping $A^+ \in \mathscr{A}$ by its matrix $(a_{dn}) \in \mathbb{R}^{D\times N}$, the 1-norm of A^+may be easily evaluated by (see [43])

$$\|A^+\|^1 = \max_{1\leqslant n\leqslant N} \sum_{d=1}^{D} |a_{dn}|. \tag{6.28}$$

We now turn our attention to the representation of $\mathscr{A}$. It follows from the Gauss-Jordan reduction process, that using elementary row operations on the matrix $[A, I_{\mathscr{S}}]$, this matrix may be converted to the form

$$\text{rref}[A, I_{\mathscr{S}}] = \begin{bmatrix} I_{\mathscr{W}} & E \\ 0 & \end{bmatrix}, \tag{6.29}$$

where E is a matrix of elementary row operations such that EA has a Hermite normal form[3]. It follows that any left inverse of A^+ may be represented in the form

$$A^+ = P \circ [I_{\mathscr{W}}, L] \circ E \tag{6.30}$$

where, P is a row permutation matrix that can be taken as unity and L is any $\mathbb{R}^{D\times(N-D)}$ matrix. Thus, the components a^+_{dn} of A^+ depend by the equation above only on the choice of the matrix L.

Returning now to the estimate (6.27) for K, we will seek $\min_{A^+\in\mathscr{A}} \|A^+\|^1$ using the representation above and varying the matrix L. The minimization problem can be set as a linear programming problem by introducing the slack variables $y_{dn} > 0$, $t > 0$. The estimate for K is now obtained as $\min_{A^+\in\mathscr{A}} \|A^+\|^1 = \min t$ under the constraints

$$-y_{dn} \leqslant a^+_{dn}(L) \leqslant y_{dn}, \quad d = 1, \ldots, D;\ n = 1, \ldots, N, \tag{6.31}$$

$$\sum_{d=1}^{D} y_{dn} \leqslant t. \quad . \tag{6.32}$$

[3]See [44] for a detailed exposition of the subject.

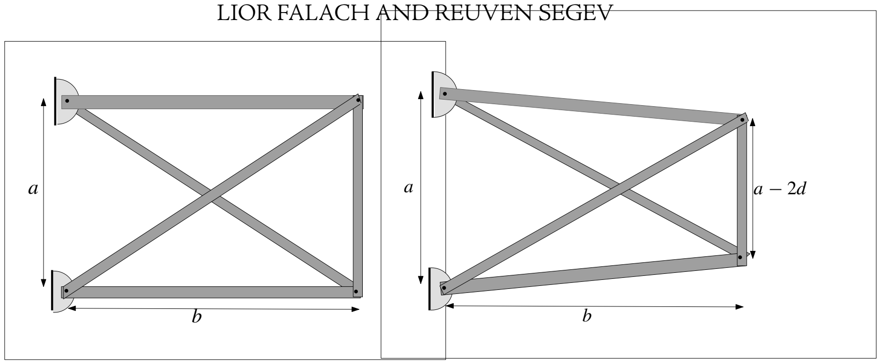

FIGURE 7.1. Rectangular and trapezoidal trusses

If a norm preserving extension $B\colon \mathscr{S} \to \mathscr{W}$ existed, then

$$\|B\|^1 = \|A^{-1}\| = K \tag{6.33}$$

implies that the procedure above would give K and not just an upper bound. In general, linear mappings cannot be extended isometrically. Nevertheless, in practice, using this method we obtained very accurate estimates for K, *e.g.*, less than 1% for example 7.1, in comparison with other methods described in Section 7. In fact, we were not able to find a counterexample where the upper bound for K obtained by this method deviated significantly from the computed values for K obtained using methods for the approximation of K.

7. Examples

This section presents a number of simple examples i order to illustrate the application of the theory and methods described in the preceding material. The computations involved with the examples used MATLAB. In particular, we used the optimization packages YALMIP (available at [45]) and the SeDuMi program (available at [46]). For simplicity, in some examples presented below the 1-norm was used for velocity vectors in $\mathbb{R}^3$. This implies that the ∞-norm should be used for force vectors in $\mathbb{R}^3$. A force vector maximizing the analog of Equation (2.6) is directed at 45° to one of the coordinate axes. As a result, for such situations a worst case loading distribution will consist of force vectors at the nodes that are so directed. In addition, in all the examples, the yield stress was normalized so $s_Y = 1$.

