Euclidean Special Relativity*

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Abstract

New four coordinates are introduced which are related to the usual space-time coordinates. For these coordinates, the Euclidean four dimensional length squared is equal to the interval squared of the Minkowski space. The Lorentz transformation, for the new coordinates, becomes an SO(4) rotation. New scalars (invariants) are derived.

A second approach to the Lorentz transformation is presented. A mixed space is generated by interchanging the notion of time and proper time in inertial frames. Within this approach the Lorentz transformation is a 4 dimentional rotation in an Euclidean space, leading to new possibilities and applications.

Keywords: Special relativity, Euclidean 4-space-time, mixed space, Lorentz transformation.

I. INTRODUCTION

One of the main problems of special relativity is whether there exists a physical four dimensional space-time or are space and time different entities for which Minkowski space

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is a convenient coordinate system. In this paper an alternative coordinate system will be derived.

Let us introduce the following notation

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^5 = c\tau, \tag{1}$$

where c is the light velocity, t is the time, x, y, z are the space coordinates and τ is the proper time. Special relativity is based on the invariance of the interval squared

$$\left(\Delta x^5\right)^2 = \left(\Delta x^0\right)^2 - \left(\Delta x^1\right)^2 - \left(\Delta x^2\right)^2 - \left(\Delta x^3\right)^2, \tag{2}$$

with respect to Lorentz transformations. Minkowski has proposed to look at the coordinates x^0, x^1, x^2, x^3 as the four dimensional space time, where the interval, Eq.(2), was serving as the distance. This approach has led to the revolutionary successes of Einstein's relativity. However there are many problems related to Minkowski's space-time. The metric is non-Euclidean and when many particles are concerned there is a problem of defining a common time.

Let us mention the possibility that a mathematically defined space is not necessarily equivalent to a physical space. For example, we may find for some problems toroidal coordinates to be convenient. This does not mean that the world is a torus.

Let us pose the following fundamental question, what are the necessary conditions for a space to be physical? As physics is mostly concerned with measurement I would suggest the following condition:

Definition 1 *A physical space is such that one can repeat a measurement in it.*

Otherwise we are dealing with an abstract space. Abstract space has the meaning of convenient coordinates. One can not repeat measurements for the same space-time coordinates, as time flows constantly forward during experiments. Therefore, according to the above definition, space-time is not a physical space but only an abstract space in which convenient coordinates can be defined. Time can not be included into physical spaces. Is this the reason why time in quantum mechanics is a parameter and not an observable?

Lorentz transformations are conveniently described in Minkowski's space with the length defined by Eq.(2). A more meaningful name would be the Minkowski coordinates. One may also consider the Euclidean space (Euclidean coordinates) described by the coordinates x^0, x^1, x^2, x^3 with the distance squared

$$(dl)^2 = \left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2, \tag{3}$$

in which events transform according to the Lorentz transformation, even though this description will lack the elegance of the tensor calculus that characterizes the Minkowski space.

We shall suggest in Sec. 2 another four dimensional space (or rather coordinates) for which the invariant interval will be Euclidean. But before that, let us first rewrite Eq.(2) in the following way

$$-\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^5\right)^2 = 0. \tag{4}$$

Now one may appreciate the notation used in Eq.(1) and consider Eq.(4) as defining a 5-dimensional hyperplane in the 5-dimensional space with coordinates x^0, x^1, x^2, x^3, x^5. Eq.(4) is more general then Eq.(2). Eq.(2) defines a subspace in the 5-dimensional hyperspace for which the projection in the x^5 coordinate is constant. Therefore there are events in the 5-dimensional hyperspace which are beyond special relativity.

