Anharmonicity of flux lattices and thermal fluctuations in layered superconductors

T. Ruth Goldin and Baruch Horovitz
Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel

I. INTRODUCTION

The properties of Abrikosov flux lattices in layered superconductors are of considerable interest in view of numerous experiments on high-T_c compounds. For magnetic field perpendicular to the layers, the flux lattice can be considered as a two-dimensional (2D) point vortices in each superconducting layer which are stacked one on top of the other. Each point vortex, or a pancake vortex, represents a singularity of the superconducting order parameter, i.e., the superconductor’s phase in a given layer changes by 2\pi around the vortex. Two pancake vortices are coupled by their magnetic field as well as by the Josephson tunneling between nearest layers. The fluctuations of the displacements of pancake vortices are manifestly by a variety of experiments, and affect phase transitions such as melting transition of the flux lattice and layer decoupling, i.e., vanishing of the interlayer Josephson coupling on long scales. For anisotropic layer systems a harmonic expansion was studied by Glazman and Koshelev in Ref. 9. This expansion is, however, nontrivial, since it involves expanding the nonlinear Josephson coupling J cos \phi_{n,n+1} where \phi_{n,n+1} is the relative phase of neighboring layers. When two pancake vortices which are one on the top of the other are separated by a distance 2\rho (parallel to the layers) then \phi_{n,n+1} has large variations in a circle of radius \rho between these pancake vortices. This effect has led GK (Ref. 9) to anticipate a \rho^2 ln \rho term in the energy expansion when \rho >> \xi where \xi is the in-layer coherence length. This term was also found by Kramer for a single vortex line and independently by us.11

In this work we present the detailed energy expansion. In addition to the difficulty at short scales, leading to the \rho^2 ln \rho term, we find that the convergence parameter of the expansion vanishes as 1/ln E_J when the Josephson coupling E_J \to 0. Thus the E_J=0 elastic constants are not recovered by a E_J \to 0 limit. We then show (Sec. III D) how to define effective elastic constants and apply our results (Sec. IV) to thermal averages of \langle \rho^2 \rangle. In particular we find \langle \rho^2 \rangle \sim T/ln T for \lambda_J \ll a, where \lambda_J is the Josephson length and a is the flux-line spacing. The expansion parameter indicates the type of elastic constants are not recovered by a E_J \to 0 limit.
volves integrating over \( \phi_n(\mathbf{r}) \) and \( \tilde{A}(\mathbf{r}, z) \), subject to a gauge condition. Since \( \tilde{A}(\mathbf{r}, z) \) is a Gaussian field (choosing the axial gauge \( A_z(\mathbf{r}, z) = 0 \), \( \tilde{A}(\mathbf{r}, z) = [\mathbf{A}(\mathbf{r}, z), 0] \)) we can shift \( \mathbf{A} \rightarrow \mathbf{A} + \delta \mathbf{A} \) where \( \mathbf{A}(\mathbf{r}, z) \) now satisfies the \( x, y \) components of

\[
\mathbf{A} = \frac{d}{2 \pi} \sum_{n} \left[ \Phi_0 \mathbf{v} \phi_n(\mathbf{r}) - \mathbf{A}(\mathbf{r}, z) \right] \delta(z - nd)
\]

(2)

and then fluctuations in \( \delta \mathbf{A} \) decouple from those of \( \phi_n(\mathbf{r}) \). The partition sum at temperature \( T \) is now

\[
Z = \int \mathcal{D} \phi_n(\mathbf{r}) \exp[-\mathcal{F}/T]
\]

(3)

with \( \tilde{A}(\mathbf{r}, z) \) in Eq. (1) given by the solution of Eq. (2). Note that since Eq. (2) is gauge invariant under \( \mathbf{A} \rightarrow \mathbf{A} - (\Phi_0/2\pi) \mathbf{v} \chi (\mathbf{r}, nd) \) and \( \phi_n(\mathbf{r}) \rightarrow \phi_n(\mathbf{r}) - \chi (\mathbf{r}, nd) \) one can in fact choose any gauge.

We now decompose \( \phi_n(\mathbf{r}) \) to

\[
\phi_n(\mathbf{r}) = \phi_n^0(\mathbf{r}) + \sum_{\mathbf{r}'} s_n(\mathbf{r}') \alpha(\mathbf{r} - \mathbf{r}'),
\]

(4a)

\[
\theta_n(\mathbf{r}) = \phi_n^0(\mathbf{r}) - \phi_{n-1}^0(\mathbf{r}),
\]

(4b)

where \( \phi_n^0(\mathbf{r}) \) is the nonsingular part of \( \phi_n(\mathbf{r}) \), \( \alpha(\mathbf{r}) = \arctan(y/x) \) with \( \mathbf{r} = (x, y) \), \( s_n(\mathbf{r}) = 1 \) at pancake vortex sites and \( s_n(\mathbf{r}) = 0 \) otherwise. The sum in Eq. (4a) is then a sum on \( \mathbf{r}' \) being the vortex positions on the \( n \)th layer.

Solving Eq. (2) for \( \theta_n(\mathbf{r}) \) and \( s_n \), substituting in Eq. (1) yields

\[
\mathcal{F} = \mathcal{F}_v + \mathcal{F}_j + \mathcal{F}_f,
\]

where \( \mathcal{F}_v \) is the vortex-vortex interaction via the 3D magnetic field, \( \mathcal{F}_j \) is interlayer Josephson coupling term and \( \mathcal{F}_f \) is an energy due to fluctuations of the nonsingular phase:

\[
\mathcal{F}_v = \frac{1}{2} \sum_{n, n'} \sum_{r, r'} s_n(\mathbf{r}) G_{q}(\mathbf{r} - \mathbf{r'}; n - n') s_{n'}(\mathbf{r'}),
\]

(5a)

\[
\mathcal{F}_j = -E_J \sum_n \int d^2 \mathbf{r} \left[ \cos\left( \theta_n(\mathbf{r}) + \sum_{\mathbf{r}'} s_n(\mathbf{r}') \right) 
\right.

- s_{n-1}(\mathbf{r'}) \alpha(\mathbf{r} - \mathbf{r'}) - 1 \right],
\]

(5b)

\[
\mathcal{F}_f = \frac{1}{2} \int_{BZ} \frac{d^2 q d k}{(2\pi)^3} \left( d a^2 \right)^2 \{ [q^2 c_{11}(q, k) + k^2 c_{44}(q, k)] |u'(q, k)|^2 + [q^2 c_{66}(q, k) + k^2 c_{66}'(q, k)] |u''(q, k)|^2 \}
\]

(7)

where \( k^2 = (4d^2) \sin^2(kd/2) \), \( a^2 \) is the area of a unit cell \( (a^2 = \Phi_0/BZ) \) and \( f_{BZ} \) is for \( q \) integration over the Brillouin zone. We assume below that \( d \ll a \lambda_{ab} \) as is the case for high-\( T_c \) compounds.\(^{1,2}\)

Note that for \( q = 0 \) there should be no distinction between transverse and longitudinal \( c_{44}'(0, k) = c_{44}''(0, k) \), however for \( q \neq 0 \), \( c_{44}(q, k) \) and \( c_{44}''(q, k) \) may differ.