7.1. Truss Examples.

We consider two possible designs of a truss structure, a rectangular design and a trapezoidal design as shown in Figure 7.1.

7.1.1. *The Influence of Geometry on the Load Capacity Ratio.* In order to demonstrate the influence of geometry on the load capacity ratio of the truss structures, we computed the load capacity ratio of the structures for the values $a = 1$m, $d = 0.2$m and for values of b varying from 0.5m to 5m. The cross section area of all members is assumed to be uniform and its value S was set as unity. The

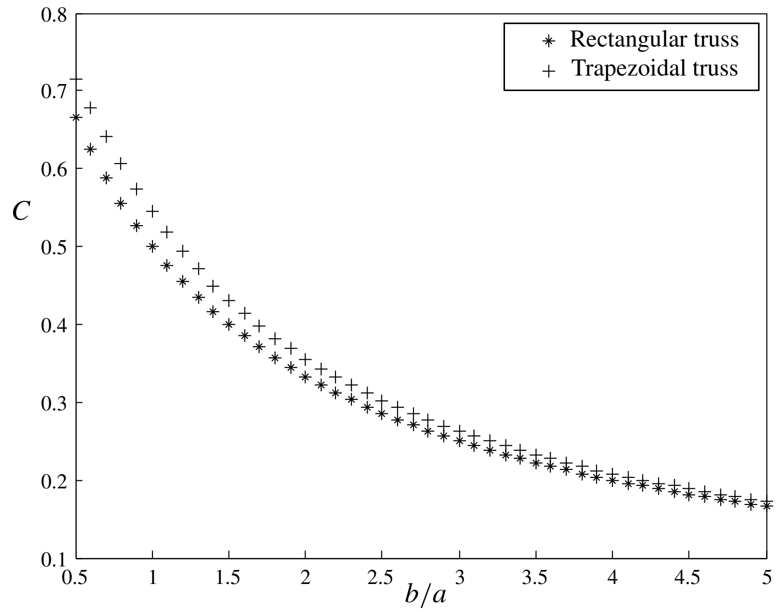

FIGURE 7.2. Load capacity ratio for trusses

results are shown in Figure 7.2. It follows from the computations that the trapezoidal structure is somewhat stronger than the rectangular one. As expected, both structures become very sensitive to vertical loading as the ratio b/a increases.

7.1.2. *The Influence of the Members' Design on the Load Capacity Ratio.* In order to demonstrate the influence of the cross section areas of the truss's members on the strength of the truss, we computed the load capacity ratio of the trusses for varying values of the cross section area S_1 of the diagonal members. The length of the trusses was taken $b = 1$m and the other values remain as in the previous example.

As the computations indicate, heavier diagonal elements may strengthen the structure, but only up to a certain point. Above the value $S_1/S = 1$, heavier diagonal elements will not contribute to the load capacity ratio of the structure even though they might contribute to the factor of safety for some particular loadings.

7.2. **Frame Examples.** Since a frame may be loaded by both external forces and couples, it would be somewhat artificial to mix the two kinds of loadings in the evaluation of the load capacity ratio. Thus, in the present example we consider only the load capacity ratio of the frame for external forces by using the result of Subsection 3.3 for the subspace $\mathscr{M}^*$ containing loads with zero external couples. It follows that the space $\mathscr{M}$ contains virtual displacements with zero rotations of the nodes.

