Magnetic helicity tensor for an anisotropic turbulence

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The evolution of the magnetic helicity tensor for a nonzero mean magnetic field and for large magnetic Reynolds numbers in an anisotropic turbulence is studied. It is shown that the isotropic and anisotropic parts of the magnetic helicity tensor have different characteristic times of evolution. The time of variation of the isotropic part of the magnetic helicity tensor is much larger than the correlation time of the turbulent velocity field. The anisotropic part of the magnetic helicity tensor changes for the correlation time of the turbulent velocity field. The mean turbulent flux of the magnetic helicity is calculated as well. It is shown that even a small anisotropy of turbulence strongly modifies the flux of the magnetic helicity. It is demonstrated that the tensor of the magnetic part of the $\alpha$ effect for weakly inhomogeneous turbulence is determined only by the isotropic part of the magnetic helicity tensor. [S1063-651X(99)05506-3]

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I. INTRODUCTION

The magnetic helicity $A^{(i)}$, $H$ is a fundamental quantity in magnetohydrodynamics because it is conserved in the limit of infinite electrical conductivity of the medium, where $H = \nabla \times A^{(i)}$ is the magnetic field and $A^{(i)}$ is the magnetic vector potential. In addition, the topological properties of magnetic field are determined by the magnetic helicity (see, e.g., [1,2]). In developed magnetohydrodynamic turbulence the mean magnetic helicity $(a \cdot H)$ is conserved as well in the limit of infinite magnetic Reynolds numbers and zero mean magnetic field, where $H$ and $a$ are fluctuations of the magnetic field and the magnetic vector potential, respectively (see, e.g., [1,2]). The magnetic helicity tensor $\chi_{ij} = \langle a_i(x) h_j(x) \rangle$ determines the tensor of the magnetic part of the $\alpha$ effect. The latter is of fundamental importance in view of magnetic dynamo (see, e.g., [1–3]). In spite of the great importance of this quantity, a dynamics of the magnetic helicity tensor for an anisotropic turbulence is poorly understood.

In the present paper the equation for the magnetic helicity tensor for an anisotropic turbulence and a nonzero mean magnetic field, and for large magnetic Reynolds numbers is derived. It is shown that the isotropic and anisotropic parts of the magnetic helicity tensor have different characteristic times of evolution. The time of variation of the isotropic part of the magnetic helicity tensor is much longer than the correlation time of the turbulent velocity field. On the other hand, the anisotropic part of the magnetic helicity tensor changes for the correlation time of the turbulent velocity field. This anisotropic part is determined only by the turbulent magnetic diffusion tensor. The mean turbulent flux of the magnetic helicity is calculated as well. It is shown that even small anisotropy of turbulence strongly modifies the flux of the magnetic helicity.

II. THE EQUATION FOR THE MAGNETIC HELICITY: SIMPLE APPROACH

First, we derive an equation for the magnetic helicity for an anisotropic turbulence by a simple consideration. The induction equation for the magnetic field $H$ is given by

$$\frac{\partial H}{\partial t} = \nabla \times (v \times H - \eta \nabla \times H),$$

where $H = B + h$, $B = \langle H \rangle$ is the mean magnetic field, $v = \nabla + u$, $v = \langle v \rangle$ is the mean fluid velocity field, and $\eta$ is the magnetic diffusion due to electrical conductivity of fluid. The equation for the vector potential $A^{(i)}$ follows from the induction equation (1),

$$\frac{\partial A^{(i)}}{\partial t} = v \times H - \eta \nabla \times (\nabla \times A^{(i)}) + \nabla \varphi,$$

where $H = \nabla \times A^{(i)}$, $A^{(i)} = A + a$, and $A = \langle A^{(i)} \rangle$ is the mean vector potential, $\varphi = \tilde{\Phi} + \phi$ is an arbitrary scalar function, and $\Phi = \langle \varphi \rangle$. Now we multiply Eq. (1) by $a$ and Eq. (2) by $h$, add them, and average over the ensemble of turbulent fields. This yields an equation for the magnetic helicity $\chi = \langle a_p(x) h_p(x) \rangle$:

$$\frac{\partial \chi}{\partial t} = -2 \langle u \times h \rangle \cdot B - 2 \eta \langle h \cdot (\nabla \times h) \rangle - \nabla \cdot \tilde{F},$$

where $F_p = V_p - \chi_{pp} V_p + (a \times u) 