III. ELASTIC CONSTANTS OF THE FLUX LATTICE

A. Magnetic coupling

We consider first the case with no Josephson coupling, \( E_J = 0 \). The vortex-vortex interaction then has the form
\[ F_v = \frac{1}{2} \sum_{n,n'} \left( 1 - \delta_{n,n'} \right) \int_{-\pi/d}^{\pi/d} \frac{d^2qdk}{(2\pi)^2} G_v(q,k)e^{iq(R_{ij} - R_{ij} + u''_{ij} - u''_{ij}')} e^{ik(n-n')d} \]
\[ + \frac{1}{2} \sum_{n,n'} \left( 1 - \delta_{n,n'} \right) \int_{-\pi/d}^{\pi/d} \frac{d^2qdk}{(2\pi)^2} G_v(q,k)e^{iq(u''_{ij} - u''_{ij}')} e^{ik(n-n')d}. \]

The first and second terms can be expanded with respect to \( \mathbf{q} \cdot \mathbf{u} \) since in absence of the \( l = l' \), \( n = n' \) term they converge. It is important not to decompose the \( (1 - \delta_{n,n'}) \) or \( (1 - \delta_{n,n'}) \) factors until all integrals converge; the \( l = l' \), \( n = n' \) terms produce then the integral terms in the following elastic matrix:

\[ F_v = \frac{1}{2} \int_{BZ} \int_{-\pi/d}^{\pi/d} \frac{d^2qdk}{(2\pi)^2} \phi_{ij}(q,k)u_i(q,k)u_j(q,k) \]

\[ \phi_{ij}(q,k) = \left( \frac{1}{d}\right)^2 \sum_{Q} \left[ G_v[(\mathbf{Q} - \mathbf{q}),(\mathbf{Q} - \mathbf{q})^i(\mathbf{Q} - \mathbf{q})^j - G_v(\mathbf{Q},0)Q^iQ^j] \right] \]
\[ - \left( \frac{1}{d}\right)^2 \sum_{Q} \left[ G_v[(\mathbf{p} - \mathbf{q}),(\mathbf{p} - \mathbf{q})^i(\mathbf{p} - \mathbf{q})^j - G_v(p,0)p^i p^j] \right] \]
\[ + \frac{k_{ij}}{2} \left( \frac{1}{d}\right)^2 \sum_{Q} \left[ G_v(p,k) - G_v(p,0) \right], \]

where \( \mathbf{Q} \) are 2D reciprocal vectors of the hexagonal lattice.

Considering \( \mathbf{q} \to 0 \) we use the symmetry of the hexagonal lattice

\[ \sum_{i,j} g(\mathbf{Q})Q_iQ_j = \frac{1}{2} \sum_{Q} g(Q)Q^2, \] (8a)

\[ \sum_{i,j} g(Q)Q_iQ_j = 0, \] (8b)

\[ \sum_{i,j,m} g(Q)Q_i Q_j Q_m = \frac{1}{8} \left( \delta_{ij}\delta_{jm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} \right) \sum_{Q} g(Q)Q^4, \] (8c)

and separate the \( Q = 0 \) and \( \Sigma_{Q \neq 0} \) parts. We consider flux-line spacing \( a \gg d \), so that the \( Q \) sums involve many terms \( \sim (a/d)^2 \) and the sums can be approximated by integrals

\[ \sum_{Q \neq 0} \sim \frac{2}{Q^2} \int_{Q=0}^{\infty} QdQ, \] (9)

where \( Q^2 = 4\pi B/\Phi_0 = 4\pi/a^2 \), i.e., \( \pi Q^2 \) is the area of a Brillouin zone.

Note that for \( q \to 0 \):

\[ G_v(q,k) \to \frac{\Phi_0^2 d^2}{4\pi} \left[ \frac{1}{1 + \lambda_{ab}^2(q^2 + k^2)} + \frac{k^2}{q^2 + \lambda_{ab}^2(q^2 + k^2)} \right]. \] (10)

The first term of Eq. (10) contributes to the compression moduli \( c_{11} \) while the second term to the longitudinal part of the tilt moduli \( c_{44}^{10} \). Therefore the \( E_J = 0 \) compression \( c_{11} \), shear \( c_{66} \) and tilt \( c_{44}^{10} \) moduli for \( q \ll Q_0 \) are

\[ c_{11}(q,k) = \frac{B^2}{4\pi} \left[ \frac{1}{1 + \lambda_{ab}^2(q^2 + k^2)} + \frac{3}{16} \left( \sum_{Q \neq 0} \frac{1}{Q} \frac{\partial}{\partial Q} \right) Q^2 G_v(Q,k) - \frac{1}{(2\pi)^2} \frac{1}{p} \frac{\partial}{\partial p} \left( p^2 G_v(p,k) \right) \right] \]
\[ + \frac{3}{16d^2a^2} \left( \sum_{Q \neq 0} \frac{1}{Q} \frac{\partial}{\partial Q} \right) Q^2 G_v(Q,k) - \frac{1}{(2\pi)^2} \frac{1}{p} \frac{\partial}{\partial p} \left( p^3 \frac{\partial}{\partial p} G_v(p,k) \right) \]
\[ = \frac{B^2}{4\pi} \left[ \frac{1}{1 + \lambda_{ab}^2(q^2 + k^2)} - \frac{B\Phi_0}{(8\pi\lambda_{ab})^2} \right], \] (11a)

\[ c_{66}(q,k) = \frac{1}{d^2a^2} \left( \sum_{Q \neq 0} \frac{1}{Q} \frac{\partial}{\partial Q} \right) Q^2 G_v(q,k) - \frac{1}{(2\pi)^2} \frac{1}{p} \frac{\partial}{\partial p} \left( p^3 \frac{\partial}{\partial p} G_v(p,k) \right) = \frac{B\Phi_0}{(8\pi\lambda_{ab})^2}, \] (11b)
ANHARMONICITY OF FLUX LATTICES AND THERMAL...

\[ c_{44}^{(0)}(q,k) = \frac{B^2}{4\pi} \left( \frac{1}{1 + k_{ab}^2(q^2 + k_z^2)} + c_{44}^{(0)}(q,k) \right), \]

\[ c_{44}^{(r,0)}(q,k) = \frac{1}{2} \left( \frac{1}{d a^2} \right)^2 \frac{1}{k_z} \sum_{\mathcal{Q} \in 0} [G_c(Q,k) - G_c(Q,0)] \mathcal{Q}^2 = \frac{2B\Phi_0}{(8\pi k_{ab}^2)^2} \left( \frac{1 + k_z^2/\mathcal{Q}_0^2}{1 + \frac{k_z^2}{k_z}} \right). \]

with $1/\xi$ a cutoff on the $\mathcal{Q}$ summation.