If in the 5-dimensional space we restrict ourselves to a constant projection in the x^0 coordinate, the resulting subspace of the x^1, x^2, x^3, x^5 coordinates can be Euclidean

$$\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^5\right)^2 = \left(dx^0\right)^2. \tag{5}$$

Montanus [1], [2] has suggested to look at the 4 coordinates x^1, x^2, x^3, x^5 as forming the four dimensional space-time, and suggested to look at the left hand side of Eq.(5) as the Euclidean distance squared in this space. It is consistent with the Lorentz transformation only when Eq.(5) is exactly satisfied. There exists an invariant in this space, namely the square of the four-velocity

$$v^\mu = \frac{dx^\mu}{dt} = \frac{c dx^\mu}{dx^0} = v_\mu, \quad (\mu = 1, 2, 3, 5), \tag{6}$$

which is according to Eq.(4)

$$v_\mu v^\mu = \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2 + \left(\frac{dx^5}{dt}\right)^2 = c^2. \tag{7}$$

It is interesting to note that the 4-velocity addition for v^μ is linear, but it happens in 4-space dimensions (it is non-linear in Minkowski's subspace of 3 space dimensions). The fourth space dimension corrected the seemingly impossible situation in 3 dimensions.

Montanus [3] has suggested to look at v^μ of Eq.(6) as a $SO(4)$ four vector with respect to rotations in the x^1, x^2, x^3, x^5 4-space. Differentiating Eq.(7) one may find

$$a_\mu v^\mu = 0, \quad a^\mu = \frac{dv^\mu}{dt}, \quad a^\mu = a_\mu, \tag{8}$$

therefore also a^μ can be considered as a $SO(4)$ four vector with respect to rotations in the x^1, x^2, x^3, x^5 4-space, always orthogonal to v^μ. In this way Montanus was successful to develop a $SO(4)$ tensor calculus for the x^1, x^2, x^3, x^5 4-space. It is not completely consistent with Lorentz invariance, as the x^5 coordinate may change, in contrary to the requirements of the Lorentz invariance. $SO(4)$ rotations will leave in Eq.(4) the time interval invariant. On the other hand Eq.(7) (contrary to Eq.(5), which acted on a 5-dimensional hyperplane) acts in 4-dimensional space and is consistent with Lorentz invariance, in the sense that the absolute value of the 4-velocity is the same in each inertial frame and equal to the velocity of light.

The $SO(4)$ tensor calculus which is based on the invariance of time interval in Eq.(5) may be applied to Feynman path integrals in which trajectories are defined for a fixed time interval.

It is interesting to note that Eq.(7) is a restriction in the x^1, x^2, x^3, x^5 4-space, i.e. it defines a subspace in which Lorentz transformations may occur for one particle states.

In the next section we shall find Euclidean four coordinates for which $SO(4)$ rotations will be consistent with the Lorentz transformations.

II. EUCLIDEAN METRIC

Let us introduce the q^λ coordinates defined by

$$dq^\lambda = \frac{dx^5}{dx^0}dx^\lambda = \frac{d\tau}{dt}dx^\lambda, \quad \lambda = 0,1,2,3,5, \tag{9}$$

where the x^λ $(\lambda = 0,1,2,3,5)$, were defined in Eq.(1). Multiplying Eq.(4) by $\left(dx^5/dx^0\right)^2$ we obtain

$$\left(dx^5\right)^2 = \left(dq^0\right)^2 = \left(dq^1\right)^2 + \left(dq^2\right)^2 + \left(dq^3\right)^2 + \left(dq^5\right)^2. \tag{10}$$

Now the Lorentz transformation can be defined as a rotation in the four dimensional space q^1, q^2, q^3, q^5 and can be described with the help of the group of proper rotations, the $SO(4)$ group.

We shall consider the action (being a scalar), in order to show that such a procedure is feasible.

A. The action

Let us start from the relation between the Lagrangian L and the Hamiltonian (energy) H

$$L = \mathbf{p}\cdot\frac{d\mathbf{x}}{dt} - H. \tag{11}$$

Remembering that the action S is related to the Lagrangian via

$$\frac{dS}{dt} = L, \tag{12}$$

we obtain

$$dS = \mathbf{p}\cdot d\mathbf{x} - \frac{H}{c}d\left(ct\right). \tag{13}$$

If $\mathbf{p}$ is the relativistic momentum, dS is a scalar under Lorentz transformation. It is obtained as the scalar product of two four vectors in Minkowski's space, which will be denoted as

$$p^\mu \equiv \left(p^0, \mathbf{p}\right), \quad x^\mu \equiv \left(x^0, \mathbf{x}\right), \quad p^0 = \frac{H}{c}, \quad x^0 = ct \tag{14}$$

and we shall distinguish them as Minkowski space four vectors.

