On Partial Equilibrium in a Queuing System with Two Servers

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1. INTRODUCTION
This paper will deal with the following question: what happens in a service system where the service is supplied by two competitive firms? By a service system we mean any system where customers have to wait in lines before they are served. As we will show, we cannot apply the usual results of economic theory to service systems. For example, in ordinary economic theory two firms supplying the same good will always sell it at the same price. This result is not always true for the case of service and waiting lines. We will discuss the conditions under which two service firms selling at the same price are not in equilibrium.

A discussion of queuing systems with single server (that is, a monopoly) are found in the works of Edelson [1], Naor [3], Knudsen [2] and others. However, there is no mention in the literature of a general equilibrium in a queuing system with more than one server. Between these two extreme cases, the general equilibrium and the determinate single-firm model, lie the partial equilibrium cases, and this paper deals with one of these.

2. TWO FIRMS, TWO PRICES
When a good is sold at different prices, price-equalization forces come into play. These forces do not operate when the firms supply services and the customers may have to wait in queue.

Suppose that only one firm reduces its price. Some of the customers will now prefer to receive the service at a lower price, but this causes the queue at the low-price firm to get longer as an increasing number of customers turn to it. In the high-price firm, on the other hand, the queue becomes shorter and waiting time is reduced. Customers who sufficiently prefer a shorter waiting time will not go to the low-priced firm even though the price is lower. At some stage this process of customers transferring from the high-price firm to the low-price firm will come to an end. We show later that the point where the process ends can be an equilibrium.

Before turning to the detailed model, let us clarify the difference between our model and a model of two firms selling different goods. The waiting for a service is clearly part of the service itself. If there is a different waiting time in each firm, then the service produced by each firm is a different commodity. But if the commodity is not the same in both firms, then of course the price need not be the same. The main feature of our problem is that we cannot say a priori whether there will be two commodities or only one: that is, whether or not the waiting time will be the same in both firms is determined by the system itself.

A similar problem was discussed by Smithies [6] in his paper about duopolistic spatial competition. In both models there are two firms, two prices, and partition of the customers between the two firms. In Smithies' model the customer is affected by freight rates, while in our model the customer considers also the waiting cost. The main difference is that in our model the price is the only independent variable (waiting time is a dependent variable), while in Smithies' model both price and location are independent variables. This difference,
and the fact that waiting time is not the same as freight cost, does not allow us to compare the results of the two models.

But in one respect a similarity can be found. We will show that usually both firms charge different prices and the streams of customers are not the same, but if no customer leaves without service—the prices are the same, and each firm serves one-half of the customers. This last result is equivalent to the case in Smithies' model, where one firm located at each quartile of the market and serves exactly one-half of the customers, and the prices are the same. Smithies shows the conditions that are necessary for this result. But, under general conditions, the firms are not at each quartile, and the prices may differ.

Assume a system with a service rendered by two firms. In both firms the quality of the service itself is the same and the service time is distributed exponentially with the same mean service-time parameter. Each firm fixes its service price so as to maximize profit. We assume that there is no co-operation between the firms. (Later in the paper we shall relax this assumption.)

There is a Poisson stream of customers arriving at the firms. Each customer has three alternatives—to choose firm 1, to choose firm 2, or to leave without receiving service. The choice must be made by considering the cost of waiting, the expected waiting time in each firm, and the price of the service in each firm.

Two computer centres which compete in the same market is an example of such a model. The expected waiting time and the prices in both centres are known to all the customers and potential customers. A potential customer is included in the stream of customers even though he never bothers to show up at either line because of the high prices and the long expected waiting time. By assumption the customer's choice is made according to the prices and the expected waiting time and is not dependent on the current waiting line. That is, we assume a high transaction cost of moving if the customer is already at one of the centres.