Note that $c_{44}^{(0)}(q,k) \neq c_{44}^{(r,0)}(q,k)$ even for $q \to 0$ due to the singular form of the vortex-vortex interaction $G_c(q,k)$ [Eq. (10)]. At $q = 0$ the $c_{44}$ terms combine into $\frac{1}{2}k_z^2[\mathcal{L}(0,k) + c_{44}^{(0)}(0,k)]$ which can also be verified by direct expansion for $u(0,k)$. As shown below, a finite Josephson coupling restores the equality $c_{44}^{(0)}(q,k) = c_{44}^{(r,0)}(q,k)$ at $q \to 0$.

**B. Josephson coupling: “naïve” expansion**

We consider now the contribution of the Josephson coupling Eq. (5b) to the elastic constants by a conventional expansion, reproducing the results of G-K.

The singular part of Josephson phase difference in the interlayer Josephson coupling term can be written as (see Fig. 1)

\[ \psi^n_i(r) = \alpha(r - R^n_i - \rho_i^n) - \alpha(r - R^n_i + \rho_i^n). \]

Here we defined

\[ R^n_i = R_i + \frac{u^n_i + u_i^{n-1}}{2}, \]

\[ \rho_i^n = \frac{u^n_i - u_i^{n-1}}{2}, \]

\[ v^n_i(r) = r - R^n_i. \]

The usual way for treatment of the cosine term in Eq. (5b) is by using a “naïve” double expansion:

(i) Expansion of the cosine with respect to the phase difference

\[ \cos\left( \sum \psi^n_i(r) \right). \]

(ii) Expansion of the singular phase difference $\psi^n_i(r)$ with respect to $\rho_i^n$.

Each pair of vortices, displaced by $u^n_i$ and $u_i^{n-1}$, respectively, defines a “$\rho$ circle” in space where $|v^n_i(r)| < \rho_i^n$, see Fig. 1. Within a $\rho$ circle $\psi^n_i(r)$ has a $2\pi$ discontinuity and therefore cannot be expanded. The expansion (ii) is reasonable only in the region far from the $\rho$ circle where $|v^n_i(r)| \gg |\rho_i^n|$.

Within the approximations (i) and (ii) we can write

\[ \sum_n \int d^2r \left( \cos \theta^n(r) + \sum_n \psi^n_i(r) \right) - 1 \]

\[ \approx \frac{1}{2} \sum_n \int d^2r \left[ \cos \theta^n(r) + \sum_n \psi^n_i(r) \right]^2 \]

\[ \approx \frac{1}{2d} \int d^2q dk \left( |\theta(q,k)|^2 + \theta^*(q,k)\mathcal{B}(q,k) + c.c. + |\mathcal{B}(q,k)|^2 \right), \]

where we define

\[ \mathcal{B}(q,k) = d \sum_n \int d^2r e^{iqr + iknd} \sum_l B^n_l(r) \]

\[ = - \frac{4\pi i d [\hat{z} \times \hat{q}] \cdot \rho(q,k)}{q}. \]

\[ B^n_l(r) = -2\nabla \alpha(r - R^n_i) \cdot \rho_i^n = \frac{2[\rho^n_i \times v^n_i(r)_z]}{v^n_i(r)^2}, \]

and use the Fourier transform

\[ \int d^2r \nabla \alpha(r) e^{iqr} = \frac{2\pi i [\hat{z} \times \hat{q}]}{q}. \]

Note, that merely the use of the expansion (ii) leads to an error of order $\rho^2$ since the difference $(\psi^n_i(r))^2 - (B^n_l(r))^2$ is of order 1 in the $\rho$ circle with area $\sim \rho^2$.

Combining Eq. (14) with $\mathcal{F}_f$ of Eq. (5c) yields

\[ \mathcal{F}_f + \mathcal{F}_j = \frac{1}{2} \int_{-\pi/d}^{\pi/d} d^2q dk \left( G^{-1}(q,k) + E_f/4d \right) |\theta(q,k)| \]

\[ - \theta^0(q,k)^2 + \frac{E_f}{2d} \int_{-\pi/d}^{\pi/d} d^2q dk \]

\[ \times \left[ 4\pi i d [\hat{z} \times \hat{q}] \cdot \rho(q,k)^2 \right] \frac{2\pi i [\hat{z} \times \hat{q}]}{q^2 + \eta^2} + O(\rho^4). \]

Here we introduced

\[ \psi^n_i(r) \approx -2\nabla \alpha(r - R^n_i) \cdot \rho_i^n. \]
\[ \theta^0(q,k) = -\frac{\eta^2 B(q,k)}{q^2 + \eta_k^2}, \]
\[ \eta_k^2 = 4 \lambda_j^{-2} \left( \sin^2 \frac{k d}{2} + \frac{d^2}{4 \lambda_{ab}} \right), \]

where the Josephson length is
\[ \lambda_j = \frac{\Phi_0 d}{4 \lambda_{ab} \sqrt{E_j d \pi^2}}. \]

Since \( d \ll \lambda_{ab} \), typically \( \eta_k \approx 2/\lambda_j \) for most \( k \) averages below.

The last term in Eq. (16) contributes to the longitudinal \( c_{44}^l \) and transverse \( c_{44}^t \) part of the tilt moduli. Rewriting the integrand in the form
\[ \int_0^\infty d^2q g(q,k) = \sum_n \int_0^\infty d^2q g(q + Q_n,k), \]
using the symmetry of the hexagonal lattice Eqs. (8) as well as Eq. (9) with an upper cutoff \( 1/\xi \), the tilt moduli [including the magnetic contribution, Eqs. (11c), (11d)] can be written as
\[ c_{44}^l(q,k) = c_{44}^{l0} - \frac{2 B \Phi_0}{(8 \pi \lambda_j)^2} \ln \xi \left[ Q_2^2 + (1 + \lambda_{ab}^2 k_z^2)/\lambda_j^2 \right]. \]

(18a)

\[ \Phi = \Phi_\nu + \frac{1}{2} \sum_n \int d^2q d^2k \left[ G_{q}^{-1}(q,k)[|\theta^0(q,k)|^2 + |\epsilon(q,k)|^2 + \theta^0(q,k)\epsilon^*(q,k) + c.c.] \right. \]
\[ -E_J \sum_n \int d^2r \left\{ \cos[\theta^0(r) + \sum_l \psi^l(r)] - 1 - \frac{1}{2} \epsilon^0(r)^2 - \epsilon^0(r) \sin[\theta^0(r) + \sum_l \psi^l(r)] \right\} + O(\epsilon^4, \epsilon^2 \rho^2). \]

(19)

We show below that terms of order \( \epsilon^2 \{ \cos[\theta^0(r)] + \sum \psi^l(r) \} \) contribute the \( O(\epsilon^2 \rho^2) \) correction to the free energy after integration over \( r \).