(\langle a \times v \rangle - \langle h \rangle \hat{\gamma} \nabla \times B)$

(see, e.g., [4,5]), where $\hat{\gamma} = \delta_{mn} - n_{pp} n_{mn} / 2$, $n_{mn} = \eta \delta_{mn} + \eta_{\tilde{\gamma}mn}$, $\eta_{\tilde{\gamma}mn} = \langle \tau u_m u_n \rangle$, $(U)_n = - \nabla_m \eta_{\tilde{\gamma}mn} / 2$ is the velocity caused by the turbulent diamagnetism, $\hat{\alpha} = \alpha^{(i)} + \alpha^{(B)}$, and the tensors $\alpha^{(i)}_{mn}$ and $\alpha^{(B)}_{mn}$ are given by

$$\alpha^{(i)}_{mn} = \left[ \varepsilon_{mji}(\tau h_j(x) \nabla_u u_m(x)) + \varepsilon_{nji}(\tau h_j(x) \nabla_u u_n(x)) \right] / 2,$$

$$\alpha^{(B)}_{mn} = \left[ \varepsilon_{mji}(\tau h_j(x) \nabla_u h_m(x)) + \varepsilon_{nji}(\tau h_j(x) \nabla_u h_n(x)) \right] / (2 \mu_0 \rho).$$

Substituting Eq. (4) into Eq. (3) we obtain after simple manipulations an equation for the magnetic helicity:
\[
\frac{\partial \chi}{\partial t} = -2 \eta \left( \frac{\partial^2 \chi}{\partial x_p \partial y_p} \right)_{r=0} + 2 \eta \hat{\alpha}_{mn} B_m (\nabla \times \mathbf{B})_n - 2 \hat{\alpha}_{mn} B_m B_n
\]
\[-\nabla \cdot \mathbf{F}, \quad (7)
\]
where we used an identity \( (\mathbf{h} \cdot (\nabla \times \mathbf{h})) = (\partial^2 \chi / \partial x_p \partial y_p)_{r=0} \), and \( r = y - x \). The second and third terms in Eq. (7) describe the magnetic field and the magnetic helicity, and the hydrodynamic helicity are the sources of the magnetic helicity. The first term in Eq. (7) determines the relaxation of the magnetic helicity with the characteristic time \( T \) which depends on the molecular magnetic diffusion \( \eta \). This time is given by
\[
T^{-1} = \frac{2 \eta}{\chi} \left( \frac{\partial^2 \chi}{\partial x_p \partial y_p} \right)_{r=0}. \quad (8)
\]
The characteristic relaxation time \( T \) of the magnetic helicity is \( T \sim \tau_0 \text{Rm} \) (Rm denotes ‘magnetic Reynolds number’), i.e., it is much longer than the correlation time \( \tau_0 = l_0 / u_0 \) of the turbulent velocity field, where \( u_0 \) is the characteristic turbulent velocity in the maximum scale of turbulent motions \( l_0 \). The last term in Eq. (7) describes the turbulent flux \( \mathbf{F} \) of the magnetic helicity which will be calculated in Sec. III. Equation (7) in the case of an isotropic turbulence coincides with that derived in [6] (see also [7,8]).

III. THE EQUATION FOR THE MAGNETIC HELICITY TENSOR: METHOD OF PATH INTEGRALS

In this section we derive an equation for the magnetic helicity tensor. To this purpose we use a method of path integrals (see, e.g., [5,9-11]). This method allows us to derive the equation for the tensor \( \chi_{ij} = \langle a_i(x) h_j(y) \rangle \).

\[
\frac{\partial \chi_{ij}}{\partial t} = -2 \eta \left( \frac{\partial^2 \chi_{ij}}{\partial x_p \partial y_p} \right)_{r=0} + \frac{\partial}{\partial R_p} (e_{jpl} \alpha_{lsp}^{(v)} x_s - V_{p} \chi_{ij} - \chi_{ij} \alpha_{lsp}^{(v)} x_p) + \frac{\partial V_{ij}}{\partial R_p} x_p
\]
\[-a_{ij} \chi_{ij} + 2 \alpha_{ij}^{(v)} h_{ij} - \alpha_{ij}^{(g)} h_{ij} + e_{i sp} \epsilon_{p h j} + \epsilon_{i sp} h_{ij} + I_{ij}
\]
\[
(9)
\]
(9) (for details, see Appendix A), where \( \mathbf{R} = (x + y)/2 \),
\[I_{ij} = \alpha_{ij}^{(g)} B_i B_j - \alpha_{ij}^{(g)} B_i B_j + e_{i sp} \chi_{ij} \]
\[= 2 e_{i sp} \chi_{ij} \hat{\eta}_{p l} \hat{\eta}_{l p} \hat{\alpha}_{ij}^{(g)} B_{l p} B_{l p} + I_{ij}
\]
\[
(10)
\]
(10).

\[h_{ij} = (h_i(x) h_j(x)), \quad \hat{\eta}_{ij} = \langle u_i(x) u_j(x) \rangle, \quad S_i = \langle u_i(x) b(x) \rangle, \quad \phi_j = \langle \phi_j(x) h_j(x) \rangle, \quad I_{ij} = e_{i sp} \alpha_{l sp}^{(g)} ((\partial a_i / \partial h_j) h_j + \partial \phi_j / \partial R_i - (\partial h_i / \partial h_j) \phi_j), \quad \text{and} \ \mathbf{b} = \nabla \cdot \mathbf{u} \].