We assume that the cost of waiting is a linear function of waiting time (here defined as the sum of queuing time and service time) and that each customer has a different waiting-cost function, so that each customer is characterized by the parameter $C$ which denotes the cost of waiting one unit of time. We assume that there exists a distribution function which shows the probability that the waiting cost parameter does not exceed $C$.

We shall use the following notation:

\[
R = \text{the reward from receiving the service (assumed equal for all customers)}
\]

\[
1/V = \text{expected service time}
\]

\[
W_i = \text{expected waiting time at firm } i \ (i = 1, 2)
\]

\[
C = \text{waiting cost per unit of time (it is assumed that } C > 0)
\]

\[
P_i = \text{the price of the service at firm } i \ (i = 1, 2)
\]

\[
U_i = \text{the net gain of a customer receiving the service from firm } i
\]

\[
F(C) = \text{the distribution of the } C \text{ parameter among customers (assumed continuous)}
\]

\[
f(C) = \text{the density function corresponding to } F(C)
\]

\[
\lambda = \text{the Poisson parameter of the flow of potential customers. For simplicity we assume } \lambda = 1
\]

\[
\lambda_i = \text{the Poisson parameter of arrivals at firm } i.
\]

The customer's net gain at firm $i$ is:

\[
U_i = R - W_i C - P_i \quad (i = 1, 2)
\]  \(\ldots(1)\)

The customer will prefer firm 2 when $U_2 > U_1$, that is, when $(P_1 - P_2)/(W_2 - W_1) > C$ (from $R - W_2 C - P_2 > R - W_1 C - P_1$), and vice versa for firm 1. He will prefer to leave without service when the net gain is negative in both firms, that is, according to (1), when $C > (R - P_1)/W_i$ for $i = 1, 2$. 


Let us summarize these conditions. The customer prefers firm 2 when his waiting cost satisfies

\[ 0 < C \leq (P_1 - P_2)/(W_2 - W_1) \quad \text{and} \quad C \leq (R - P_2)/W_2. \]  

(2A)

He prefers firm 1 when

\[ C > (P_1 - P_2)/(W_2 - W_1) \quad \text{and} \quad C \leq (R - P_1)/W_1 \]  

(3A)

and he leaves without service when

\[ C > (R - P_1)/W_1 \quad \text{and} \quad C > (R - P_2)/W_2. \]  

(4A)

We assume that both firms are operating and that at least one customer arrives at each. It follows that:

\[ (P_1 - P_2)/(W_2 - W_1) < (R - P_1)/W_1 \]  

(5)

Under this assumption, the above conditions are reduced to the following three cases:

The customer prefers firm 2 when:

\[ C \leq (P_1 - P_2)/(W_2 - W_1). \]  

(2)

He prefers firm 1 when:

\[ (P_1 - P_2)/(W_2 - W_1) < C \leq (R - P_1)/W_1 \]  

(3)

and he leaves without service when:

\[ C > (R - P_1)/W_1. \]  

(4)

When (2) holds (i.e. \( U_2 > U_1 \)) and according to (5), we have \( C < (R - P_1)/W_1 \) which implies \( U_1 > 0 \) and hence \( U_2 > U_1 > 0 \). Thus (2) is a sufficient (and of course necessary) condition for the customer to prefer firm 2. However, when (4) holds (i.e. \( U_1 < 0 \)), then from (5), (2) does not hold and \( U_2 < U_1 < 0 \). Thus (4) is a sufficient (and necessary) condition for the customer to leave without service.

In this model, the choice of firm depends not only on the prices in each firm but also on the expected waiting time. But the expected waiting time and the customer’s decisions are interdependent.

Given any price pair \( P_1^*, P_2^* \), the customer-flow to each firm will settle down to two Poisson processes with expected waiting times \( W_1^*, W_2^* \). We are concerned only with the expected waiting time at equilibrium.