Let us now consider Eq.(4) and use it for a free particle of rest mass m moving with a velocity $\mathbf{v}$. Let us divide both sides of Eq.(2) by (the Lorentz invariant) dx^5 and multiply both sides by mc, one obtains

$$mcdx^5 = mc\frac{dx^0}{dx^5}dx^0 - mc\frac{d\mathbf{x}}{dx^5}\cdot d\mathbf{x} = Edt - \mathbf{p}\cdot d\mathbf{x} = -dS, \tag{15}$$

where E and $\mathbf{p}$ are the free particle total energy and momentum respectively given by the well known relations:

$$E = \frac{mc^2}{\gamma}, \quad \mathbf{p} = \frac{md\mathbf{x}}{\gamma dt} = \frac{m\mathbf{v}}{\gamma}, \quad \gamma = \frac{dx^5}{dx^0} = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}. \tag{16}$$

We shall bring Eq.(15) to a Euclidean form. First we rewrite Eq.(15) as

$$mc\frac{dx^0}{dx^5}dx^0 = mc\frac{d\mathbf{x}}{dx^5}\cdot d\mathbf{x} + mcdx^5 \tag{17}$$

then multiply both sides by $\left(\frac{dx^5}{dx^0}\right)^2$ and obtain

$$mcdx^5 = mc\frac{d\mathbf{x}}{dx^0}\cdot\frac{d\mathbf{x}}{dx^0}dx^5 + mc\left(\frac{dx^5}{dx^0}\right)^2 dx^5, \tag{18}$$

where the left hand side is Lorentz invariant. Dividing by the invariant dx^5, Eq.(18) can be rewritten as

$$\left(mc\frac{dx^0}{dx^5}\right)\frac{dx^5}{dx^0} = mc\frac{d\mathbf{x}}{dx^0}\cdot\frac{d\mathbf{x}}{dx^0} + mc\frac{dx^5}{dx^0}\frac{dx^5}{dx^0}, \tag{19}$$

or

$$\frac{E}{c}\frac{dx^5}{dx^0} = \frac{dx^5}{dx^0}\mathbf{p}\cdot\frac{d\mathbf{x}}{dx^0} + \left(mc\frac{dx^5}{dx^0}\right)\frac{dx^5}{dx^0} = \tilde{\pi}_\mu \tilde{y}^\mu. \tag{20}$$

The left hand side of Eq.(20) is a Lorentz scalar, while the right hand side is a scalar product of two $SO(4)$ Euclidean vectors, which we shall denote (using this time square brackets) in the following way

$$\tilde{\pi}^\mu \equiv \left[\frac{dx^5}{dx^0}\mathbf{p},\; mc\frac{dx^5}{dx^0}\right] = [m\mathbf{v},\; mc\gamma] = \tilde{\pi}_\mu, \tag{21}$$

$$\tilde{y}^\mu \equiv \left[\frac{d\mathbf{x}}{dx^0},\; \frac{dx^5}{dx^0}\right] = \left[\frac{\mathbf{v}}{c},\; \gamma\right],\; \mu = 1, 2, 3, 5, \tag{22}$$

please note the difference of notation (and order) in comparison with the notation of Minkowski vectors of Eq.(14).

There are some new interesting features related to the Euclidean 4-space treatment. First in Eq.(20) we found that $\frac{E}{c}\frac{dx^5}{dx^0}$ is a scalar. We can see it clearly in the invariant equation

$$\tilde{\pi}_\mu \tilde{\pi}^\mu = \left(\frac{dx^5}{dx^0}\right)^2 \mathbf{p}^2 + m^2c^2\left(\frac{dx^5}{dx^0}\right)^2 = \left(\frac{E}{c}\frac{dx^5}{dx^0}\right)^2. \tag{23}$$

In order to define a least action principle for the Euclidean 4-space treatment, let us note that in Eq.(15) the action is proportional to the proper time. We may use this relation and require that

Definition 2 *The least action principle is the Fermat-like principle of least proper time.*