Equation (9) is derived for the case \( \nabla \cdot \mathbf{h} = 0 \). We use here the \( \hat{\eta} \)-correlated in time random process to describe a turbulent velocity field. The results remain valid also for the velocity field with a finite correlation time, if the second-order correlation functions of the magnetic field and the magnetic helicity vary slowly in comparison with the correlation time of the turbulent velocity field (see, e.g., [9,12]). We also take into account the dependence of the momentum relaxation time on the scale of turbulent velocity field: \( \tau_0 = 2 \tau_0 k / k_0^{1-p} \), where \( p \) is the exponent in the spectrum of kinetic turbulent energy, \( k \) is the wave number, \( k_0 = l_0^{-1} \). The equation for \( \chi = \chi_{pp} \) follows from Eq. (9):
\[
\frac{\partial \chi}{\partial t} = -2 \eta \left( \frac{\partial^2 \chi}{\partial x_p \partial y_p} \right)_{r=0} + 2 \hat{\alpha}_{mn} B_m (\nabla \times \mathbf{B})_n - 2 \alpha_{mn}^{(v)} B_m B_n
\]
\[-\nabla \cdot \mathbf{F}, \quad (11)
\]
(see Appendix A), where hereafter \( \nabla \cdot \mathbf{F} = \partial / \partial R_p \), and we used the gauge condition for the mean vector potential \( \hat{\eta}_{ip} \hat{\eta}_{j p} A_s = 0 \). For an isotropic turbulence (\( \hat{\eta}_{mn} = \hat{\eta}_{m n} / 3 \)) the gauge condition is given by \( \nabla \cdot \mathbf{A} = 0 \). The last term in Eq. (11) describes the turbulent flux of the magnetic helicity \( F_p = e_{plm} \chi_{tm} \alpha_{lm}^{(v)} + V_{m} \chi_{mp} - (3/4) V_{p} \chi \). The mean turbulent flux of the magnetic helicity depends on the tensor of hydrodynamic helicity \( \alpha_{ij}^{(v)} \) and the mean fluid velocity \( \mathbf{V} \). Comparison of Eq. (11) (which was derived by the path integral method) with Eq. (7) (which was obtained by the simple consideration) shows that these two approaches arrive at the similar equation after the change \( \alpha_{mn}^{(v)} \rightarrow \alpha_{mn} \). Note that the mean turbulent flux of the magnetic helicity \( \mathbf{F} \) cannot be calculated by the simple consideration.