Let us now define the functions that determine the expected waiting time. From the literature on queueing theory we can use the standard result for expected waiting time in a Poisson process: \( W_i = (V - \lambda_i)^{-1} \) (for example see Saaty [4, p. 342]). \( 1/V \) is the expected service time and \( \lambda_i \) is the parameter of the Poisson stream of arrivals at firm \( i \). All the customers whose waiting cost satisfies (2) will prefer 2, and the relevant fraction of the customer stream is exactly \( F[(P_1 - P_2)/(W_2 - W_1)] \). Accordingly, the parameter of the Poisson stream at firm 2 is

\[ \lambda_2 = F[(P_1 - P_2)/(W_2 - W_1)]. \]  

(6)

(The right-hand side is here multiplied by \( \lambda \), omitted because we have assumed that \( \lambda = 1 \).) According to (3) the fraction of the stream that prefers firm 1 is:

\[ \lambda_1 = F[(R - P_1)/W_1] - F[(P_1 - P_2)/(W_2 - W_1)]. \]  

(7)

Using \( W_i = (V - \lambda_i)^{-1} \) and putting

\[ (P_1 - P_2)/(W_2 - W_1) = \alpha, \quad (R - P_1)/W_1 = \beta, \]

we get the following expressions for the expected waiting time in each firm:

\[ W_1 = [V - F(\beta) + F(\alpha)]^{-1} \]  

(8)

\[ W_2 = [V - F(\alpha)]^{-1}. \]  

(9)
Equations (8) and (9) define the expected waiting time in each firm as a function of the service price.

We have so far assumed that \( P_1 > P_2 \). But if the price is the same in both firms, some difficulties arise. First, \( \alpha \) is not defined when \( P_1 = P_2 \) for it then follows that the expected waiting times must be the same, i.e. \( W_1 = W_2 \), so that the denominator of the expression is zero. Second, in the preceding discussion we could distinguish between the customers of the two firms according to their waiting cost. With equal prices we cannot make this distinction. Exactly half of the total stream arrives at each firm.

In order to avoid these difficulties, we can look at the limiting values of \( \alpha \) as \( P_1 \to P_2 \). Now \( W_1 \) must be close to \( W_2 \) as \( P_1 \) approaches \( P_2 \), and for the expected waiting times to be similar, the arrival rates of each stream must also be similar. That is, \( \lambda_1 \to \lambda_2 \) as \( P_1 \to P_2 \).

Using the \( \alpha, \beta \) notation, the \( \lambda_i \) stream parameters are

\[
\lambda_1 = F(\beta) - F(\alpha) \quad \text{...(7')}
\]

\[
\lambda_2 = F(\alpha) \quad \text{...(6')}
\]

In the limit when \( \lambda_1 \to \lambda_2 \), we have:

\[
F(\alpha) = \frac{1}{2}F(\beta) \quad \text{...(10)}
\]

This equation implicitly gives the limit of \( \alpha \) as \( P_1 \to P_2 \).

3. EQUAL OR UNEQUAL PRICES

Assume that the firms' costs are constant and independent of the stream of customers. The profit function of firm 1 is \( \pi_1 = P_1 \lambda_1 \) for \( \lambda_1 \) is the expected number of arrivals at firm 1 per unit of time and hence, \( P_1 \lambda_1 \) is the expected profit of firm 1 per unit of time. Firm 1 assumes that the service price of firm 2 is constant, and chooses its optimum price \( P_1 \). But \( \lambda_1 \) is also a function of \( P_1 \), and we formulate the following maximum problem, where the constraints denote the relationship between \( \lambda_1 \) and \( P_1 \):

\[
\max_{P_1, W_1, \lambda_1} \pi_1 = P_1 \lambda_1 \quad \text{...(11)}
\]

subject to (6)-(9) and with \( P_2 \) assumed constant. Similarly, for firm 2 we get

\[
\max_{P_2, W_1, \lambda_1} \pi_2 = P_2 \lambda_2 \quad \text{...(12)}
\]

subject to the same constraints and with \( P_1 \) assumed constant.