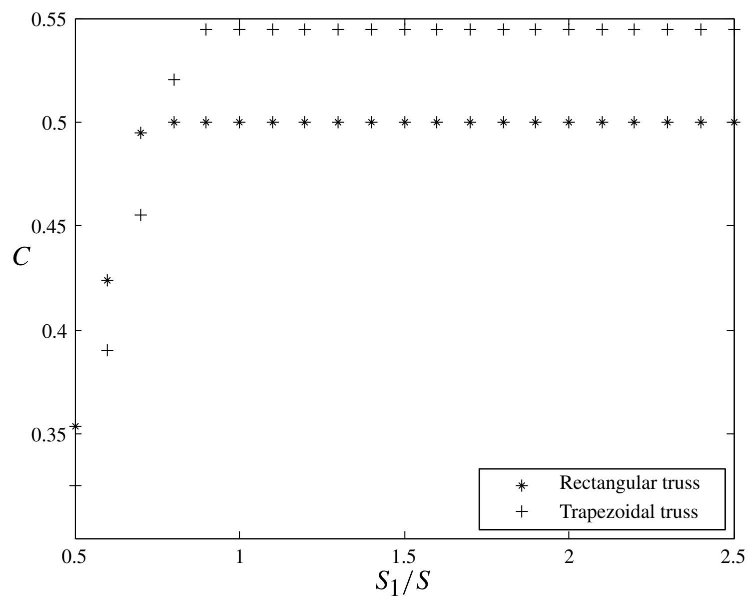

FIGURE 7.3. Load capacity ratio of trusses for various ratios of cross section areas S_1/S.

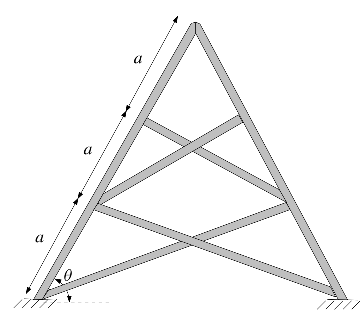

FIGURE 7.4. A power pylon like structure modeled as a frame

7.2.1. *A power pylon.* As an example of a frame structure we chose to consider a power pylon like structure as illustrated in Figure 7.4. For simplicity, the cross section of the members was taken as square with unit side length. Each of the beams that make up the structure was subdivided into 7 beam elements as mentioned in Subsection 5.2.[4].

The results of the analysis are presented graphically in Figure 7.5 . The left-hand side illustration, (a), shows a collapse load, a distribution of external force

[4]Further subdivision did not change the results significantly.

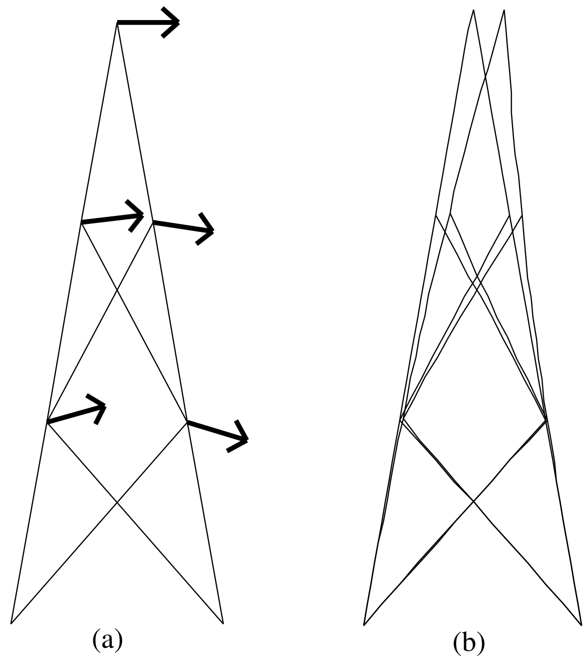

Figure 7.5. Results for power pylon frame model: (a) worst loading, (b) maximizing virtual displacement.

of minimal norm that will cause collapse[5], and the right-hand side illustration, (b), shows the corresponding maximizing virtual displacement (see Subsection 6.3). The value obtained for the stress sensitivity is 29.

7.2.2. *A simple frame.* As a second example we consider a simple frame illustrated in Figure 7.6. The cross section parameters were taken as $Q = 10$ and $S = 1000$. Each of the beams that make up the structure was subdivided into 5 elements as mentioned in Subsection 5.2.

The analysis was performed for two cases: a shallow frame and a tall frame. The results of the analysis for the shallow frame are presented graphically in Figure 7.7 and the results for the tall frame are shown in Figure 7.8. For the flat frame, the number of elements was 30 and for the tall frame we used 45 elements. The value of K was $1.70 \cdot 10^{12}$ for the shallow frame and $1.68 \cdot 10^{13}$ for the tall frame. As intuition would suggest, the two structures are sensitive to different modes of loadings

7.3. **Plane Stress Example.** Consider the plane stress mechanical system consisting of a rectangle supported on its sides with a crack penetrating into its top side as illustrated in Figure 7.9. The system is assumed to be heterogeneous so

[5]As can be observed in the illustration, we used here the Euclidean norm for vectors in the plane.