B. Special case

Let us consider the special case with

$$\frac{dx^5}{dx^0} = const, \tag{24}$$

from Eq.(9) we have

$$q^\lambda = \frac{dx^5}{dx^0}x^\lambda, \quad \lambda = 1, 2, 3, 5. \tag{25}$$

Let p^j be a Lorentz 4-vector ($j = 0, 1, 2, 3$), then the corresponding $SO(4)$ vector will be $\tilde{p}^\lambda$ ($\lambda = 1, 2, 3, 5$)

$$\tilde{p}^\lambda = \frac{dx^5}{dx^0}p^\lambda, \quad \lambda = 1, 2, 3, 5, \tag{26}$$

where p^5 has to be properly defined. One may consider the equality of scalar products as a way of defining p^5. For this purpose one should introduce the metric. For the Lorentz group the metric is

$$g_{jk} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{27}$$

while for the SO(4) group the metric is simply the unit matrix. Thus equality of scalar products lead to (repeated indices mean a summation)

$$S = g_{jk}p^j p^k = \tilde{p}_\lambda \tilde{p}^\lambda, \quad \tilde{p}_\lambda = \tilde{p}^\lambda, \quad j,k = 0,1,2,3, \quad \lambda = 1,2,3,5. \tag{28}$$

From Eqs. (2) and (9) one may infer that

$$(p^5)^2 = g_{jk}p^j p^k = S, \quad \Rightarrow p^5 = \pm\sqrt{g_{jk}p^j p^k}. \tag{29}$$

1. An example

Let us consider a particle of mass m moving with a constant velocity $\mathbf{v}$. The energy momentum (Lorentz) four vector is

$$p^0 = \frac{E}{c}, \quad p^1 = p_x, \quad p^2 = p_y, \quad p^3 = p_z, \quad g_{jk}p^j p^k = m^2c^2 = (p^4)^2, \quad p^4 = \pm mc, \tag{30}$$

The transition to the $SO(4)$ four vector goes as follows

$$\frac{dx^5}{dx^0} = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \tag{31}$$

and the $SO(4)$ four vector is

$$\tilde{p}^\lambda = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} p^\lambda, \quad \lambda = 1,2,3,5. \tag{32}$$

On the basis of Eqs.(9-10) one may define

$$\tilde{p}^0 = p^5. \tag{33}$$

Now the scalar products take the form

$$p_k p^k = p^5 p^5, \quad k = 0, 1, 2, 3, \tag{34}$$

$$\tilde{p}_\lambda \tilde{p}^\lambda = \tilde{p}^0 \tilde{p}^0, \quad \lambda = 1, 2, 3, 5, \tag{35}$$

Explicitly Eq.(34) is the well known relation

$$\left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = m^2 c^2, \tag{36}$$

while Eq.(35) is equivalent to it, but in Euclidean relativity it takes the form

$$\left(1 - \frac{\mathbf{v}^2}{c^2}\right)\left(\mathbf{p}^2 + m^2 c^2\right) = m^2 c^2. \tag{37}$$

C. The Euclidean 4-velocity

In the Euclidean 4-space q^1, q^2, q^3, q^5, the $SO(4)$ 4-velocity $\tilde{v}^\lambda$ can be defined as follows

$$\tilde{v}^\lambda = \frac{dq^\lambda}{d\tau} = \frac{d\tau}{dt}\frac{dx^\lambda}{d\tau} = \frac{dx^\lambda}{dt}, \quad \lambda = 1, 2, 3, 5, \tag{38}$$

where the last result was obtained using Eq.(9). Thus the projection of the Euclidean 4-velocity on the 3-dimensional space q^1, q^2, q^3, is the nonrelativistic velocity $\mathbf{v} = d\mathbf{x}/dt$

$$\tilde{v}^\lambda \equiv [\mathbf{v}, c\frac{d\tau}{dt}]. \tag{39}$$

The $SO(4)$ invariant 4-velocity squared is equal to

$$\tilde{v}_\lambda \tilde{v}^\lambda = \mathbf{v}^2 + c^2 \left(\frac{d\tau}{dt}\right)^2, \tag{40}$$

which for free particles, using Eq.(31), is equal to

$$\tilde{v}_\lambda \tilde{v}^\lambda = \mathbf{v}^2 + c^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right) = c^2. \tag{41}$$

Thus all free particles move with the velocity of light in the the 4-space q^1, q^2, q^3, q^5.