The tensor \( \chi_{ij} \) can be presented in the form \( \chi_{ij} = \chi \delta_{ij} / 3 + \mu_{ij} \), where the anisotropic part of the magnetic helicity tensor \( \mu_{ij} \) has the following properties: \( \mu_{pp} = 0 \), and \( \mu_{ij} = \mu_{ji} \). For the calculation of the second spatial derivative \( (\partial^2 \chi_{ij} / \partial x_p \partial y_p)_{r=0} \) we use the tensor \( \chi_{ij}(k_1^{(1)}, k_2^{(1)}) \) in \( \mathbf{k} \) space:
\[
\chi_{ij}(k_1^{(1)}, k_2^{(1)}) = -5(k_{pp} \delta_{ij} - k_{ij}) (\chi_{ij}^{(5)} - k_{mn} \mu_{mn} / 2k_{pp})
\]
\[-\mu_{im} k_{mj} - k_{im} \mu_{mj} + k_{pp} \mu_{ij}
\]
\[+ k_{mn} \mu_{mn} \delta_{ij}) / 8 \pi k^2, \quad (12)
\]
where
\[
\chi_{ij}(x,y) = \int \chi_{ij}(k_1^{(1)}, k_2^{(1)}) \exp(i(k_1^{(1)} x + k_2^{(1)} y)) d k_1^{(1)} d k_2^{(1)}
\]
and \( k_j = k_j^{(2)} \). The tensor \( \chi_{ij}(k_1^{(1)}, k_2^{(1)}) \) satisfies the identities \( k_{ij} \chi_{ij}(k_1^{(1)}, k_2^{(1)}) = 0 \) and \( \chi_{ij}(k_1^{(1)}, k_2^{(1)}) k_j^{(2)} = 0 \). These identities correspond to the conditions \( \nabla \cdot \mathbf{a} = 0 \) and \( \nabla \cdot \mathbf{h} = 0 \), respectively. Using Eqs. (8), (12), and (B1) (see Appendix B) we rewrite Eq. (11) for the magnetic helicity in the form
\[
\frac{\partial \chi}{\partial t} + \chi / T + \nabla \cdot \mathbf{F} = 2 \alpha_{mn}^{(v)} B_m B_n - 2 \hat{\eta}_{mn} B_m (\nabla \times \mathbf{B})_n
\]
\[-\nabla \cdot (\mu_{ij} e_{jpl} \alpha_{lsp}^{(v)} + V_{s} \mu_{sp}), \quad (13)
\]
Equation (13) implies that the characteristic relaxation time \( T \) of the isotropic part of the magnetic helicity tensor is \( T \sim \tau_0 \text{Rm} \), i.e., it is much longer than the correlation time \( \tau_0 = l_0 / u_0 \) of the turbulent velocity field. Equations (9) and (13) yield the equation for the tensor \( \mu_{ij} \):
\[ \eta_{ij}^* \mu_{pi} + 8 \eta_{ip}^* \mu_{pj} - 3 \mu_{ij} - 3 \delta_{ij} \eta_{pm}^* \mu_{mp} = (7/10)(3 \eta_{ij}^* - \delta_{ij}) \chi + O(t_\rho / T), \]  

where \( \eta_{ij}^* = \eta_{ij} / \eta_{pp} \). We neglected here small terms \(~(t_\rho / T)\) and \(~t_\rho B^2\). It follows from Eq. (14) that the anisotropic part of the magnetic helicity tensor is determined only by the turbulent diffusion. Therefore, the characteristic time of the anisotropic part \( \mu_{ij} \) of the magnetic helicity tensor is of the order of \( t_\rho \), i.e., it is very small. Solving Eq. (14) in the complete set of the eigenfunctions of the matrix \( \eta_{ij}^* \) we obtain \( \mu_{ij} = 0 \) when \( i \neq j \), \( \mu_{11} = \mu_1 \chi \), \( \mu_{22} = \mu_2 \chi \), and \( \mu_{33} = -(\mu_1 + \mu_2) \chi \), where \( \eta_{ij}^* = 0 \) when \( i \neq j \), and \( \eta_{11}^* = \eta_1 \), \( \eta_{22}^* = \eta_2 \), \( \eta_{33}^* = 1 - (\eta_1 + \eta_2) \), and

\[
\eta_1 = 1/3 + \epsilon_1, \quad \text{and} \quad \eta_2 = 1/3 + \epsilon_2.
\]

In the case of one preferential direction \((\epsilon_1 = \epsilon_2 = \epsilon = 0)\) we obtain \( \mu_1 = \mu_2 = 7/30 \). When \( \epsilon = 0 \) the anisotropic part of the magnetic helicity tensor \( \mu_{ij} = 0 \). In the case of one preferential direction (say, in the direction \( \mathbf{e} \)), Eqs. (13) and (14) yield

\[
\frac{\partial \chi}{\partial t} + \chi / T + \nabla_p (\nabla_p^{\text{eff}} \chi) + 2 \alpha_{mn} B_m B_n - 2 \eta_{mn} B_m (\nabla \times \mathbf{B})_n = 0,
\]

where \( \nabla^{\text{eff}} = 23V/30 + 7(e \cdot \mathbf{v})e/10 - 7(e \times \mathbf{d})/15 \), and the vector \( D_m = \alpha_{mn} e_n \). Equation (15) implies that even small anisotropy of turbulence \((\text{Rm}^{-1} \ll \epsilon \ll 1)\) strongly modifies the flux of the magnetic helicity.

For a weakly inhomogeneous turbulence the magnetic part of the \( \alpha \) tensor is given by

\[
\alpha_{mn}^{(B)}(\mathbf{r} = 0) = 2 \chi \eta_{T\mu_0} \delta_{mn} = \alpha^{(B)} \delta_{mn}
\]

(see Appendix C), where \( \alpha^{(B)} = 2 \chi (9 \eta_{T\mu_0} \rho) \) and \( \chi = \chi(\mathbf{R}) \). This implies that the tensor for the magnetic part of the \( \alpha \) effect for weakly inhomogeneous turbulence is determined only by the isotropic part of the magnetic helicity tensor. Thus, the evolutionary equation for the magnetic part of the \( \alpha \) effect in this case is given by

\[
\frac{\partial \alpha_{mn}^{(B)}}{\partial t} + \frac{\alpha_{mn}^{(B)}}{T} + \frac{1}{\rho} \nabla_p (\nabla_p^{\text{eff}} \alpha_{mn}^{(B)}) \rho = -4 \frac{3}{9 \eta_{T\mu_0} \rho} [\alpha_{mn}^{(v)} B_m B_n - \eta_{mn} B_m (\nabla \times \mathbf{B})_n],
\]

where we used Eqs. (15) and (16).