The expansion is most efficient when the term linear in \( \epsilon(q,k) \) vanishes. This determines \( \theta^0(q,k) \) to be the solution of
\[ \theta^0(q,k) = -\frac{\eta^2}{q} \cdot d \sum_n \int d^2r \sin \left( \theta^0(r) + \sum_l \psi^l(r) \right) \times e^{iqr + i\kappa nd}. \]

(20)

To solve Eq. (20) we introduce the functions
\[ D^0_l(r) = e^{i[\theta^0(r) + \psi^l(r)]} \]
\[ \delta^0_l(r) = \theta^0(r) + C^l(r), \]
where \( C^l(r) \) is defined as
\[ C^l(r) = \frac{2 \rho^l \times \nu^l(r)}{u^l(r)^2 + (\rho^l)^2}. \]

(23)

\[ e^{i\theta^0_l(r)} = \frac{2 \rho^l \times \nu^l(r)}{[(\nu^l(r)^2 + (\rho^l)^2)^2 - 4(\nu^l(r) \cdot \rho^l)^2]^{1/2}} \]

(24)

for both \( \nu^l \ll \rho^l \) and \( \nu^l \gg \rho^l \), the difference between imaginary part of \( D^0_l(r) \), \( \Im D^0_l(r) \), and \( \delta^0_l(r) \) is only on the \( \rho \) circle, so that
\[ \int d^2r \Im D^0_l(r) - \delta^0_l(r) \sim O(\rho^2). \]

(25a)
\[ \int d^2 \mathbf{r} \sum_{l \neq l'} \text{Im} \, D_l^r(\mathbf{r}) \text{Im} \, D_{l'}^r(\mathbf{r}) = \int d^2 \mathbf{r} \sum_{l \neq l'} \delta_l(\mathbf{r}) \delta_{l'}(\mathbf{r}) + O(p^3/a). \]  

(25b)

We show now that an expansion in \( \rho^2 \) is possible if the following expansion parameters are small

\[ \chi = \frac{2d}{\pi a^2} \int_{1/a}^{2d} \frac{d^2 q}{q^2} \int_{-\pi/a}^{\pi/a} dk \langle |\rho_n(q, \mathbf{k})|^2 \rangle \ll 1, \quad \text{if} \quad \lambda_J \gg a, \quad (26a) \]

\[ \langle \epsilon \rangle \equiv \langle\langle |\rho_n|^2\rangle/\lambda_J^2 \rangle \ll 1, \quad \text{if} \quad \lambda_J \ll a, \quad (26b) \]

where \( \langle \rho^2 \rangle \) is an average of \( \rho^2 \) which is diagonal in \( q, k \). The case of thermal average is evaluated in Sec. IV. The parameter \( \chi \) controls the expansion of the sine term in Eq. (20) and is evaluated in Appendix B, while \( \langle \epsilon \rangle = O(\rho^2) \) results from the solution of Eq. (20) which is to leading order in \( \rho \), so that the term linear in \( \epsilon \) in Eq. (19) survives and leads to higher-order corrections.

We claim then that the solution of Eq. (20) [compare with \( \theta^{\nu,0}(\mathbf{r}) \) from the naive expansion]

\[ \theta(\mathbf{q}, k) = \sum_l \theta_l(\mathbf{q}, k) = -\frac{\eta_k^2 \Sigma_j C_l(\mathbf{q}, k)}{q^2 + \eta_k^2}, \]

(27)

so that with Eq. (22)

\[ \delta_l(\mathbf{r}) = \frac{d}{\pi} \sum_n \int_{-\pi/a}^{\pi/a} dk \int_0^\infty dq \frac{q^2 \rho_n^2 K_n(q \rho_n^q)J_n(q v_n^q)}{q^2 + \eta_k^2} \times [\mathbf{v}_l^q \times \mathbf{P}_l^m], e^{ik(n-m)d}, \]

(28)

where \( \delta_l(\mathbf{r}) \) is the real part of \( D_l^r(\mathbf{r}) \). Substituting in Eq. (20) shows that Eq. (27) is indeed the solution for Eq. (20), i.e., it is the optimal \( \theta_l^{\nu,1} \). Furthermore we have from Eq. (B4), using Eq. (25b)

\[ \int d^2 \mathbf{r} \cos \left[ \theta^{\nu,1}(\mathbf{r}) + \sum_l \psi_l^r(\mathbf{r}) \right] - 1 = \int d^2 \mathbf{r} \sum_l \text{Re} \, D_l^r(\mathbf{r}) - \frac{1}{2} \sum_{l \neq l'} \text{Im} \, D_l^r(\mathbf{r}) \text{Im} \, D_{l'}^r(\mathbf{r}) \right] [1 + O(\chi)] \]

\[ = \int d^2 \mathbf{r} \sum_l \text{Re} \, D_l^r(\mathbf{r}) - \frac{1}{2} \sum_{l \neq l'} \delta_l(\mathbf{r}) \delta_{l'}(\mathbf{r}) \right] [1 + O(\chi)], \]

(30)

where \( \text{Re} \, D_l^r(\mathbf{r}) \) is the real part of \( D_l^r(\mathbf{r}) \). Substituting in Eq. (19) we obtain

\[ \mathcal{F} = \mathcal{F}_0 + \frac{1}{2} \sum_{l, m} \int \frac{d^2 q dk}{(2\pi)^2} \left[ G_{l}^{-1}(q,k) + E_{l}/d \right] \epsilon(q,k)^2 + \frac{E_{l}}{2d} \int \frac{d^2 q dk}{(2\pi)^2} \frac{|C(q,k)|^2}{q^2 + \eta_k^2} \]

\[- E_{l}/2 \sum_{l, m} \int d^2 \mathbf{r} \left[ \cos \left[ \theta^{\nu,1}(\mathbf{r}) + \psi_l^r(\mathbf{r}) \right] + \frac{1}{2} (\delta_l(\mathbf{r}))^2 - 1 \right] \left[ 1 + O(\epsilon, \epsilon^2, \Delta) \right] \],

(31)

where \( C(q,k) = \Sigma_j C_l(q,k) \).

The balance between the first \( |\epsilon(q,k)|^2 \) term in Eq. (31) and the \( O(\epsilon) \) term leads to \( \langle \epsilon \rangle \sim \langle \rho^2 \rangle \). The \( O(\rho^2) \) term depends on the distribution of \( \epsilon(q,k) \); for thermal average it has a comparable value (Sec. IV).

We have identified two types of expansion parameters. The first one, \( \chi \), is related to the convergence of the \( l \) summation of singular vortex phases while the second one, \( \epsilon \), is related to the response of the nonsingular phase. For weak Josephson coupling \( \lambda_J \gg a \) we find \( \chi \gg \langle \epsilon \rangle \) so that the expansion parameter is \( \chi \), while for \( \lambda_J \ll a \) we find from Eq. (B6) \( \chi \ll \langle \epsilon \rangle \) so that the expansion parameter is \( \langle \epsilon \rangle \).