D. The action in the Euclidean 4-space

Let us define the Euclidean 4-momenta

$$\tilde{\pi}_\lambda = \frac{mcd\tilde{q}_\lambda}{dx^{\mathbf{5}}}. \tag{42}$$

The action S for a free particle satisfies

$$mcdx^{\mathbf{5}} = \frac{mcd\tilde{q}_\lambda}{dx^{\mathbf{5}}} d\tilde{q}^\lambda = \tilde{\pi}_\lambda d\tilde{q}^\lambda = -dS. \tag{43}$$

The contributions of the electromagnetic field can be included by introducing the Euclidean electromagnetic 4-potentials $\tilde{A}_\lambda$

$$\tilde{A}_\lambda = \frac{dx^0}{dx^{\mathbf{5}}} A_\lambda, \quad \lambda = 1, 2, 3, 5, . \tag{44}$$

The action may be now recasted in a form, similar to that of the minimal coupling interaction

$$-dS = \left(\tilde{\pi}_\lambda - \frac{e}{c}\tilde{A}_\lambda\right) d\tilde{q}^\lambda. \tag{45}$$

III. MIXED SPACES

The Lorentz transformation has been derived in a four dimensional space-time with non-Euclidean metric. This causes some calculational and conceptual problems, especially when applied to quantum field theory. [4] [5] In this section a different approach is presented by using the mixed interval method. In this method the Lorentz transformation is described by 4-rotations in a four dimensional Euclidean space which we will call the "mixed space". These results have a different structure from results which one would have obtained by using the Lorentz transformation in the Minkowski space. Below we define the mixed space.

Let us consider two inertial reference systems S and S' with coordinate axes XYZ and $X'Y'Z'$ and time axes T and T' respectively. Let $x_1y_1z_1t_{1,} \quad x_2y_2z_2t_2$, be two events in S and $x'_1y'_1z'_1t'_{1,} \quad x'_2y'_2z'_2t'_2$, the corresponding events in S'. Let us denote:

$$\begin{aligned}
(\Delta\mathbf{r})^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\
&= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 , \\
(\Delta\mathbf{r}')^2 &= (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 \\
&= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 ,
\end{aligned} \tag{46}$$

and

$$\Delta t = t_2 - t_1, \quad \Delta t' = t_2' - t_1'. \tag{47}$$

Lorentz invariance requires the invariance of the interval s_{12}, where

$$s_{12}^2 = c^2 (\Delta t)^2 - (\Delta\mathbf{r})^2 = c^2 (\Delta t')^2 - (\Delta\mathbf{r}')^2 , \tag{48}$$

and c is the velocity of light. The four dimensional geometry described by the quadratic forms of Eq.(48) is non-Euclidean. Let us define the "mixed interval" m_{12} :

$$m_{12}^2 = c^2 (\Delta t)^2 + (\Delta\mathbf{r}')^2 = c^2 (\Delta t')^2 + (\Delta\mathbf{r})^2 , \tag{49}$$

which is a different way of expressing Eq.(48) and the Lorentz invariance. Eq.(49) is equivalent to Eq.(48). Eq.(49) is given in the mixed spaces $XYZT'$ and $X'Y'Z'T$, it has difficulties of interpretation, but have the advantage that the four-dimensional geometry of its quadratic forms is Euclidean.

A. The Lorentz Transformation

Let us derive the Lorentz transformation for the case where system S' moves along the $X(X')$ axis with velocity v with respect to system S. If at times $t = t' = 0$, the two systems coincide, the transformation corresponding to Eq.(49) is a simple rotation in the XT' $(X'T)$ plane:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct' \end{pmatrix} \tag{50}$$

and the inverse relation is:

$$\begin{pmatrix} x \\ y \\ z \\ ct' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ct \end{pmatrix} \tag{51}$$

