Nonlinear waves (zonons) in zonostrophic turbulence

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Overview:

- A two-dimensional flow on the surface of a rotating sphere presents a simple model of planetary turbulence (or drift waves turbulence in plasma theory).

- With finite Rossby deformation radius, the flow is known as geostrophic turbulence (Charney).

- Even in its simplified, barotropic version (infinite Rossby radius), the commingling of strong nonlinearity, strong anisotropy and Rossby waves gives rise to complicated dynamics.

- In flows with small-scale forcing, the inherent anisotropic inverse energy cascade may lead to the regime of zonostrophic turbulence.

- It is distinguished by an anisotropic spectrum and stable systems of alternating zonal jets.

- Another important attribute of zonostrophic turbulence is a new class of nonlinear waves - zonons.

- Zonons may form coherent structures observable in physical space (solitons).
2D turbulence on the surface of a rotating sphere (BVE)

The flow is forced on small scales and linearly damped on large scales

\[
\frac{\partial \zeta}{\partial t} + J(\psi, \zeta + f) = \nu \nabla^2 p \zeta - \lambda \zeta + \xi,
\]

\(\zeta = \Delta \psi \) - vorticity; \(\psi\) - stream function; \(f = 2\Omega \sin \theta\) - Coriolis parameter; \(\Omega\) - angular velocity of the sphere’s rotation; \(\theta\) - latitude; \(\phi\) - longitude; \(\nu\) - hyperviscosity coefficient; \(\lambda\) - linear friction coefficient which sets the large-scale friction wave number \(n_{fr}\).

In plasma theory, this model describes drift waves turbulence in nonuniform, finite \(\beta\) plasma with infinite gyroradius

The small-scale forcing \(\xi\) acts on the scales \(n_\xi^{-1}\) and pumps energy into the system at a constant rate. This energy feeds the inverse cascade at a rate \(\varepsilon\).

**Beta-plane** approximates a curved spherical surface by a tangential plane, \(\beta\) - gradient of Coriolis parameter \((f = f_0 + \beta y, y \text{- northward, } x \text{- eastward})\)

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (\nabla^{-2} \zeta, \zeta)}{\partial (x, y)} + \beta \frac{\partial (\nabla^{-2} \zeta)}{\partial x} = \nu \nabla^2 p \zeta - \lambda \zeta + \xi
\]
**Turbulence and Rossby waves: The basics**

Conservation of potential vorticity on a rotating sphere leads to generation of Rossby-Haurwitz waves (RHWs) with the dispersion relation

\[ \omega_R(m,n) = -\frac{2\Omega}{R} \frac{m}{n(n+1)} \]

(spherical harmonics decomposition, \(m\) – zonal, \(n\) – total wave-number)

On a beta-plane: \( \omega_R(k) = -\beta k_x/k^2 \)

**RHW** are solutions of barotropic vorticity equation (BVE) on a rotating sphere without nonlinear term.

Fully nonlinear BVE without rotation describes classical 2D turbulence with inverse energy cascade.

Variation of Coriolis parameter with latitude (beta-effect) introduces anisotropy and Rossby waves which give rise to complicated dynamics.

Characteristic feature of such dynamics – generation of zonal jets.
Forced 2D turbulence - simulations

In order to study nonlinear dynamics, we performed DNS of BVE on a rotating sphere. Spectral model is employed

\[ \Psi(\mu, \phi, t) = \sum_{n=1}^{N} \sum_{m=-n}^{n} \psi_n^m(t) Y_n^m(\mu, \phi) \]

- R-truncation; R133 and R240 resolutions
- Random energy injection with the constant rate \( \varepsilon \) at about \( n_\xi = 100 \)
- Very long-term integrations in a steady-state to compile long records for statistical analysis
- Analyze anisotropic spectrum

\[ E(n) = \frac{n(n+1)}{4R^2} \sum_{m=-n}^{n} \langle |\psi_n^m|^2 \rangle = E_Z(n) + E_R(n) \]

zonal \( (m=0) \) residual
Processes of turbulence on β-plane/rotating sphere

1. How is energy delivered to \( k_x \to 0 \) modes?
2. How much energy those modes retain?

For answers we look at spectral energy transfer

\[
\left[ \frac{\partial}{\partial t} + 2\nu_0 k^2 \right] \Omega(k, t) = T_\Omega(k, t)
\]

\[
T_\Omega(k, t) = \int \int_D T(k, p, q, t) dp dq.
\]