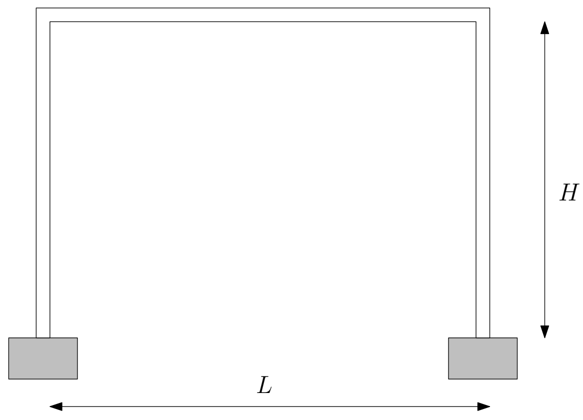

FIGURE 7.6. A simple frame structure

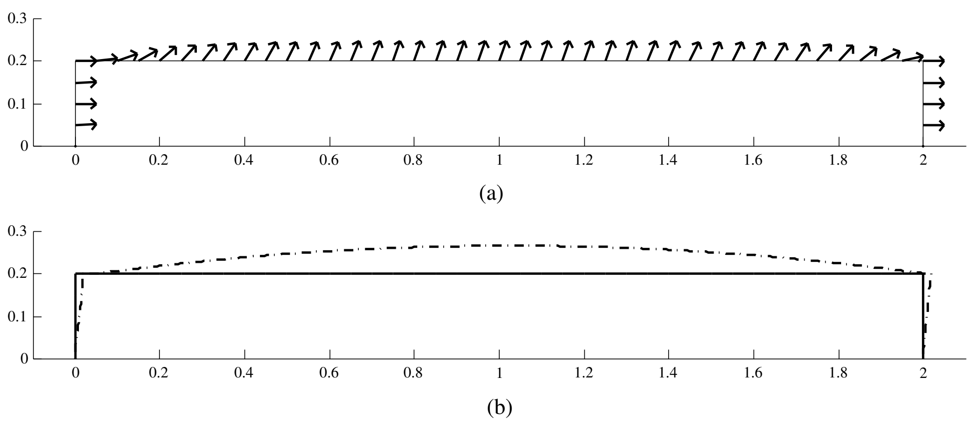

FIGURE 7.7. Results for shallow frame model: (a) worst loading, (b) maximizing virtual displacement.

that the outer circumference is composed of material whose yield stress is 5 times higher than the core.

The restricted load capacity ratio was computed for all load distributions on the top. The load capacity ratio was evaluated using a model comprising of 200 triangular finite elements. The value obtained was $C = 0.013$. A collapse load[6] and a maximizing virtual displacement are illustrated in Figure 7.10.

7.4. **Plane Strain Example.**

7.4.1. *The Plain Strain Test Problem.* In order to check the implementation of our plain strain model and verify that the incompressibility condition is imposed correctly, we ran a test problem having an analytic solution (see [35]) and whose

[6]Here again the forces are in 45° to the axes as we used the ∞-norm for force vectors.

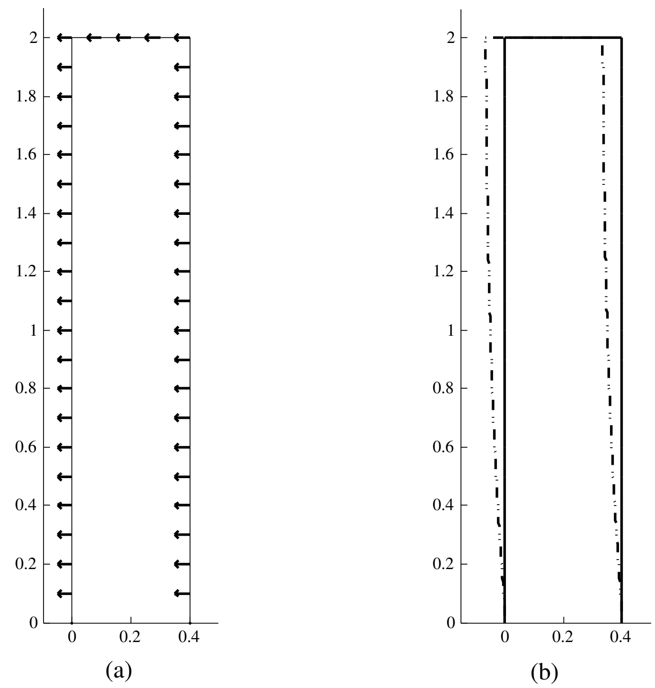

Figure 7.8. Results for the tall frame model: (a) worst loading, (b) maximizing virtual displacement.