The angle α can be evaluated from the motion of the origin of the coordinates of system S' with respect to S, in this case $x' = 0$, $x = vt$ and from Eq.(51) we obtain:

$$x = vt = x' \cos\alpha - ct \sin\alpha,$$

from which we find:

$$\sin\alpha = -\frac{v}{c} \tag{52}$$

and from Eq.(52):

$$\cos\alpha = \sqrt{1 - \frac{v^2}{c^2}}. \tag{53}$$

The form of the Lorentz transformation in the Minkowski space of Eq.(48) can be obtained by extracting ct' from Eq.(50):

$$ct' = x \tan\alpha + ct/\cos\alpha,$$

and substituting it into the first of Eqs. (50):

$$x' = x\cos\alpha + ct' \sin\alpha = x\cos\alpha + (x\tan\alpha + ct/\cos\alpha)\sin\alpha = x/\cos\alpha + ct\tan\alpha.$$

Thus we obtain:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 1/\cos\alpha & 0 & 0 & \tan\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \tan\alpha & 0 & 0 & 1/\cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \tag{54}$$

$$= \begin{pmatrix} \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} & 0 & 0 & -vc\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -vc\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} & 0 & 0 & \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \tag{55}$$

The above procedure can be generalized to a general transformation as given by Eq.(49), which is a rotation in a four dimensional Euclidean space and will be given in Sec. 4.

B. Velocity, acceleration and higher order time derivative additions

The advantage of the mixed interval and the mixed space shows up when we deal with time derivatives in calculating the laws of velocity addition, acceleration addition and higher order time derivatives additions. For simplicity let us consider Eq.(51) and let us take its n-th time derivative for $n \geq 2$:

$$\frac{d^n}{dt^n}\begin{pmatrix} x \\ y \\ z \\ ct' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \frac{d^n}{dt^n}\begin{pmatrix} x' \\ y' \\ z' \\ ct \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \frac{d^n}{dt^n}\begin{pmatrix} x' \\ y' \\ z' \\ 0 \end{pmatrix}, \quad n \geq 2 \tag{56}$$

as $\frac{d^n t}{dt^n} = 0$ for $n \geq 2$, and where $\sin\alpha$ and $\cos\alpha$ are given in Eqs. (52) and (53) respectively. For n larger than one, we can consider from Eq.(56) only the three dimensional relation:

$$\frac{d^n}{dt^n}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d^n}{dt^n}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad n \geq 2. \tag{57}$$

The transition to time derivatives in the prime system can be achieved by solving Eq.(56) for $n = 1$, for which the last equation is:

$$\frac{d\,(ct')}{dt} = \sin\alpha\frac{dx'}{dt} + c\cos\alpha = \sin\alpha\frac{dx'}{dt'}\frac{dt'}{dt} + c\cos\alpha,$$

from which we can extract

$$\eta' \equiv \frac{dt'}{dt} = \frac{\cos\alpha}{1 - \sin\alpha\frac{dx'}{cdt'}}. \tag{58}$$

From Eq.(58) obtain

$$\frac{d}{dt} = \frac{dt'}{dt}\frac{d}{dt'} = \eta'\frac{d}{dt'} \tag{59}$$

Using Eqs. (57-59) we obtain for the accelerations the simple relation:

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dt}\eta'\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\eta'\frac{d\eta'}{dt'}\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + (\eta')^2\frac{d^2}{dt'^2}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right]. \tag{60}$$

Next derivation leads to:

$$\frac{d^3}{dt^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\eta'\frac{d}{dt'}\left(\eta'\frac{d\eta'}{dt'}\right)\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right.$$

$$\left. + 3\,(\eta')^2\frac{d\eta'}{dt'}\frac{d^2}{dt'^2}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + (\eta')^3\frac{d^3}{dt'^3}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right] \tag{61}$$

and so on.

C. Generalizations

In the following we will describe the Lorentz transformation in terms of orthogonal matrices, therefore there will be no need to distinguish between covariant and contravariant indices. We will use the notation:

$$x_0 \equiv ct, \quad x_1 \equiv x, \quad x_2 \equiv y, \quad x_3 \equiv z, \quad x_4 \equiv ict \equiv ix_0. \tag{62}$$

Roman letters will be used in three dimensional summation indices , and Greek letters in four dimensional indices. Repeated indices convention for summation will be understood. Let us introduce the following notation, suitable for the mixed space operations:

$$\overline{x}_a \equiv x_a, \quad \overline{x}_0 \equiv x'_0, \quad \overline{x}'_a \equiv x'_a, \quad \overline{x}'_0 \equiv x_0, \quad a = 1,2,3. \tag{63}$$