Each firm chooses an optimum price for every price of the other firm, and a price-changing process takes place in the two firms. The process ends, with the firms' equilibrium, when neither firm wants to change its own price. Let \( P_1^*, P_2^* \) be the prices chosen at this point.

The main question of this paper is whether \( P_1^* = P_2^* \) is consistent with equilibrium. We analyse this question as follows: first, the partial derivatives \( \partial \pi_1 / \partial P_1 \) and \( \partial \pi_2 / \partial P_2 \) are found; second, the values of these partial derivatives are calculated at points where \( P_1 = P_2 \). \( P_1^* = P_2^* \) are equilibrium prices if

\[
\frac{\partial \pi_1}{\partial P_1} \leq 0 \quad \text{for all } P_1 \geq P_1^* \quad \text{...(13)}
\]

\[
\frac{\partial \pi_2}{\partial P_2} \geq 0 \quad \text{for all } P_2 \leq P_2^*
\]

These are the general conditions for equilibrium prices. In the special case where the demand functions are continuous and without kinks, the conditions must hold with strict equality (see Shubik [5, p. 155]).

We will show that the demand function is not continuous at this point, so that the only way to write the conditions for a Nash equilibrium point is by inequalities like (13).
If \( \partial \pi_1 / \partial P_1 > \partial \pi_2 / \partial P_2 \) whenever \( P_1 = P_2 \), then conditions (13) for equilibrium with equal prices do not hold and the system is not in equilibrium when both firms sell at the same price.

Let us begin with \( \partial \pi_2 / \partial P_2 \). From (12) and (6) we get the profit function of firm 2:

\[
\pi_2 = P_2 f'(x) \tag{14}
\]

whose partial derivative with respect to \( P_2 \) is

\[
\frac{\partial \pi_2}{\partial P_2} = f'(x) - P_2 f'(x) \left[ \frac{W_2 - W_1}{W_2 - W_1} + \frac{\partial W_2}{\partial P_2} - \frac{\partial W_1}{\partial P_2} \right]. \tag{15}
\]

In order to obtain the factor \( \partial W_2 / \partial P_2 - \partial W_1 / \partial P_2 \) in terms of the parameters, differentiate (8) and (9) with respect to \( P_2 \). Substitution in (15) gives:

\[
\frac{\partial \pi_2}{\partial P_2} = f'(x) - P_2 \left[ \frac{W_2 - W_1}{f'(x)} + W_2^2 \alpha + \frac{W_1^2 \alpha}{\gamma} \right]^{-1}, \tag{16}
\]

where \( \gamma = 1 + W_1 f'(x) \). If \( P_2 = 0 \), then \( \partial \pi_2 / \partial P_2 = f'(x) > 0 \).

As \( P_2 \to P_1 \) (when \( P_2 \leq P_1 \)), \( W_2 \to W_1 \). Hence

\[
\lim_{P_2 \to P_1} \frac{\partial \pi_2}{\partial P_2} = f'(x) - \frac{P_2}{W^2} \left( \frac{1}{\alpha + \alpha / \gamma} \right), \tag{17}
\]

where \( W = W_1 = W_2 \) (in the limit).

For firm 1, the derivation is similar and we get

\[
\lim_{P_1 \to P_2} \frac{\partial \pi_1}{\partial P_1} = f'(x) - \frac{P_1}{W^2} \left[ f'(x) W \alpha + 1 \right]/\gamma. \tag{18}
\]

We can now look at the relationship between the two derivatives (17) and (18). Using \( \gamma \) defined as above, the last derivative can be written:

\[
\lim_{P_1 \to P_2} \frac{\partial \pi_1}{\partial P_1} = f'(x) - \frac{P_1}{W^2} \left( \frac{1}{\alpha + \alpha / \gamma} \right) + \frac{P_1}{W^2} \frac{f'(x) W (\beta - \alpha)}{(\gamma \alpha + \alpha)}. \tag{19}
\]

and using (17),

\[
\lim_{P_1 \to P_2} \frac{\partial \pi_1}{\partial P_1} = \frac{\partial \pi_2}{\partial P_2} + \frac{P_1}{W} \frac{f'(x) (\beta - \alpha)}{\alpha (\gamma + 1)}. \tag{20}
\]

Now \( \beta > \alpha \), hence the right-hand argument is positive. Therefore

\[
\partial \pi_1 / \partial P_1 \geq \partial \pi_2 / \partial P_2 \tag{21}
\]

and the equality holds only if \( f'(x) = 0 \).