Consider now the function \( \delta^*(\mathbf{r}) = \Sigma_l \delta_l(\mathbf{r}) \). Since \( q \approx 1/p^m_\nu \) (due to \( K_1 \) function) for \( p^m_\nu \ll \lambda_J \), the dominant integral range with \( q \approx 1/p \) has \( q^2 + \eta_k^2 \approx q^2 \). Hence \( \delta^*(\mathbf{r}) - C^*(\mathbf{r}) = O(\rho/\lambda_J^2) \) and the last term in Eq. (31) for \( \rho \ll \lambda_J \) can be replaced by
\[ \int d^2 r \left( \cos \psi_1^n(r) - \frac{1}{2} C_1^n(r)^2 \right) = - \pi \ln[4 e] (\rho_1^n)^2. \]  

(32)

It is straightforward to see that the integral is convergent and therefore must be proportional to \((\rho_1^n)^2\); the coefficient can be found after some algebra.

The contribution to the second term in Eq. (31) from different flux lines can be written in the form

\[
\sum_{l \neq l'} \int_{-\pi/d}^{\pi/d} \int_{-\pi/d}^{\pi/d} d^2 qdk \left( q^2 C_l(q,k) C_{l'}(q,k) \right) \frac{q^2 + \eta_q^2}{q^2 + \eta_q^2} = 2d^2 \sum_{n, n'} \sum_{l \neq l'} \rho_1^n \cdot \rho_1^{n'} \int_{-\pi/d}^{\pi/d} dk K_0(\eta_q | R_l^n - R_{l'}^{n'}|) e^{i k (n-n')} d
\]

\[
= 4\pi d \sum_n (\rho_1^n)^2 \ln \left( \frac{\lambda_j}{\rho_1^n} \right) + 4\pi d \sum_{n \neq n'} \rho_1^n \cdot \rho_1^{n'} e^{-|n-n'| |\lambda_j/\lambda_j| |n-n'|} + O(\rho_1^4). \tag{34}
\]

The last line is obtained by introducing \(x = q \sqrt{\rho_1^n} \rho_1^{n'}\) and writing the integral (34) as

\[
\int_{-\pi/d}^{\pi/d} \int_{-\pi/d}^{\pi/d} dx \frac{x}{x^2 + \eta_q^2 \rho_1^n \rho_1^{n'}} e^{i k (n-n')} + \int_{-\pi/d}^{\pi/d} dx \frac{x^2 \left[ K_1(x \sqrt{\rho_1^n} \rho_1^{n'}) K_1(x \sqrt{\rho_1^n} \rho_1^{n'}) - 1/x^2 \right]}{x^2 + \eta_q^2 \rho_1^n \rho_1^{n'}} e^{i k (n-n')} d \]

\[
= \int_{-\pi/d}^{\pi/d} dx \frac{x^3 K_1(x \sqrt{\rho_1^n} \rho_1^{n'}) K_1(x \sqrt{\rho_1^n} \rho_1^{n'}) - 1/x^2}{x^2 + \eta_q^2 \rho_1^n \rho_1^{n'}} e^{i k (n-n')} d.
\]

In the last two terms one can put all \(\rho_1^n \to 0\) since both integrals converge. After separating the \(n = n'\) and \(n \neq n'\) terms and integrating over \(k\) the result Eq. (34) is obtained. In Appendix C we consider displacement of a single pancake vortex in one flux line and demonstrate the agreement between a numerical exact evaluation of Eq. (31) and the analytic expansion (see Fig. 2). We also show in Appendix C that for the single pancake displacement an expansion in \(\rho\) is possible also for \(\lambda_j < \rho < a\).

We have shown in Eq. (34) that in general there are \((\rho_1^n)^2 \ln \rho_1^n\) terms in the energy expansion, confirming the anticipation by GK. Thus, strictly speaking the elastic constants are ill defined. However, \(\ln \rho_1^n\) is a slowly varying function so that replacing it by an average value \(\ln \rho\) should yield the main nonlinear correction. At finite temperatures \(\rho = (\rho^2)^{1/2}\) would be a thermal average. This procedure is tested for the single pancake displacement (Appendix C) and is found to be in a good agreement with the exact thermal average.

D. Effective elastic constants

Effective elastic constants are obtained by replacing anharmonic terms in \(\rho_1^n\) by an average value \(\rho\), which is to be determined self-consistently, e.g., by a thermal average \(\rho = (\rho^2)^{1/2}\). We introduce the effective singular phase difference \(\psi_{1,\text{eff}}(r)\) which leads to effective \(C_{1,\text{eff}}(r)\) functions

\[
\sin \psi_{1,\text{eff}}(r) = \frac{2[\rho_1^n \times \hat{v}_1^n(r)]_z}{[(v_1^n(r)^2 + \rho^2) - 4(v_1^n(r) \cdot \rho_1^n)^2 \rho^2]^{1/2}},
\]

\[
\sin \psi_{1,\text{eff}}(r) = \frac{2[v_1^n(r) \times \rho_1^n]}{v_1^n(r)^2 + \rho^2},
\]

\[
\sin \psi_{1,\text{eff}}(r) = \frac{4\pi i d \hat{z} \cdot \hat{q}}{q} \rho K_1(\rho q).
\]

(35)
This approximation simplifies significantly all computations of averages because the energy can be written in the harmonic form

\[
\mathcal{F} = \mathcal{F}_v + \frac{1}{2} \int \frac{d^2qdk}{(2\pi)^3} (G_f^{-1}(q,k) + E_J/d) |\theta(q,k) - \theta^{\text{eff}}(q,k)|^2 \\
+ \frac{1}{2}(4\pi)^2E_Jd \int_0^{\pi/d} d^2qdk(2\pi)^3 \frac{[\hat{z} \times \hat{q}] \cdot \rho(q,k)]^2 q^2p^2K_1^2(qp)}{q^2 + \eta_k^2} \\
+ \frac{\pi E_J}{d} \int_0^{\pi/d} d^2qdk \int_{-\pi/d}^{\pi/d} d^2qdk (2\pi)^3 d\alpha^2 \ln(4e)|\rho(q,k)|^2,
\]

(36)

where we define

\[
\theta^{\text{eff}}(q,k) = -\frac{\eta_k^2 C^{\text{eff}}(q,k)}{q^2 + \eta_k^2}.
\]

To derive Eq. (36) the proper expansion is used with the \(\sin \theta^{\text{eff}}(r)\) term. The result is similar to replacing \(\rho\) by \(\tilde{\rho}\) into the coefficients of anharmonic terms in Eq. (31).