The Lorentz transformation is a rotation in the mixed space and is represented with the orthogonal matrix R

$$\overline{x}'_\mu = R_{\mu\lambda}\overline{x}_\lambda, \quad \mu, \lambda = 0,1,2,3 \tag{64}$$

From Eq.(64) one can find the Lorentz transformation in Minkowski's space if he notes that:

$$x'_a = R_{ab}x_b + R_{a0}x'_0, \quad x_0 = R_{0a}x_a + R_{00}x'_0, \quad a,b = 1,2,3, \tag{65}$$

which can be solved, in a similar way which led to Eq.(54), and we get for the Lorentz transformation in Minkowski's space:

$$\begin{aligned} &x'_\mu = L_{\mu\lambda}x_\lambda, \quad \mu, \lambda = 1,2,3,4, \\ &L_{ab} = R_{ab} - R_{a0}R_{0b}/R_{00}, \quad L_{a4} = -iR_{a0}/R_{00}, \\ &L_{4a} = -iR_{0a}/R_{00}, \quad L_{44} = 1/R_{00}, \end{aligned} \tag{66}$$

where L is an orthogonal matrix. In this way one can parametrize the Lorentz transformation L of the Minkowski space, in terms of the four dimensional rotation matrix R of Eq.(64). One can check that Eq.(54) is a particular case of Eq.(65). Using Eqs. (63) and (64) one can define tensors in the mixed space.

IV. SUMMARY AND CONCLUSIONS

Two four dimensional spaces for which the invariant interval is Euclidean were introduced. In the first case 4 new coordinates were introduced, while in the second case mixed space was used.

The new coordinates are related to the Minkowski coordinates as in Eq.(9)

$$dq^{\lambda} = \frac{d\tau}{dt} dx^{\lambda}, \quad \lambda = 1, 2, 3, 5,$$

and satisfy Eq.(10)

$$(cd\tau)^2 = \left(dq^1\right)^2 + \left(dq^2\right)^2 + \left(dq^3\right)^2 + \left(dq^5\right)^2 .$$

Now Lorentz invariance is a pure rotation in the q^1, q^2, q^3, q^5 space and a tensor calculus can be developed according to the $SO(4)$ group. An interesting scalar of this group is $\left(\frac{E}{c}\frac{d\tau}{dt}\right)^2$ as given by Eq.(23). The $SO(4)$ 4-velocity as given by Eq.(39) is

$$\tilde{v}^{\lambda} \equiv [\mathbf{v}, c\frac{d\tau}{dt}],$$

where $\mathbf{v}$ is the nonrelativistic velocity $\mathbf{v} = d\mathbf{x}/dt$. For free particles $\tilde{v}_{\lambda}\tilde{v}^{\lambda} = c^2$, i.e. they are moving in the Euclidean 4-space with the velocity of light.

The least action principle for the q^1, q^2, q^3, q^5 space was discussed and derived in subsection 2.4, Eq.(43) and Eq.(45).

The above results seem to be very intriguing, more work in this direction is now in progress.

Using the mixed space presentation we have shown that new possibilities exist for the treatment of the Lorentz transformation. Equation (49) allowed us to use a mixed 4 space-time as a basis for an Euclidean metric and to obtain in a simple way, not only the Lorentz transformation but also the addition formulas for accelerations and higher order time derivatives. Moreover, for accelerations and higher order time derivatives, the transformation follows a simple, three dimensional procedure (Eqs. (57-61)). Using Eqs. (65) and (66) one can connect the parameters of the group [6] of four dimensional rotations SO(4) and the equivalent parameters of the group of Lorentz transformations. This procedure allows to introduce new parametrization of the Lorentz transformations in terms of the parameters of pure 4-rotations.

The introduction of the mixed spaces may open new outlooks and interpretations of transformed spaces and simplified procedures for the evaluation of the transformation formulae. It may give meaning to such notions as events in future or past times, or two time equations, as the mixed intervals are described with times belonging to events in different frames.

At present this work is being continued in order to explore the full potential of the presented methods.

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