

## Tangling clustering instability for small particles in temperature stratified turbulence

T. Elperin, N. Kleeorin, M. Liberman, and I. Rogachevskii

Citation: *Phys. Fluids* **25**, 085104 (2013); doi: 10.1063/1.4816643

View online: <http://dx.doi.org/10.1063/1.4816643>

View Table of Contents: <http://pof.aip.org/resource/1/PHFLE6/v25/i8>

Published by the AIP Publishing LLC.

---

### Additional information on Phys. Fluids

Journal Homepage: <http://pof.aip.org/>

Journal Information: [http://pof.aip.org/about/about\\_the\\_journal](http://pof.aip.org/about/about_the_journal)

Top downloads: [http://pof.aip.org/features/most\\_downloaded](http://pof.aip.org/features/most_downloaded)

Information for Authors: <http://pof.aip.org/authors>

### ADVERTISEMENT



## Running in Circles Looking for the Best Science Job?

Search hundreds of exciting  
new jobs each month!

<http://careers.physicstoday.org/jobs>

physicstodayJOBS



## Tangling clustering instability for small particles in temperature stratified turbulence

T. Elperin,<sup>1,a)</sup> N. Kleeorin,<sup>1,2,3,b)</sup> M. Liberman,<sup>2,4,c)</sup> and I. Rogachevskii<sup>1,2,3,d)</sup>

<sup>1</sup>The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, P. O. Box 653, Beer-Sheva 84105, Israel

<sup>2</sup>Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, 10691 Stockholm, Sweden

<sup>3</sup>Department of Radio Physics, N. I. Lobachevsky State University of Nizhny Novgorod, Russia

<sup>4</sup>Moscow Institute of Physics and Technology, Dolgoprudnyi 141700, Russia

(Received 4 February 2013; accepted 3 July 2013; published online 13 August 2013)

We study tangling clustering instability of inertial particles in a temperature stratified turbulence with small finite correlation time. It is shown that the tangling mechanism in the temperature stratified turbulence strongly increases the degree of compressibility of particle velocity field. This results in the strong decrease of the threshold for the excitation of the tangling clustering instability even for small particles. The tangling clustering instability in the temperature stratified turbulence is essentially different from the inertial clustering instability that occurs in non-stratified isotropic and homogeneous turbulence. While the inertial clustering instability is caused by the centrifugal effect of the turbulent eddies, the mechanism of the tangling clustering instability is related to the temperature fluctuations generated by the tangling of the mean temperature gradient by the velocity fluctuations. Temperature fluctuations produce pressure fluctuations and cause particle accumulations in regions with increased instantaneous pressure. It is shown that the growth rate of the tangling clustering instability is by  $\sqrt{\text{Re}} (\ell_0/L_T)^2/(3\text{Ma})^4$  times larger than that of the inertial clustering instability, where  $\text{Re}$  is the Reynolds number,  $\text{Ma}$  is the Mach number,  $\ell_0$  is the integral turbulence scale, and  $L_T$  is the characteristic scale of the mean temperature variations. It is found that depending on the parameters of the turbulence and the mean temperature gradient there is a preferential particle size at which the particle clustering due to the tangling clustering instability is more effective. The particle number density inside the cluster after the saturation of this instability can be by several orders of magnitude larger than the mean particle number density. It is also demonstrated that the evaporation of droplets drastically changes the tangling clustering instability, e.g., it increases the instability threshold in the droplet radius. The tangling clustering instability is of a great importance, e.g., in atmospheric turbulence with temperature inversions. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4816643>]

### I. INTRODUCTION

Formation of spatial inhomogeneities in the number density distribution of small inertial particles in a turbulent flow (also called particle clustering or preferential concentration) has attracted considerable attention in the past decades.<sup>1–5</sup> The enhanced number density of particles inside the cluster may affect the particle interactions, their dynamics, and collisions. The dynamics of particle

<sup>a)</sup>Electronic mail: [elperin@bgu.ac.il](mailto:elperin@bgu.ac.il). URL: <http://www.bgu.ac.il/me/staff/tov>.

<sup>b)</sup>Electronic mail: [nat@bgu.ac.il](mailto:nat@bgu.ac.il).

<sup>c)</sup>Electronic mail: [misha.liberman@gmail.com](mailto:misha.liberman@gmail.com). URL: <http://michael-liberman.com/>.