Using Eqs. (9), (17) the second term in Eq. (36) can be written as

\[
\int_0^{\pi/d} d^2qdk \int_{-\pi/d}^{\pi/d} d^2qdk \frac{[\hat{z} \times \hat{q}] \cdot \rho(q,k)]^2 q^2p^2K_1^2(qp)}{q^2 + \eta_k^2} \\
+ \frac{1}{2} \frac{1}{Q_0^2} \int_0^{\pi/d} d^2qdk \int_{-\pi/d}^{\pi/d} d^2qdk \int_{-\pi/d}^{\pi/d} d^2qdk \frac{x^2K_1^2(x)dx}{q^2 + \eta_k^2 \tilde{\rho}^2},
\]

(37)

where the last integral has the analytic fit

\[
\int_0^{\pi/d} d^2qdk \int_{-\pi/d}^{\pi/d} d^2qdk \frac{x^2K_1^2(x)dx}{x^2 + \eta_k^2 \tilde{\rho}^2} = -\frac{1}{2} \ln \left(\frac{\tilde{\rho}^2 \eta_k^2 (1 + Q_0^2 \eta_k^{-2})}{\rho^2 \eta_k^2 + 1}\right).
\]

The effective free energy of the vortex lattice can now be written in the harmonic form with effective transverse, \(c_{44}^{\text{eff}}(q,k)\), and longitudinal, \(c_{44}^{\text{eff}}(q,k)\), tilt moduli

\[
c_{44}^{\text{eff}}(q,k) \equiv \frac{1}{4\pi^2} \int_0^{\pi/d} d^2qdk \int_{-\pi/d}^{\pi/d} d^2qdk \frac{\sin^2(kd/2)}{q^2c_{11}^c + k^2c_{44}^c} + \frac{\sin^2(kd/2)}{q^2c_{66}^c + k^2c_{44}^c},
\]

(39)

For weak magnetic fields where \(\lambda_\parallel \ll a\) the Josephson contribution to \(c_{44}^{\text{eff}}\) and \(c_{44}^{\text{eff}}\) is dominant and the latter dominate the integrals, leading to

\[
\bar{\rho}^2 \approx 4 \frac{T}{\lambda^2} \ln \left(\frac{e\tau \ln(4e)}{T}\right) \left[1 + O\left(\frac{\lambda_j^2}{a^2} \frac{a^2}{\lambda_{ab}^2}\right)\right].
\]

(40)

where \(\tau = \Phi_0^2d/(4\pi^2\lambda_{ab}^2)\) and \(T \ll \tau\) is assumed. The expansion parameter for \(\lambda_\parallel \ll a\) [Eq. (26b)] together with Eq. (40) yields, in fact, the expansion condition \(T \ll \tau\). We also find by numerical integration that the displacement average is \(\bar{\rho}^2 \approx \rho^2\).

We note that \(\bar{\rho}^2\) is nonlinear in \(T\) due to the \(\ln T\) factor in Eq. (40). Thus data on \(\bar{\rho}^2\), e.g., by a Debye-Waller term in neutron scattering may probe the \(\ln T\) factor in weak fields, i.e., \(\lambda_\parallel \ll a\).
For strong magnetic fields where $\lambda_{ab} \gg \lambda_J \gg a$ the Josephson contribution to the last terms in Eq. (38) can be ignored, i.e., $c_{44}' \approx c_{44} \approx c_{44}^{ab}$, leading to

$$
\tilde{\rho}^2 = \frac{T}{\tau a^2} \ln \frac{9 \pi \lambda^2}{32 a^2} \left[ 1 + O \left( \frac{a^2}{\lambda_J^2 \lambda_{ab}^2} \right) \right],
$$

where $k = \pi/d$ dominates the integral; here also $\tilde{\rho}^2 \approx \tilde{\rho}^2$. In this case the thermal average of $\tilde{\rho}^2$ has the usual linear temperature dependence

We proceed to evaluate the expansion parameters Eqs. (26) which determine the validity range of our expansion. For $\lambda_J \gg a$ we need $\chi$ of Eq. (26a),

$$
\chi = T \frac{2 \pi^2 d^2}{a^4} \int_{\lambda_J}^{\pi d} \int_{-\pi d}^{\pi d} \frac{d^2 q k}{(2 \pi)^3} \frac{k^2}{q^4 \{c_{66} q^2 + c_{44}''(q) k^2\}},
$$

while for $\lambda_J \ll a$ we evaluate $\langle \epsilon \rangle$ directly with Eq. (40). We note that the thermal average yields $\langle \epsilon^2 \rangle = (2T/\pi) \ln \lambda_J/\xi$ so that $\langle \epsilon^2 \rangle \approx \langle \epsilon \rangle$.

We find then that the effective harmonic expansion is valid at temperatures below $T^0$,

$$
T^0 = \frac{1}{2} T_d \ln (2 \lambda_J/\lambda_{ab}), \quad a, \lambda_{ab} \ll \lambda_J,
$$

$$
T^0 = \frac{1}{2} \tau \lambda J (\pi \lambda_J), \quad a \ll \lambda_J \ll \lambda_{ab},
$$

$$
T^0 = \frac{1}{2} \tau (\ln (\lambda_J/\xi), \quad \lambda_J \ll a.
$$

Here we defined$^{4,5}$ the decoupling temperature $T_d = \pi a^2 \ln (a d)/(4 \pi \lambda_{ab}^2)$ for the range $a, \lambda_{ab} \ll \lambda_J$. We note that the form of $\chi$ Eq. (26a) involves precisely the fluctuations that lead to the decoupling transition.$^{4,5}$ It is therefore expected that $T^0$ is related to $T_d$ for the case when $\chi$ is the relevant expansion parameter, i.e., $a \ll \lambda_J$. For both cases of Eqs. (43a), (43b) the decoupling temperature is indeed close to $T^0$ (even for $a \ll \lambda_J \ll \lambda_{ab}$ where the decoupling transition becomes first order$^9$). For $a \gg \lambda_J$ we expect that fluctuations of the non-singular part of the Josephson phase $\langle \epsilon(q, k) \rangle$ dominate so that the low temperature instability involves melting rather than decoupling.

V. CONCLUSION

We present in this work a proper expansion for defining elastic constants. Both deficiencies of the naive expansion, when corrected, lead to interesting physical consequences. The first difficulty is that a simple expansion at short scales, the $\rho$ circle in Fig. 1, is not possible. The proper expansion shows an anharmonic $\tilde{\rho}^2 \ln \rho$ term so that, strictly speaking, $c_{44}$ is ill defined for displacements $\rho > \xi$ [for $\rho < \xi$ the Josephson coupling $E_J$ should be modified by the reduced order parameter in the vortex core, an effect which is neglected in the Lawrence-Doniach model, Eq. (1)].

We find that effective elastic constants can be defined by replacing $\ln \rho$ by $\ln \tilde{\rho}$, where $\tilde{\rho}$ is thermal average $\tilde{\rho} = (\rho^2)^{1/2}$. This leads to replacing $\xi$ of the naive expansion by $\tilde{\rho}/4e$ in the effective $c_{44}$, Eqs. (38). Since $\langle \rho^2 \rangle \sim T$ this effect can show up as a ln $T$ factor in a direct measurement of $c_{44}$. Furthermore, when $\lambda_J \ll a$ the Josephson contribution dominates in the tilt moduli and $\langle \rho^2 \rangle \sim T/\ln(\pi T)$. This temperature dependence may be observable via a Debye-Waller factor in neutron scattering.