**IV. DISCUSSION**

We have shown here that an anisotropy of a fluid flow strongly modifies the turbulent transport of the magnetic helicity. In particular, even small anisotropy of turbulence significantly changes the mean flux of the magnetic helicity. It is given by \( \mathbf{F} = \mathbf{V}^{\text{eff}} \chi \). Indeed, if we consider, e.g., a small anisotropy of turbulence: \( \epsilon \sim \text{Rm}^{-\beta} \) (where \( \beta < 1 \)), then the vector \( D_m = \alpha_{mn}^{(v)} e_m / 3 + O(\text{Rm}^{-\beta}) \). When the mean velocity \( \mathbf{V} \) is normal to the vector \( \mathbf{e} \) (which is typical for astrophysical applications) we obtain \( \mathbf{V}^{\text{eff}} \sim 23V/30 \). Therefore, a very small anisotropy \( \sim \text{Rm}^{-\beta} \) changes the mean flux of the magnetic helicity to 25%. This result is associated with an existence of a small parameter \( \text{Rm}^{-1} \) which is the ratio of the relaxation times of anisotropic and isotropic parts of the magnetic helicity tensor. Note that the mean magnetic field is the main source of the magnetic helicity. For zero mean magnetic field the magnetic helicity is very small [15].

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**APPENDIX A: DERIVATION OF THE EQUATION FOR THE MAGNETIC HELICITY TENSOR**

We use a method of path integrals (and modified Feynman-Kac formula) (see, e.g., [5,9–11]). The solution of the induction equation (1) with the initial condition \( \mathbf{H}(t = t_0, \mathbf{x}) = H_0(\mathbf{x}) \) is given by the Feynman-Kac formula \( \mathbf{H}(t, \mathbf{x}) = \mathbf{M} \{ G_{ij}(t, t_0) H_{0}[\xi_i(t, t_0)] \} \), where the function \( G_{ij} \) is determined by the equation \( dG_{ij}(t, t_0) / ds = N_{is} G_{ij} \) with the initial condition \( G_{ij} = \delta_{ij} \) for \( s = t_0 \). Here \( \mathbf{M} \{ \cdot \} \) is a mathematical expectation over the ensemble of Wiener paths, \( t_s = t + s \), and \( N_{is} = \partial \xi_i / \partial \xi_j - \delta_{ij} k \), and the Wiener path \( \xi_i(t, t_0) \) is given by \( \xi_i = \int_0^t v_i(t, \xi_i) ds + \sqrt{2} \eta w(t) \), where \( \mathbf{w} \) is a Wiener process. This method allows us to get \( \mathbf{H}(t + \Delta t, \mathbf{x}) \):

\[
\mathbf{H}(t + \Delta t, \mathbf{x}) = \mathbf{H}(t, \mathbf{x}) + M \left( q_i(\Delta t + p_i(\mathbf{x})(\Delta t)^2 \right)
\]

(see Appendix in [5]), where \( Q_{in} = H_j \nabla N_{ij} \), \( - (\nabla_n H_i)(\nabla_n v_m) \), and

\[
q_i = H_m \nabla v_i - v_m \nabla H_i - b H_i + \eta w_i \nabla v_m \nabla v_m H_i (\Delta t)^{-1},
\]

\[
p_i = (1/2) H_n \left( \nabla_m v_i \nabla v_m - v_m \nabla v_m \nabla v_m - \nabla v_m \right)
\]

Now we use the following identity \( [\nabla \times (\eta \nabla \times \mathbf{H})]_k = \nabla \nabla \nabla \eta_{ki} - H_j \nabla \eta_{nk} - \eta_{ki} \nabla \eta_{n} \), where \( \eta = \eta_{ij} \)

Now we use the following identity \( [\nabla \times (\eta \nabla \times \mathbf{H})]_k = \nabla \nabla \nabla \eta_{ki} - H_j \nabla \eta_{nk} - \eta_{ki} \nabla \eta_{n} \), where \( \eta = \eta_{ij} \)