<sup>d)</sup>Electronic mail: [gary@bgu.ac.il](mailto:gary@bgu.ac.il). URL: <http://www.bgu.ac.il/~gary>.

collisions is relevant to many phenomena in nature such as the raindrop formation and atmospheric aerosols dynamics,<sup>6-9</sup> as well as to numerous industrial processes involving, e.g., sprays in diesel and jet engines.<sup>10</sup> Formation of clusters with enhanced number density of particles may increase the rate of particle collisions and coalescence.<sup>6,8</sup> This can significantly modify the size and velocity distributions of the droplets in the spray and affect the combustor performance.

Clustering of inertial particles in a turbulent flow is caused by the centrifugal effect, which implies that the inertial particles are locally accumulated in regions between the turbulent eddies. These regions have a low vorticity, high strain rate, and maximum fluid pressure. Therefore, turbulent vortices act as small centrifuges that push heavy particles to the boundary regions between the eddies by the inertial forces creating concentration inhomogeneities. This effect is known as the inertia-induced particle clustering.<sup>11</sup> The inertial particle clustering in a turbulent flow has been studied analytically,<sup>12-20</sup> numerically,<sup>21-28</sup> and experimentally.<sup>2,29-33</sup>

In these study we distinguish between two types of particle clustering. The first type is source clustering related to the source term in the equation for fluctuations of the particle number density. Most of analytical studies of preferential concentration are related to the source inertial clustering. Another type of particle clustering is associated with a spontaneous breakdown of their homogeneous spatial distribution due to the clustering instability.<sup>15,17</sup> The clustering instability can be of great importance in different practical applications involving particle mixing and transport.

In the temperature stratified turbulence the particle clustering is affected by turbulent thermal diffusion.<sup>34,35</sup> This phenomenon causes accumulation of the inertial particles in the vicinity of the mean temperature minimum and results in the formation of inhomogeneous mean particle number density distributions. Turbulent thermal diffusion is a purely collective phenomenon occurring in temperature stratified turbulence and resulting in the appearance of a non-zero mean effective velocity of particles in the direction opposite to the mean temperature gradient. A competition between the turbulent thermal diffusion and turbulent diffusion determines the conditions for the formation of large-scale particle concentrations in the vicinity of the mean temperature minimum. The phenomenon of turbulent thermal diffusion has been studied analytically,<sup>34-40</sup> investigated by means of direct numerical simulations,<sup>41</sup> and detected in the laboratory experiments in stably and unstably stratified turbulent flows,<sup>42,43</sup> and also observed in atmospheric turbulence.<sup>40</sup>

Particle clustering in the temperature stratified turbulence can be much more effective than the inertial particle clustering in isothermal turbulence.<sup>44</sup> The reason for this is that the mean temperature gradient in turbulent flow is a strong source of the temperature fluctuations which are correlated with the fluctuations of fluid velocity and pressure. The pressure fluctuations increase fluctuations of the particles number density and enhance the rate of formation of the particle clusters. Moreover, tangling of the mean gradient of particle number density (formed by the turbulent thermal diffusion) generates additional fluctuations of particle concentrations and contributes to the particle clustering.

The steady-state regime of the tangling clustering (i.e., the source tangling clustering) in temperature stratified turbulence without excitation of instability has been recently studied experimentally and theoretically in Ref. 44. It was demonstrated that in the laboratory stratified turbulence the source tangling clustering is much more effective than a pure inertial clustering that has been observed in isothermal turbulence. In particular, in the experiments in oscillating grid isothermal turbulence in air without imposed mean temperature gradient, the inertial clustering is very weak for solid particles with the diameter  $\approx 10 \mu\text{m}$  and Reynolds numbers based on turbulent length scale and rms velocity,  $\text{Re} = 250$ . In the experiments<sup>44</sup> the correlation function for the inertial clustering in isothermal turbulence is significantly less localized than that for the tangling clustering in non-isothermal turbulence. The source tangling clustering was studied in Ref. 44 for inertial particles with small Stokes numbers and with the material density that is much larger than the fluid density.

The goal of the present paper is to investigate theoretically another regime of the tangling clustering, i.e., to study the tangling clustering instability in the temperature stratified turbulence. In this paper we show that the tangling mechanism in the temperature stratified turbulence strongly increases the degree of compressibility of particle velocity field and considerably enhances the growth rate of the tangling clustering instability. For small particles the tangling clustering instability may result in the formation of small-scale particle clusters with the number density of particles exceeding the ambient average particle number density by several orders of magnitude.