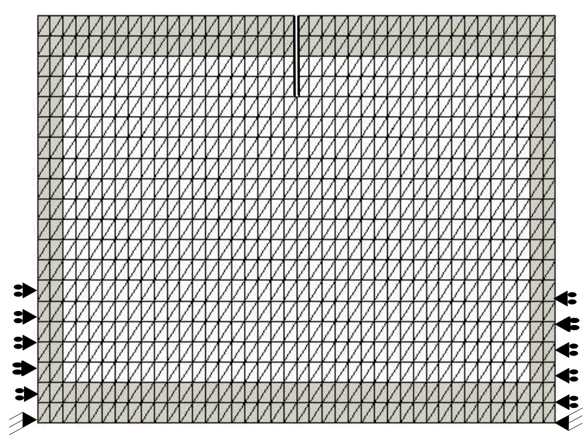

Figure 7.9. Plane stress system

limit analysis is studied numerically in [34]. Thus, we computed the factor of safety, or optimal stress, for the mechanical system illustrated in Figure 7.11.

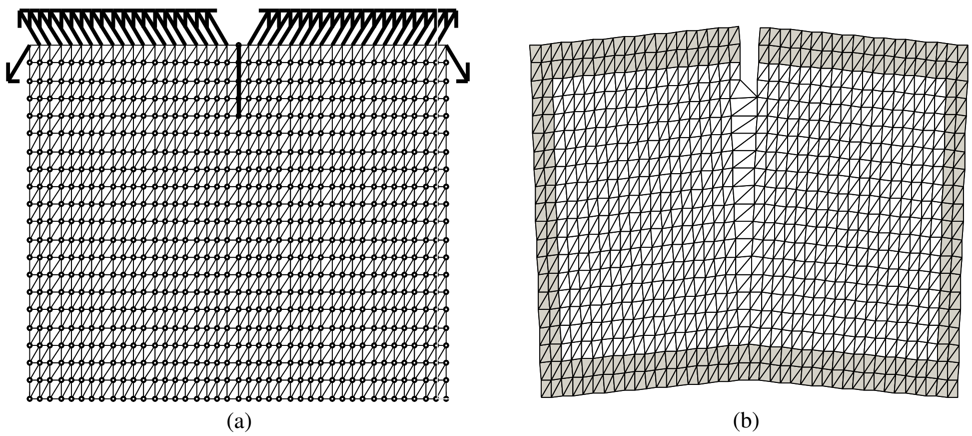

FIGURE 7.10. Plane stress example: (a) worst loading, (b) maximizing virtual displacement.

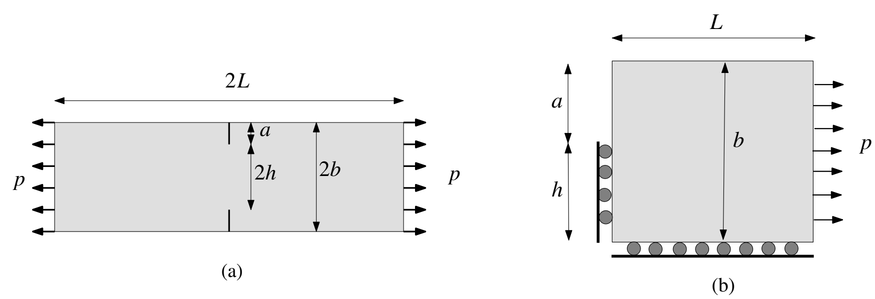

FIGURE 7.11. The test problem: (a) system, (b) model analyzed.

We examined three crack sizes: $a = h/3$, $a = h/2$, and $a = 2h/3$. Using 5184 triangular finite elements, the maximum deviation from the analytic solution was 2.7% for the case $a = 2h/3$. The maximizing virtual displacement for this case is illustrated in Figure 7.12.

7.4.2. *Plane Strain Example.* Consider the dam-like plane strain mechanical system illustrated in Figure 7.13. As indicated by the illustration, the left-hand side of the bottom is detached from the support.