The second deficiency of the naive expansion is that a $E_J$ independent term is generated from the Josephson term when $q \rightarrow 0$. This difficulty relates to the expansion parameter $\chi$ of Appendix B—the summation on flux lines converges only beyond a scale $\sim \lambda_J$ so that $\chi \sim \tilde{\rho}^2 \ln \lambda_J$. When $E_J \rightarrow 0$ the range where the $\rho$ expansion is valid, $\chi \ll 1$, vanishes as $1/\ln E_J$. Thus at $q \rightarrow 0$ a long-range effect of many flux lines invalidates the $\rho$ expansion. In practice one needs $\lambda_J \gg \lambda_{ab}$ for this effect to be noticeable, and the harmonic expansion is then limited to $T < \frac{3}{4} T_d/\ln(2 \lambda_J/\lambda_{ab})$.

The $q \rightarrow 0$ difficulty is in fact resolved by either a self-consistent harmonic approximation$^4$ or by a renormalization-group method.$^5$ In both cases the cosine function is not expanded although $\psi^J(r) = -2 \nabla a(r - R^J) \cdot \rho^J$ leading to a decoupling temperature $T_d$. For $T < T_d$ $E_J$ is renormalized to a finite value $J_R \rightarrow T(E_J, T)^{1/(1 - T/T_d)}$ which can be expanded when $T < T_d/\ln(T/\xi E_J)$, equivalent to our expansion parameter. For $T > T_d$ the renormalized $E_J$ vanishes and the $\rho$ expansion is clearly invalid.

The expansion parameter is related to $T_d$ only for $\lambda_J \gg a$, while for $\lambda_J \ll a$ the expansion is valid for $T < \frac{1}{2} \pi \ln(\lambda_J/\xi)$. We expect that in the latter case Josephson fluxons with width $\lambda_J$ can form loops in between layers and lead to melting of the flux lattice. Thus for $\lambda_J \ll a$ the dominant instability is melting, while for $\lambda_J \gg a$ it is decoupling; in the latter case the lattice at $T > T_d$ (held by magnetic coupling) melts at a higher temperature.

In recent experiments on Bi-Sr-Ca-Cu-O (Ref. 12) the phase diagram has shown a number of low-temperature phases (related to disorder$^3$), while at $T > 40 \text{ K}$, where thermal fluctuations dominate, the transition to a vortex-liquid phase is of two types: (i) At $B < 500 \text{ G}$ a first-order transition with no further transitions at higher temperatures, and (ii) at $500 \text{ G} < B < 900 \text{ G}$ a first order transition followed by another transition where surface barriers are reduced. For Bi-Sr-Ca-Cu-O $\lambda_J$ is estimated$^3$ $800 - 2400 \text{ Å}$, while at $B = 500 \text{ G}$, $a = 2000 \text{ Å}$. It is consistent to consider that the $B < 500 \text{ G}$ transition as melting ($\lambda_J < a$), while at $B > 500 \text{ G}$ decoupling dominates ($\lambda_J > a$) with melting at a higher temperature.

ACKNOWLEDGMENTS

This research was supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities.

APPENDIX A: TUTORIAL EXAMPLE

Let consider two superconducting layers with Josephson coupling between them and only one pancake vortex on each layer. The Lawrence-Doniach free energy in the simple case of $e \rightarrow 0$ has the form:

$$
\mathcal{E} = \int d^2r \sum_{n=1,2} \left[ \nabla \phi_n(r)^2 - \lambda_n^2 \right] \times [\cos(\phi_2(r) - \phi_1(r) - 1)].
$$

(A1)
We now decompose the superconducting phase $\phi^p(r)$ to the nonsingular $\phi^0_n(r)$ and singular part:

$$\phi_n(r) = \phi^0_n(r) + \alpha(r - R_n),$$

where $\alpha(r) = \arctan(y/x)$ and $R_n$ is the vortex position on the $n$th layer. Define new $\theta(r) = [\phi^0_n(r) - \phi^0_{n+1}(r)]$ and $\psi(r) = \alpha(r - R_2) - \alpha(r - R_1)$ to write the free energy in the form:

$$E = \int d^2r \left( \frac{1}{2} [\nabla \theta(r)]^2 - \lambda J^2 [\cos(\theta(r) + \psi(r)) - 1] \right) + E_0,$$

where the $E_0$ part contains magnetic interaction between vortices (we do not consider it here) and $\frac{1}{2} \int d^2r [\nabla (\phi^0_n(r) + \phi^0_{n+1}(r))]^2$ part which can be integrated out.

We can perform an expansion of the cosine term with respect to the nonsingular phase $\theta(r)$ since it can be shown self-consistently that $|\theta(r)|^2 - \lambda J^2 \rho^2 \ll 1$ for small relative displacements of vortices:

$$\rho = [R_2 - R_1]/2.$$

So we can write:

$$E = \frac{1}{2} \int d^2r \left[ (q^2 + \lambda J^2) \theta(r) + \frac{C(q)}{1 + q^2 \lambda J^2} \right] - \lambda J^2 \int d^2r \left[ \cos(\theta(r) + \frac{1}{2} (C(r))^2 - 1) \right] + O(\theta^2 \rho^2, \theta^4),$$

where

$$\sin \psi(r) = \frac{2[\rho \times v]_z}{((\rho^2 + v^2)^{1/2} - 4 \rho \cdot v}.$$

$v = r - (R_1 + R_2)/2$.

After shifting the square of $\theta$ and substituting $\sin \psi(r)$ by $C(r) = 2[\rho \times v]_z/(\rho^2 + v^2)$ we obtain

$$E = \frac{1}{2} \int d^2q \left( q^2 + \lambda J^2 \right) \left[ \frac{C(q)}{1 + q^2 \lambda J^2} \right] - \lambda J^2 \int d^2r \left[ \cos \psi(r) + \frac{1}{2} (C(r))^2 - 1 \right] + O(\theta^2 \rho^2, E_0).$$

(A3)

where the $\theta [\sin \psi - C]$ term contributes a $O(\theta^2 \rho^2)$ correction to the energy.

It can be calculated analytically that

$$\int d^2r \left[ \cos \psi(r) + \frac{1}{2} (C(r))^2 - 1 \right] = - \pi \ln[4e \rho^2],$$

(A4)

$$\int \frac{d^2q}{(2\pi)^2} \frac{q^2 \frac{1}{2} (C(q))^2}{1 + q^2 \lambda J^2} = - \frac{1}{8 \pi} \rho^2 \ln \frac{\lambda J^2 \rho^2}{1 + \lambda J^2 \rho^2} + O(\lambda J^2 \rho^2).$$

(A5)

This shows the presence of the anharmonic term $\rho^2 \ln \rho$ in the energy expansion for two superconducting layers. We obtain in Sec. III C this anharmonicity in the more general case of a vortex lattice in a layered superconductor.

**APPENDIX B: PARAMETER OF EXPANSION**

Let us introduce a 2D lattice with $l$ the unit-cell index and use definitions Eq. (13c) and Eq. (21) for $v_l^a$ and $D_l^a(r)$.