This identity can be derived as follows. Consider the vector \( E_k = \nabla \nabla \nabla \eta_{ki} - H_j \nabla \eta_{nk} - \eta_{ki} \nabla \eta_{n} \), where \( \eta_{ki} \) is an arbitrary symmetrical tensor, and we use the condition \( \nabla \cdot \mathbf{H} = 0 \). Now we change \( i \rightleftharpoons j \) and \( i \rightleftharpoons n \). This yields \( E_k = \nabla \nabla \nabla \eta_{ki} - H_j \nabla \eta_{nk} - \eta_{ki} \nabla \eta_{n} \nabla \cdot \mathbf{H} \). Using this equation we calculate the vector \( C_k = \nabla \nabla \nabla \eta_{ki} - H_j \nabla \eta_{nk} - \eta_{ki} \nabla \eta_{n} \nabla \cdot \mathbf{H} \).
Now we introduce the tensor \( \mathbf{A}^{(i)} = A^{(i)}(t) \), where \( A^{(i)}(t) = \frac{Q_i(t)}{\alpha_{i}} \), and \( \mathbf{B} = (\mathbf{H}) \). Equations (A1)–(A3) yield an equation for the vector potential \( \mathbf{A}^{(i)} \):

\[
\mathbf{A}^{(i)}(t+\Delta t, \mathbf{x}) = \mathbf{A}^{(i)}(t, \mathbf{x}) + M \left[ Q_i(t) \Delta t + P_i(t)(\Delta t)^2 \right] + \sqrt{2} \eta S_{in}(\mathbf{x}) \int_0^\Delta t \mathbf{w}_n d\sigma + \Delta t \mathbf{\nabla} \varphi.
\]

Equations (A1)–(A3) yield an equation for the mean fields \( \mathbf{A}^{(i)} \) and \( \mathbf{B} = (\mathbf{H}) \), where \( \mathbf{H} = \nabla \times \mathbf{A}^{(i)}(t) \), and \( \eta = \varepsilon_{ipn} H_{p} \mathbf{v}_{n} \),

\[
Q_i = \varepsilon_{ipn} H_{p} \mathbf{v}_{n} \mathbf{H}_{k} - \varepsilon_{ipn} H_{p} \mathbf{H}_{k} \nabla_{v} \mathbf{H}_{n} = \frac{1}{2} \int_0^\Delta t \mathbf{w}_n d\sigma + \Delta t \mathbf{\nabla} \varphi.
\]

The equation for the tensor \( \mathbf{A}^{(i)} \) follows from Eq. (A8) by the change \( \mathbf{x} \rightarrow \mathbf{y} \) and \( \mathbf{y} \rightarrow \mathbf{x} \). Now we introduce a symmetrical tensor: \( \chi^{(x)}_{ij} = (\chi^{(x)}_{ij} + \chi^{(y)}_{ij})/2 \).

Consider the case \( \varepsilon_{mn} \mathbf{H}_{n} = 0 \). Now we derive an equation for the tensor \( \chi^{(x)}_{ij} \) using Eq. (A8). The result is given by Eq. (9). For derivation of Eq. (9) we use the following identities:
\[ N_{ij}h_{ks} = \alpha_{is}^{(v)}h_{sj} - \alpha_{ks}^{(v)}h_{xk}\delta_{ij} + \varepsilon_{isp}\delta_{pj}h_{sj} + 2\tilde{\eta}_{mp}\frac{\partial^2 \chi_{mj}}{\partial x_p \partial x_i}, \]  

\[ L^{(A)}_{is}(x)h_{sj} + L^{(A)}_{is}(y)h_{js} = \alpha_{is}^{(v)}(x)h_{sj} + \alpha_{is}^{(v)}(y)h_{js} + \tilde{\eta}_{pn}\left[ \frac{\partial^2 \chi_{ij}^{(xy)}}{\partial x_p \partial x_n} + \frac{\partial^2 \chi_{ij}^{(xy)}}{\partial y_p \partial y_n} \right] \]

\[ + V_s\left( \frac{\partial \chi_{ij}^{(xy)}}{\partial x_i} + \frac{\partial \chi_{ij}^{(xy)}}{\partial y_j} \right) - V_s\left( \frac{\partial \chi_{ij}^{(xy)}}{\partial x_j} + \frac{\partial \chi_{ij}^{(xy)}}{\partial y_i} \right), \]  

\[ L^{(B)}_{js}(x)\chi_{ij}^{(xy)} + L^{(B)}_{js}(y)\chi_{ij}^{(xy)} = \varepsilon_{ijl}\left( \frac{\partial}{\partial x_p} (\alpha_{il}^{(v)} \chi_{pj}) + \frac{\partial}{\partial x_p} (\chi_{ij}^{(xy)} \alpha_{lp}) \right) \]