The restricted load capacity ratio was computed for all load distributions on the left slope of the dam. The value obtained was $C = 0.023$. A collapse load[7] and a maximizing virtual displacement are illustrated in Figure 7.13.

[7]Here the direction of forces may vary continuously as we used the Euclidean norm for vectors in the plane.

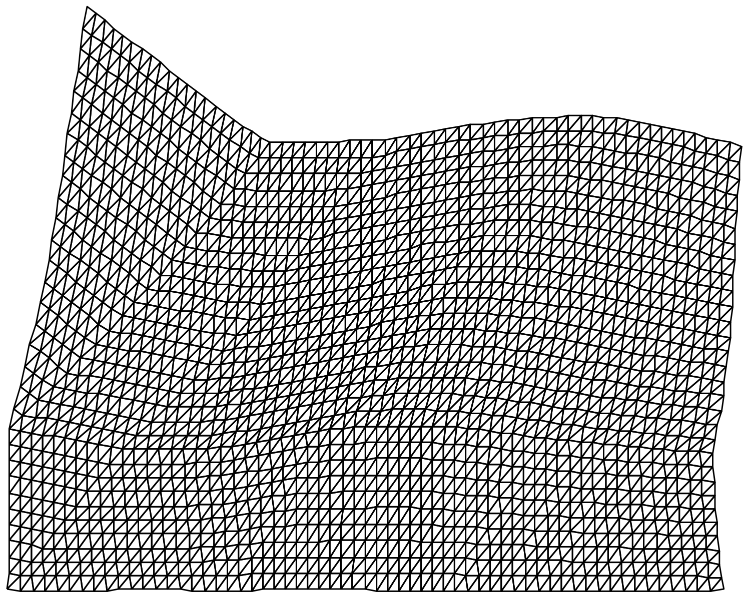

FIGURE 7.12. Maximizing virtual displacement for test problem ($a = 2h/3$)

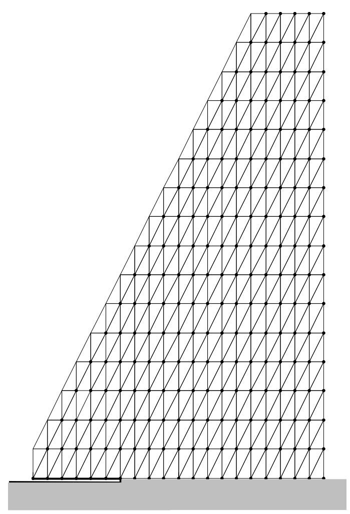

FIGURE 7.13. Plane strain system, a dam-like structure

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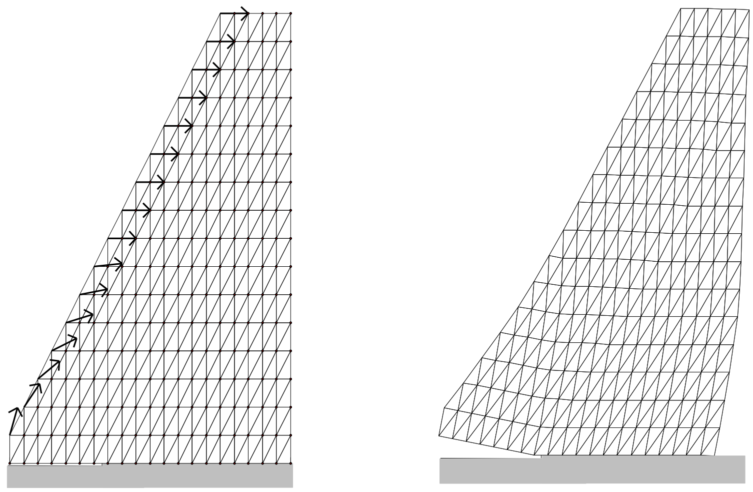

FIGURE 7.14. Plane strain example: (a) worst loading, (b) maximizing virtual displacement.