When \( P_1^* \neq P_2^* \), the two firms are in equilibrium when \( \partial \pi_i / \partial P_i = 0 \) and the second-order conditions are right. But when \( P_1^* = P_2^* \), then, as shown earlier, the firms are in equilibrium if conditions (13) are satisfied. But, as we have just seen, when \( f'(x) > 0 \), we have inequality in (21) for all \( P_i \) pairs where \( P_1 = P_2 \); that is, conditions (13) are not satisfied.

Now, \( f'(x) \) is the probability that there are customers with high waiting costs who want to receive the service but leave the queue because they have to wait too long. Thus \( f'(x) = 0 \) if all customers receive the service, while if some customers leave without being served, \( f'(x) > 0 \), and the price cannot be the same in both firms. The difficulty is that \( \beta \) depends on the price, so that we cannot say whether or not prices will in fact be equal.

Assume, however, that the system diverges (that is, \( \lambda/2 > V \)). Then if all the customers remain in the queue, the waiting time will extend to infinity, and, in such a system, \( f'(x) \) is positive. In other words, in a system where \( \lambda/2 > V \), each firm will sell at a different price.

On the other hand, if \( \lambda/2 \) is very small compared with \( V \), waiting time will be relatively short, even if all the customers receive the service. In that case, we have \( f'(x) = 1 \), so it is reasonable that \( f'(x) = 0 \) and that the price will be the same in both firms.
We can summarize these results in the following lemma:

**Lemma.** Suppose that the set on which \(f(x) > 0\) is a single interval. Then, a necessary condition for equal price equilibrium \(P\) is:

\[
R \geq W^2 F^{-1}(0.5) + WF^{-1}(1),
\]

...(22)

where \(W\) is the expected waiting time in each firm such that each firm serves one-half of the stream, and no one leaves without service.

(Note that \(\lambda = 1\), but appears here in order to make it clear that the parts of the right hand have the same unit scale.)

**Proof.** If \(P_1 = P_2 = P\), then \(f(\beta) = 0\). The condition for maximum profit in firm 1 becomes (see 19):

\[
\frac{\partial \pi_1}{\partial P_1} = F(x) - \frac{P}{W} \left( \frac{1}{2x} \right) = 0.
\]

...(23)

From the fact that \(F(x) = \lambda/2\) and \(\lambda = 1\) we obtain:

\[
P = \lambda W^2 x = \lambda W^2 F^{-1}(0.5)
\]

...(24)

\(f(\beta)\) is zero if

\[
(R - P)/W \geq F^{-1}(1).
\]

By replacing of \(P\) from (24) we obtain the necessary condition (22).

The meaning of the condition for equal prices (22) is that no customer leaves without service. That happens when even for the highest waiting cost per hour, we have that the benefit from the service \(R\) is greater than the sum of the total waiting cost \([WF^{-1}(1)]\) and the price for the service \([W^2 \lambda F^{-1}(0.5)]\).

Unfortunately, condition (22) is not sufficient for equal price equilibrium (a counter example can be found). The reason is that we cannot be sure that \(f(\beta)\) has vanished. The firms may end up with such high prices that some customers leave without service.

### 4. CO-OPERATION BETWEEN FIRMS

In this section we relax the assumption of no co-operation. The two firms now aim at maximizing the sum of their profits, that is, \(\max \pi = \pi_1 + \pi_2\). We consider here whether their prices will be equal. We first determine the necessary conditions for maximum profits under equal prices and then calculate how profits change when one price is changed.