\[ - \left( \frac{\partial V_p}{\partial x_p} + V_s \frac{\partial}{\partial x_p} \chi_{ij}^{(xy)} - \frac{\partial V_p}{\partial y_p} + V_s \frac{\partial}{\partial y_p} \chi_{ij}^{(xy)} \right) \]

\[ + \chi_{ip}^{(xy)} \frac{\partial V_l}{\partial x_p} + \chi_{ip}^{(xy)} \frac{\partial V_l}{\partial y_p} - \tilde{\eta}_{pn}\left( \frac{\partial^2 \chi_{ij}^{(xy)}}{\partial y_p \partial y_n} + \frac{\partial^2 \chi_{ij}^{(xy)}}{\partial x_p \partial x_n} \right). \]  

The tensor $\alpha_{mn}^{(v)} = (\alpha_{mn}^{(xy)} + \alpha_{mn}^{(xy)})/2$. We used here that

\[ (\partial \chi_{is}^{(xy)})/\partial x_p + \partial \chi_{is}^{(xy)}/\partial y_p \bigg|_{r=0} = 2[\nabla_p \chi_{is} - (\partial a_i/\partial x_p)h_s]. \]

The latter identity can be derived as follows:

\[ \left( \frac{\partial \chi_{is}^{(xy)}}{\partial x_p} \right)_{r=0} = \left( \frac{\partial h_s(x)}{\partial x_p} a_i(y) \right)_{r=0} \]

\[ = \left( \frac{\partial}{\partial x_p} (h_s(x) a_i(y)) - \left( \frac{\partial a_i}{\partial x_p} h_s \right) \right)_{r=0} \]

\[ = \nabla_p \chi_{is} - (\partial a_i/\partial x_p)h_s. \]

For the derivation of Eq. (11) we used the following identities:

\[ \varepsilon_{ijk}\tilde{\eta}_{lp}B_k \nabla_p B_i = - \tilde{\eta}_{im}B_l (\nabla \times B)_m - B_p \nabla_l (\tilde{\eta}_{np} \nabla_p A_i), \]

and $\tilde{\eta}_{pp} = - V_p \chi/3 + O(l_{B}^2/l_{B}^2)$, where $l_B$ is the characteristic scale of the mean magnetic field variations, $l_0$ is the maximum scale of turbulent motions, and $l_0 \ll l_B$.

**APPENDIX B: THE DERIVATION OF EQ. (13)**

We use here the two-scale approach (see, e.g., [13,14]). Indeed, let us consider, for example, a correlation function

\[ \langle u_i(x)u_j(y) \rangle = \int \langle u_i(k^{(1)}u_j(k^{(2)}) \rangle \exp[i(k^{(1)}x + k^{(2)}y)] \]

\[ \times d{k^{(1)}}/d{k^{(2)}} \]

\[ = \int \tilde{f}_{ij}(r, K) \exp(i K \cdot r) dK \]

\[ = \int f_{ij}(k, R) \exp(i k \cdot r) dk, \]

where

\[ \tilde{f}_{ij}(K, r) = \int \langle u_i(k + K/2)u_j(-k + K/2) \rangle \exp(i k \cdot r) dk, \]

\[ f_{ij}(k, R) = \int \langle u_i(k + K/2)u_j(-k + K/2) \rangle \exp(i K \cdot r) dk, \]

and $R = (x + y)/2$, $r = y - x$, $K = k^{(1)} + k^{(2)}$, $k = (k^{(2)} - k^{(1)})/2$, $R$ and $K$ correspond to the large scales, and $r$ and $k$ describe the small scales. Using Eq. (12) we obtain

\[ \eta_{mp}^{\ast} \left( \frac{\partial^2 \chi_{ij}}{\partial x_m \partial y_p} \right)_{r=0} = \tau_0 \left[ \left( \frac{3 \eta_{ij}^{\ast} - \delta_{ij}}{\chi} \right) 10 + \left( \eta_{ij}^{\ast} \mu_{pi} \right) \right. \]

\[ + \left. 8 \eta_{ij}^{\ast} \mu_{pj} - 3 \delta_{ij} - 3 \delta_{ij} \eta_{mp}^{\ast} \mu_{mp}^{\ast} / 7 \right], \]

\[ (B1) \]