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Appendix A. Motivation for the Assumption $\|\pi^*(g)\| = \|g\|$

The assumption made in Equation (3.33) can be motivated as follows. Consider the case where the projection $\pi\colon \mathscr{W} \to \mathscr{M}$ satisfies the condition

$$\|\pi(w)\| \leqslant \|w\|, \tag{A.1}$$

so noting that $\pi(v) = v$, for all $v \in \mathscr{M}$, $\pi(\overline{v}) = 0$, for all $\overline{v} \in \overline{\mathscr{M}}$, we have

$$\|v\| \leqslant \|v + \overline{v}\|, \qquad \inf_{\overline{v} \in \overline{\mathscr{M}}} \|v + \overline{v}\| = \|v\|. \tag{A.2}$$

It follows that

$$\begin{aligned}
\|\pi^*(g)\| &= \sup_{w \in \mathscr{W}} \frac{\pi^*(g)(w)}{\|w\|}, \\
&= \sup_{w \in \mathscr{W}} \frac{g(\pi(w))}{\|w\|}, \\
&= \sup_{v \in \mathscr{M}} \left\{ \sup_{\overline{v} \in \overline{\mathscr{M}}} \frac{g(v)}{\|v + \overline{v}\|} \right\}, \\
&= \sup_{v \in \mathscr{M}} \left\{ g(v) \left\{ \sup_{\overline{v} \in \overline{\mathscr{M}}} \frac{1}{\|v + \overline{v}\|} \right\} \right\}, \\
&= \sup_{v \in \mathscr{M}} \left\{ \frac{g(v)}{\|v\|} \right\},
\end{aligned}$$

and we conclude that π^* is an isometry, *i.e.*,

$$\|\pi^*(g)\| = \|g\|. \tag{A.3}$$

Equation (A.1) holds for example in the case where the spaces $\mathscr{M}$ and $\overline{\mathscr{M}}$ are "orthogonal" relative to the norm, *i.e.*, when we have

$$\|w\| = \left(\|\pi(w)\|^p + \|\overline{\pi}(w)\|^p\right)^{1/p}. \tag{A.4}$$

This condition holds for all case considered here and all traditional finite element models where we require that the external forces at a subset of the collection of all nodes vanish.

Another situation where π^* is an isometry is where the norm induced on $\mathscr{M}$ by the norm on $\mathscr{W}$, is the quotient norm on $\mathscr{M} \cong \mathscr{W}/\overline{\mathscr{M}}$ (see the appendix of [27]).

Appendix B. Review of Results for Continuous Bodies

For the sake of completeness, we review in this section the results obtained in [25] for continuous bodies loaded on their boundaries. See also [26, 27, 28] for some additional aspects of the theory for continuous bodies.

The region occupied by the solid body in its current configuration in the physical space $\mathbb{R}^3$ is denoted by Ω and it is assumed to be an open subset of $\mathbb{R}^3$ having a Lipschitz boundary $\partial\Omega = \Gamma$. It is assumed that the boundary may be written as $\Gamma = \Gamma_0 \cup \Gamma_t$, where, Γ_0 is an open subset of the boundary where the body is fixed to its support, Γ_t is an open subset of the boundary where an external traction field t may be applied to the body, $\Gamma_0 \cap \Gamma_t = \varnothing$, and $\overline{\Gamma}_0 \cap \overline{\Gamma}_t$ is a curve on the boundary. It is assumed that the traction field is an essentially bounded vector field on Γ_t so that

$$t_{\max} = \operatorname*{ess\,sup}_{y \in \Gamma_t} \left| t(y) \right| = \|t\|^{\infty},$$

where $\|t\|^{\infty}$ is the L^{∞}-norm of the vector field t. (Body forces may be incorporated into the analysis and are omitted here for the sake of simplicity.)

For a vector field w defined on Ω (interpreted as a velocity field or a virtual displacement field), let $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^{\mathsf{T}})$ be the associated stretching field (or virtual strain field depending on the interpretation). We use a norm $|\sigma(x)|$ for the values of a stress field at $x \in \Omega$ (symmetric stress matrices), and we use the dual norm (although the same notation is used) $|\varepsilon(x)|$ for the values of a strain field.

Using this notation and subject to the assumptions specified, we prove the following in [25].