Second order spectral closures yield

\[
T(k, p, q) = \Theta_{-k,p,q}(p^2 - q^2) \sin \alpha
\]

\[
\left[ \frac{p^2 - q^2}{p^2 q^2} \Omega(p) \Omega(q) - \frac{k^2 - q^2}{k^2 q^2} \Omega(q) \Omega(k) \right.
\]

\[
+ \frac{k^2 - p^2}{k^2 p^2} \Omega(p) \Omega(k) \left. \right] + \text{similar terms.}
\]

where \( \Theta_{-k,p,q} \) is triad relaxation time
Further insight: from triad relaxation time

\[ \Theta_{-k,p,q} = \frac{\mu_k + \mu_p + \mu_q}{[\mu_k + \mu_p + \mu_q]^2 + [\omega_{-k} + \omega_p + \omega_q]^2}, \]

\[ \mu_k \propto \tau_{tu}^{-1}, \]

eddy frequency scale at wave-number \( k \)

\[ \theta_{-k,p,q} \rightarrow \pi \delta(\omega_{-k} + \omega_p + \omega_q) \]

in the limit

\[ \frac{(\mu_{-k} + \mu_p + \mu_q)^2}{(\omega_{-k} + \omega_p + \omega_q)^2} \rightarrow 0 \]

Thus, at large scales, resonance condition is needed for effective energy transfer, \( \omega_{-k} + \omega_p + \omega_q \rightarrow 0 \)

Nonlinear interactions of turbulence and waves modifies the flow dynamics

Figure: Spectral energy transfer function computed from DNS of beta-plane turbulence
Development of kinetic energy spectra

**Figure:** Total (a) and nonzonal (b) energy spectra at different times.
The transitional wave number, \( n_\beta \) and Rhines’s wave number, \( n_R \)

- The characteristic time scale of turbulence is \( \tau_t = [n^3E(n)]^{-1/2} \)
- The characteristic time of RHWs is \( \tau_R = [\omega_R(n,m)]^{-1} \)
- Turbulent processes prevail on small scales where \( \tau_t < \tau_R \)
- RHWs are dominant on large scales.
- The transitional wave is at the scale with\( n_\beta \approx \tau_t \sim \tau_R \)
  leading to \( n_\beta = 0.5 (\beta^3/\epsilon)^{1/5}, \beta = \Omega/R \)
- On larger scales, we observe anisotropization of the inverse energy cascade
- In flows with large-scale drag the “final”, stationary destination of the energy front is identified with the friction wave number \( n_{fr} \) which coincides with the Rhines’s wavenumber \( n_R = (2V/\beta)^{1/2}, V \) is the rms velocity
Zonostrophic turbulence

(from Greek ζωνή - band, belt, and στροφή - turning)

Zonostrophy index:

\[ R_\beta = \frac{n_\beta}{n_R} > 2 \]

\[ E_Z(n) = C_Z (\Omega/R)^2 n^{-5}, \quad C_Z \approx 0.5 \]

\[ E_R(n) = C_K \varepsilon^{2/3} n^{-5/3}, \quad C_K \approx 4 \text{ to } 6 \]
Examples of zonostrophic turbulence - the ocean-Jupiter connection

Figure 1. (a) Composite view of the banded structure of the disk of Jupiter taken by NASA’s Cassini spacecraft on December 7, 2000 (image credit: NASA/JPL/University of Arizona); (b) zonal jets at 1000 m depth in the North Pacific Ocean averaged over the last five years of a 58-year long computer simulation. The initial flow field was reconstructed from the Levitus climatology; the flow evolution was driven by the ECMWF climatological forcing. Shaded and white areas are westward and eastward currents, respectively; the contour interval is 2 cm s⁻¹.
The zonal and residual spectra in the ocean, on giant planets and in simulations are indicative of zonostrophic turbulence.
Rossby-Haurwitz waves and turbulence

Are RHWs present in the fully nonlinear equation and, if yes, how are they affected by the nonlinearity?