Consider $r$ in the $l^*$ unit cell with $r = R_{l*} + v_{l*}^a$, so that $|v_{l*}^a| < \alpha$. Since $d_{l*}^a(r) = d_{l*}^a(r^*)$ depends on $r$ only through $v_r^a = r - R_l$, [see Eq. (28)], we can write

$$\chi_r^a = \sum_{l \neq l^*} \text{Im} \ D_l^a(r) = \sum_{l \neq l^*} \overline{d}_{l*}^a(v_r^a) + \sum_{l \neq l^*} \left[ \text{Im} \ D_l^a(r) - d_{l*}^a(r) \right]$$

$$= - \frac{d}{\pi} \sum_{l \neq l^*} \int_{-\pi/a}^{\pi/a} dk \eta_k K(\eta_k | R_l - R_{l^*}) e^{iknd} \left[ \frac{v_r^a \times \rho(k)}{|R_l - R_{l^*}|} \right] \times \left( 1 + O\left( \frac{|v_r^a|}{|R_l - R_{l^*}|} \right) \right) + O\left( \rho^2 \right),$$

$$\chi_r^a = \sum_{l \neq l^*} \text{Re} \ D_l^a = \frac{1}{2} \sum_{l \neq l^*} \left[ \overline{d}_{l*}^a(v_r^a) \right]^2 + O\left( \rho^2 \right)$$

$$= \frac{d^2}{2 \pi^2} \sum_{l \neq l^*} \left[ \int_{-\pi/a}^{\pi/a} dk \eta_k K(\eta_k | R_l - R_{l^*}) \left[ \frac{(R_l - R_{l^*}) \times \rho(k)}{|R_l - R_{l^*}|} \right] e^{iknd} \right]^2 \times \left( 1 + O\left( \frac{|v_r^a|}{|R_l - R_{l^*}|} \right) \right) + O\left( \rho^2 \right).$$

(B1)

Using the expansion
we obtain for the right-hand side of Eq. (20)

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr} = \sum_{l} \int d^2v_{ls} \left[ \sum_{l} D_l \right] e^{iqr}
\]

which for \( \lambda_j \ll a \) is used and \( \int d^a \) means integration over the unit cell.

For the real part of Eq. (B2) we obtain

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr}
\]

APPENDIX C: ONE DISPLACED VORTEX POINT
IN ONE VORTEX LINE

In the case of one displaced pancake vortex on the layer \( n=0 \), in one vortex line, \( l=0 \), so only \( \rho_0 = \rho \) are exist, we use Eq. (27) in Eq. (31) to evaluate the energy numerically. The result for the energy \( \mathcal{F}_1 \) (without magnetic part \( \mathcal{F}_2 \)) is shown by the dots in Fig. 2.

The numerical result can be fitted as:

\[
\mathcal{F}_1 = 2\pi E_J \left[ a^2 \right] \left[ 1.6\rho^2 - 1.04\rho^2 \ln \left( \frac{2\rho^2/(\lambda_j^2)}{2\rho^2/(\lambda_j^2) + 1} \right) \right].
\]  

we obtain for the right-hand side of Eq. (20)

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr} = \sum_{l} \int d^2v_{ls} \left[ \sum_{l} D_l \right] e^{iqr}
\]

which for \( \lambda_j \ll a \) is used and \( \int d^a \) means integration over the unit cell.

For the real part of Eq. (B2) we obtain

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr}
\]

Note, that the expansion parameter depends on the configuration \( \rho_0 \). We consider averages of \( [\chi^\alpha_l]^2 \) and \( \chi^\alpha_l \) which are diagonal in \( \rho(q,k) \), e.g., as in thermal average. We define the expansion parameter \( \chi = \langle [\chi^\alpha_l]^2 \rangle \) and obtain the form

\[
\chi = \frac{2\pi a^2}{\pi d} \int_{BZ} \frac{d^2q}{a^2} \int_{-\pi d}^{\pi d} dB \langle [\rho^\alpha(q,k)]^2 \rangle
\]

which for \( \lambda_j \ll a \) reduces to Eq. (26a). For \( \lambda_j \gg a \) this is the relevant expansion parameter since from Eq. (26b) \( \chi \ll \epsilon \); furthermore, the other expansion parameter

\[
\chi = \frac{2\pi}{\pi d} \int_{BZ} \frac{d^2q}{a^2} \int_{-\pi d}^{\pi d} dB \langle [\rho^\alpha(q,k)]^2 \rangle
\]

which for \( \lambda_j \ll a \) is used and \( \int d^a \) means integration over the unit cell.

For the real part of Eq. (B2) we obtain

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr}
\]

Note, that the expansion parameter depends on the configuration \( \rho_0 \). We consider averages of \( [\chi^\alpha_l]^2 \) and \( \chi^\alpha_l \) which are diagonal in \( \rho(q,k) \), e.g., as in thermal average. We define the expansion parameter \( \chi = \langle [\chi^\alpha_l]^2 \rangle \) and obtain the form

\[
\chi = \frac{2\pi a^2}{\pi d} \int_{BZ} \frac{d^2q}{a^2} \int_{-\pi d}^{\pi d} dB \langle [\rho^\alpha(q,k)]^2 \rangle
\]

which for \( \lambda_j \ll a \) reduces to Eq. (26a). For \( \lambda_j \gg a \) this is the relevant expansion parameter since from Eq. (26b) \( \chi \ll \epsilon \); furthermore, the other expansion parameter

\[
\chi = \frac{2\pi}{\pi d} \int_{BZ} \frac{d^2q}{a^2} \int_{-\pi d}^{\pi d} dB \langle [\rho^\alpha(q,k)]^2 \rangle
\]

which for \( \lambda_j \ll a \) is used and \( \int d^a \) means integration over the unit cell.

For the real part of Eq. (B2) we obtain

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr}
\]

Note, that the expansion parameter depends on the configuration \( \rho_0 \). We consider averages of \( [\chi^\alpha_l]^2 \) and \( \chi^\alpha_l \) which are diagonal in \( \rho(q,k) \), e.g., as in thermal average. We define the expansion parameter \( \chi = \langle [\chi^\alpha_l]^2 \rangle \) and obtain the form

\[
\chi = \frac{2\pi a^2}{\pi d} \int_{BZ} \frac{d^2q}{a^2} \int_{-\pi d}^{\pi d} dB \langle [\rho^\alpha(q,k)]^2 \rangle
\]

which for \( \lambda_j \ll a \) reduces to Eq. (26a). For \( \lambda_j \gg a \) this is the relevant expansion parameter since from Eq. (26b) \( \chi \ll \epsilon \); furthermore, the other expansion parameter

\[
\chi = \frac{2\pi}{\pi d} \int_{BZ} \frac{d^2q}{a^2} \int_{-\pi d}^{\pi d} dB \langle [\rho^\alpha(q,k)]^2 \rangle
\]

which for \( \lambda_j \ll a \) is used and \( \int d^a \) means integration over the unit cell.

For the real part of Eq. (B2) we obtain

\[
\int d^2r \left[ \sum_{l} D_l \right] e^{iqr}
\]
2 For a review see G. Blatter et al., Rev. Mod. Phys. 66, 1125 (1995).