Theorem B.1. (i) ***The Existence of stresses.*** *Given an essentially bounded traction field t on Γ_t, there is a collection Φ_t of essentially bounded symmetric tensor fields, interpreted physically as stress fields, that represent t in the form*

$$\int_{\Gamma_t} t \cdot w \, \mathrm{d}A = \int_{\Omega} \sigma_{ij} \varepsilon_{ij}(w) \, \mathrm{d}V, \quad \textit{for all} \quad \sigma \in \Phi_t, \; w \in C^{\infty}(\overline{\Omega}, \mathbb{R}^3). \tag{B.1}$$

(ii) ***The Existence of optimal stress fields.*** *Let*

$$s_t^{\mathrm{opt}} = \inf_{\sigma \in \Phi_t} \left\{ \operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)| \right\} \tag{B.2}$$

be the optimal maximal stress. There is an optimal stress field $\sigma_t^{\mathrm{opt}} \in \Phi_t$ for which the optimum is attained, i.e.,

$$s_t^{\mathrm{opt}} = \operatorname*{ess\,sup}_{x \in \Omega} |\widehat{\sigma}(x)|. \tag{B.3}$$

(iii) ***The expression for the optimum.*** *The optimum satisfies*

$$s_t^{\mathrm{opt}} = \sup_{w \in C^{\infty}(\overline{\Omega}, \mathbb{R}^3)} \frac{\left| \int_{\Gamma_t} t \cdot w \, \mathrm{d}A \right|}{\int_{\Omega} |\varepsilon(w)| \, \mathrm{d}V}. \tag{B.4}$$

Using a yield function for a perfectly plastic material as a norm on stress matrices, setting $s_Y = s_t^{\mathrm{opt}}$, and making additional modifications, (*iii*) is equivalent mathematically to a theorem on the limit analysis factor in the theory of plasticity (see [32, 33, 34, 4]). This equivalence implies that perfectly plastic bodies, a common mathematical model for a large class of engineering materials, are optimal in the sense defined above.

The result concerning the generalized stress concentration factor is as follows.

Theorem B.2. *Let the generalized stress concentration factor be defined by*

$$K = \sup_t \frac{s_t^{\mathrm{opt}}}{t_{\max}}. \tag{B.5}$$

Then,

$$K = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} = \|\gamma_0\|, \tag{B.6}$$

where γ_0 is the trace mapping for vector fields satisfying the boundary conditions on Γ_0.[8]

As shown in [25], this result may be adapted in order to obtain an expression for the load capacity ratio of a perfectly plastic body. We recall that a yield function for plastic materials may be expressed as a norm $|\cdot|$ on the space of matrices that is applied to the deviatoric (traceless) component of the stress. Thus, the yield function is written as $|\sigma_D(x)|$, where $\sigma_D = \sigma - \frac{1}{3}\sigma_{ii} I$ is the deviatoric component of the stress matrix. In the sequel, $LD(\Omega)_D$ denotes the space of isochoric (incompressible) vector fields w satisfying the boundary conditions on Γ_0 for which

$$\|w\| = \int_\Omega |\varepsilon(w)| \, \mathrm{d}V < \infty, \tag{B.7}$$

where $|\varepsilon(w)(x)|$ denotes the norm on isochoric strain matrices induced by the norm for stresses specifying the yield condition as in Equations (2.6) and (2.7).

For a homogeneous, perfectly plastic body whose yield stress is s_Y, the result for the load capacity ratio is given as follows.

Theorem B.3. *There is a maximal number C, the load capacity ratio, such that no collapse will occur in the body for any external traction distribution t as long as $t_{\max} \leqslant s_Y C$. The load capacity ratio satisfies*

$$\frac{1}{C} = \sup_{w\in LD(\Omega)_D} \frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} = \|\gamma_D\|, \tag{B.8}$$

where γ_D is the trace mapping for isochoric vector fields satisfying the boundary conditions on Γ_0.

[8] That is, for a continuous vector field u defined on the closure $\overline{\Omega}$, $\gamma_0(u|_\Omega) = u|_\Gamma$. The norm of γ_0 is evaluated relative to the L^1-norm for $\gamma_0(u)$ and the norm $\|u\| = \|\varepsilon(u)\|_{L^1}$ for vector fields on Ω.

We conclude that basically the generalized stress concentration factor and the load capacity ratio are given by similar expressions with the difference that for the load capacity ratio and traditional yield functions the supremum is taken over isochoric (incompressible) fields only.

As an alternative to the approach adopted in this paper, one may consider approximations to the load capacity ratio of a body by searching for optimizing vectors w in the expressions above in finite dimensional subspaces of the space $LD(\Omega)_D$.

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