where $\chi(r = 0) = \int \chi^{\ast} \exp(i K \cdot R) dK \ dK$, $\mu_{ij}(r = 0) = \int \mu_{ij}^{\ast} \exp(i K \cdot R) dK \ dK$, $\chi^{\ast} = \chi^{\ast}(k, K)$, and $\mu_{ij}^{\ast} = \mu_{ij}^{\ast}(k, K)$. In order to obtain Eq. (B1) in $r$ space we used the transformations: $ik^{(1)} \rightarrow \partial / \partial x$ and $ik^{(2)} \rightarrow \partial / \partial y$, and we assumed a weak inhomogeneity of the magnetic helicity, i.e., we neglected the terms $\sim O(K)$ in Eq. (12). We also used the realizability condition for the magnetic helicity (see, e.g., [1]), i.e., we assumed that the spectral densities $\chi^{\ast}$ and $\mu_{ij}^{\ast}$ of $\chi$ are localized in the vicinity of the maximum scale of turbulent motion $l_0$. In order to derive Eq. (B1) we used the following integrals:

\[ Y_{ijmn} = \int (k_i k_j k_m k_n / k^6) \sin \theta \, d \theta \, d \varphi \]

\[ = (4 \pi / 15) \left( \delta_{ij} \delta_{mn} + \delta_{im} \delta_{nj} + \delta_{in} \delta_{mj} \right), \]

\[ \int (k_i k_j k_i k_j, k, l^6) \sin \theta \, d \theta \, d \varphi \]

\[ = (1/7) \left( Y_{ffr} \delta_{ij} + Y_{ffr} \delta_{it} + Y_{iff} \delta_{fr} + Y_{iff} \delta_{fr} \right) \]

Equations (B1) and (11) allow us to obtain Eq. (13).

**APPENDIX C: THE MAGNETIC PART OF THE $\alpha$ EFFECT FOR WEAKLY INHOMOGENEOUS TURBULENCE**

In this appendix we derive a formula for the magnetic part of the $\alpha$ effect for weakly inhomogeneous turbulence. We show that this tensor is determined by the trace of the magnetic helicity tensor. The tensor $\alpha_{mn}^{(B)}$ for the magnetic part of the $\alpha$ effect is determined by Eq. (6). Now we calculate
\[ e_{mij} \langle \tau h_i(x) \nabla a_j(y) \rangle \]

\[ = -e_{mij} \epsilon_{ljx} \int \tau(k^{(2)}) k^{(2)}_n \langle a_q(k^{(2)}) h_j(k^{(1)}) \rangle \exp[i(k^{(1)} \cdot x + k^{(2)} \cdot y)] d\mathbf{k}^{(1)} d\mathbf{k}^{(2)} \]

\[ = \int \tau(k^{(2)}) (k^{(2)}_m k^{(1)}_n - k^{(2)}_p k^{(1)}_n) \chi_{mp} \exp[i(k^{(1)} \cdot x + k^{(2)} \cdot y)] d\mathbf{k}^{(1)} d\mathbf{k}^{(2)}, \]

where \( \chi_{mn} = \langle a_m(k^{(2)}) h_n(k^{(1)}) \rangle. \) Since \( k^{(2)} = k + K/2 \) and \( k^{(1)} = -k + K/2, \) we obtain

\[ \alpha^{(B)}_{mn}(r = 0) = \int \tau(k) [k_m k_n \chi_{pp} - k_p k_m \chi_{np} - k_m K_n \chi_{pp} + k_n K_m \chi_{np} + K_m K_n \chi_{pp} + k_m k_n \chi_{np}] \exp[ik \cdot R] d\mathbf{k} d\mathbf{K}/\mu_0 \rho, \]

(C1)

where \( \rho \) is the fluid density, and \( \mu_0 \) is the magnetic permeability. Equation (C1) implies that the main contribution to the tensor for the magnetic part of the \( \alpha \) effect is from the trace for the magnetic helicity tensor, i.e., \( \alpha^{(B)}_{mn}(r = 0) \) is within interval \( k_0 < k < k_{\chi} \), \( \chi(R) = \int \chi(k, R) dk, \) and \( k_0 = l_0^{-1}. \) The correlation time is \( \tau(k) = 2 \tau_0 (k/k_0)^{1-q}. \) The integration in the equation for \( \alpha^{(B)}_{mn}(r = 0) \) yields

\[ \alpha^{(B)}_{mn}(r = 0) \sim \frac{\chi(R) (q - 1)}{9(2-q) \eta T \mu_0 \rho} \left[ \frac{k_{\chi}}{k_0} \right]^{4 - 2q} \cdot \delta_{mn}. \]

(C2)

The realizability condition causes \( k_{\chi} = k_0, \) i.e., the magnetic helicity is localized at the maximum scale of turbulent motions (see, e.g., [1,2]). Therefore Eq. (C2) yields Eq. (16).