We start by setting \(P_1 = P_2 = P\), that is, by forcing the prices to be equal. The intensity of the stream of customers who do not leave without service is \(\lambda:\)

\[
\lambda = F[(R - P)/W].
\]

...(25)

Total profit is \(\pi = \lambda P\). Each firm receives half of the total stream of customers, and expected waiting time in each firm is therefore

\[
W = (V - \frac{\lambda}{2})^{-1}.
\]

Using (25) and putting \((R - P)/W = \beta\), we can write

\[
\max_P \pi = PF(\beta)
\]

s.t. \(W = [V - \frac{1}{2}F(\beta)]^{-1}\)

with solution:

\[
\frac{1}{2} \beta F(\beta) + \frac{F(\beta)}{WF(\beta)} - \frac{P}{W} = 0.
\]

...(27)
The \( P^* \) which satisfies (27) maximizes the joint profit of the two firms, under the constraint of equal prices.

Let us now allow the price of one of the firm to change. In order to demonstrate the effect on joint profits we have to determine the sign of the following expression:

\[
\frac{\partial \pi}{\partial P_1} = \frac{\partial \pi_1}{\partial P_1} + \frac{\partial \pi_2}{\partial P_1}, \tag{28}
\]

where the derivatives are valued at the points \( P^*_1 = P^*_2 = P^* \) which satisfy (27). The solution for the left-hand term is equation (18), while the right-hand term is given by

\[
\frac{\partial \pi_2}{\partial P_1} = \frac{P_2}{W^2} \frac{1 - [W_1 P_2 \varepsilon f(\beta)]^{\gamma}}{\alpha + \alpha/\gamma}. \tag{29}
\]

Substituting (29) and (18) in (28) yields:

\[
\frac{\partial \pi}{\partial P_1} = F(\alpha) - \frac{P_1}{W^2} \frac{W \varepsilon f(\beta) + 1}{\gamma} + \frac{P_2}{W^2} \frac{1 - [W_1 P_2 \varepsilon f(\beta)]^{\gamma}}{\alpha + \alpha/\gamma}. \tag{30}
\]

We can also obtain \( \frac{\partial \pi}{\partial P_1} \) in the form:

\[
A \frac{\partial \pi}{\partial P_1} = \frac{F(\beta)}{W f(\beta)} + \frac{\beta F(\beta)}{2} - \frac{P}{W^2} \left( 2 - \frac{\beta}{\alpha} \right). \tag{31}
\]

where \( A \) is positive.

We are looking for the sign of the derivative \( \frac{\partial \pi}{\partial P_1} \) when it is valued at the point \( P^*_1 = P^*_2 = P^* \), the optimum price for the two firms under the constraint of equal service prices. That is, \( P^* \) must satisfy (27). Then subtracting (27) from (31) gives:

\[
A \frac{\partial \pi}{\partial P_1} = - \frac{P}{W^2} \left( 1 - \frac{\beta}{\alpha} \right) > 0 \tag{32}
\]

because \( \beta > \alpha \).

Thus, \( \frac{\partial \pi}{\partial P_1} \) is always positive and the two firms can always increase their total profit by selling their services at different prices.

That is an obvious result following the assumption that different customers have different waiting costs. In the initial case, the two firms charge the same price. Then, one firm can raise its price without any change in the total number of customers. The result is higher joint profits. Suppose that the prices are the same in each firm, and the customers are divided between the firms so that those with higher waiting cost are served in firm 1. Now, let us take the one with the lowest waiting cost in firm 1 and transfer him to the second firm. A decrease in the expected waiting time in firm 1 will enable the firm to raise its price in such a way that those who were served in firm 1 continue to be served there. In conclusion, one firm raises its price, the second firm doesn't change its price, the total number of customers remains the same, and the result is higher joint profits.

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