Fourier-transform of the velocity autocorrelation function

\[ U(\omega, m, n) = \frac{n(n+1)}{4R^2} \langle |\psi^m_n(\omega)|^2 \rangle \]

\( \psi(\omega) \) is a time Fourier transformed spectral coefficient \( \psi(t) \)

Spikes of \( U(\omega,m,n) \) correspond to the dispersion relation ➔ the correlator \( U(\omega,m,n) \) is a convenient diagnostic tool for finding waves in data and in simulations
Waves in friction-dominated regime

\[ n_R = 9.2 \]
\[ n_\beta = 12.3 \]
\[ R_\beta = 1.34 \]

The filled triangles correspond to the RHWs dispersion relation.

The large-scale modes are populated by linear RHWs

A strong RHW signature is present even on scales with \( n/n_\beta > 2 \)

On the smallest scales, the RHW peaks are broadened by turbulence

Even though the flow dynamics is dominated by strong nonlinearity the flow features linear RHWs
Waves in zonostrophic turbulence

\[ n_R = 5.5 \]
\[ n_\beta = 16.2 \]
\[ R_\beta = 2.95 \]

- \( U(\omega, n, m) \) is the velocity correlator
- Filled triangles \( \Rightarrow \) RHWs dispersion relation
- Filled circles \( \Rightarrow \) zonons.

<table>
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Frequencies of RHWs and zonons as functions of m for different n

- RHWs are evident in all modes including \( n > n_\beta \).
- Along with RHWs, one discerns zonons excited by the most energetic RHWs with \( n = 4 \) and \( 5 \).
- \( \omega_z(n,m) \propto m \) and independent of \( n \) for all zonons.
- Zonons form wave packets.
- Their zonal speeds are \( c_z = \frac{\omega_z(n,m)}{m} \).
- \( c_z \) are equal to the zonal phase speeds of the corresponding master RHWs.
All zonons are “slave” waves excited by RHWs

Their dispersion relations differ from RHWs ➔ zonons should be recognized as an entity completely different from RHWs

How do the zonons appear in the physical space? The RHWs with $n = 4$ (denoted $n_E$) are the most energetic ➔ their respective packets of zonons are dominant in physical space and are easiest to observe

The zonal speed of these packets is $\omega_R(n_E,m)/m = c_{RE}$

In physical space, these wave packets are expected to form westward propagating eddies detectable in the Hovmoller diagrams

The slope of the demeaned diagrams yields a velocity of the zonally propagating eddies relative to local zonal flows

If eddies are indeed comprised of zonons, their zonal phase speed should be equal to $c_z = c_{RE}$
Zonons are Rossby wave solitons

- The Hovmoller diagrams reveal westward propagating eddies at three different latitudes at which the zonal jets have their maximum, minimum, and zero velocity.

- The slope of the diagrams yields a velocity of the zonally propagating eddies relative to local zonal flows. The figure demonstrates that $c_z = c_{RE}$ at all latitudes.

**Hovmoller diagram along a latitude**

This diagram shows that zonons propagate along a latitude with maximal shear → zonal flow forms a waveguide, in which the solitary waves propagate.
Energy exchange between jets and waves/turbulence

Energy flux from all nonzonal modes < n to mean flow

Nonlinear interactions between jets and zonons cause energy oscillations
Jets-Zonons Symbiosis

The zonal flow forms a waveguide, in which the solitary waves propagate. Zonons and mean flow (jets) continuously exchange large amount of energy.
Conclusions

- A new class of nonlinear waves in 2D turbulence with a $\beta$-effect, zonons, is presented.

- Zonons are forced oscillations excited by RHWs in other modes via non-linear interactions.

- Zonons are an integral part of the zonostrophic regime. They emerge in the process of energy accumulation in the large-scale modes, formation of the steep $n^{-5}$ spectrum and generation of zonal jets.

- Zonons have characteristic features of solitary waves. The zonal flow forms a waveguide, in which the solitary waves propagate.

- Future research should clarify zonons’ roles in planetary circulations and their relation to large oceanic eddies detected in satellite altimetry (provided that the oceanic circulation is marginally zonostrophic).
References.


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