The paper is organized as follows. The large-scale effects in particle transport in temperature stratified turbulence are discussed in Sec. II. The governing equations for analysis of instability are given in Sec. III. Solutions for the tangling clustering instability without the source term in the equation for the second moment of particle number density and with the source term are analyzed in Secs. IV and V, respectively. The instability growth rate and saturated value of the particle number density inside a cluster are determined in Secs. IV and V. In Sec. VI we take into account an effect of droplet evaporation on tangling clustering instability. Finally, in Sec. VII we draw conclusions and discuss the implications of the tangling clustering instability.

## II. PARTICLES IN TEMPERATURE STRATIFIED TURBULENCE

### A. Governing equations

Advection-diffusion equation for the number density  $n_p(t, \mathbf{x})$  of inertial particles in a turbulent flow reads<sup>45,46</sup>

$$\frac{\partial n_p}{\partial t} + \nabla \cdot (n_p \mathbf{v}_p) = D_m \Delta n_p, \quad (1)$$

where  $D_m$  is the coefficient of molecular (Brownian) diffusion and  $\mathbf{v}_p(t, \mathbf{x})$  is the instantaneous particle velocity field. We use a mean field approach in which the particle number density and velocity, the fluid temperature, density, and pressure are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. Averaging Eq. (1) over an ensemble of turbulent velocity fields we obtain an equation for the mean number density of particles

$$\frac{\partial N}{\partial t} + \nabla \cdot (N \mathbf{V}_p + \langle n \mathbf{v} \rangle) = D_m \Delta N, \quad (2)$$

where  $N = \langle n_p \rangle$  is the mean particle number density, the angular brackets imply ensemble averaging,  $\langle n \mathbf{v} \rangle$  is the turbulent flux of particles,  $\mathbf{v}(t, \mathbf{x})$  are the fluctuations of the particle velocity field, and  $\mathbf{V}_p$  is the particle mean velocity. To obtain a closed mean-field equation one needs to determine the turbulent flux of particles. The equation for fluctuations of the particle number density,  $n(t, \mathbf{x}) = n_p(t, \mathbf{x}) - N(t, \mathbf{x})$ , then reads

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v} - \langle n \mathbf{v} \rangle) - D_m \Delta n = -(\mathbf{v} \cdot \nabla) N - N \nabla \cdot \mathbf{v}. \quad (3)$$

### B. Turbulent thermal diffusion

Turbulent thermal diffusion results in formation of a nonzero gradient of the mean particle number density  $\nabla N$  in temperature stratified turbulence.<sup>34,35</sup> The physical mechanism of turbulent thermal diffusion is as follows. For the particles with the material density  $\rho_m \gg \rho$ , their velocity is determined by

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u} - \mathbf{v}}{\tau_s} + \mathbf{g}, \quad (4)$$

where  $\mathbf{u}$  is the fluid velocity field,  $\mathbf{g}$  is the acceleration of gravity,  $\tau_s = m_p/6\pi\rho\nu a_p$  is the Stokes time for the small spherical particles of the radius  $a_p$  and mass  $m_p$ ,  $\nu$  is the kinematic viscosity, and  $\rho$  is the mean fluid density. For small Stokes numbers,  $St = \tau_s/\tau_\eta \ll 1$ , solution of Eq. (4) has the following form (see, e.g., Ref. 11):

$$\mathbf{v} = \mathbf{u} - \tau_s \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{g} \right] + O(\tau_s^2), \quad (5)$$

or introducing the dimensionless units, it can be written in the dimensionless form

$$\mathbf{v} = \mathbf{u} - St \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{\tau_\eta^2 \mathbf{g}}{\ell_\eta} \right] + O(St^2), \quad (6)$$

where the distance is measured in the Kolmogorov viscous scale units,  $\ell_\eta = \ell_0/\text{Re}^{3/4}$ , and the time is measured in the Kolmogorov time scale units,  $\tau_\eta = \tau_0/\text{Re}^{1/2}$ . Here  $\text{Re} = \ell_0 u_0/\nu$  is the fluid Reynolds numbers,  $u_0$  is the characteristic turbulent velocity at the integral scale  $\ell_0$  of turbulent motions and  $\tau_0 = \ell_0/u_0$ . The terms in squared brackets in Eq. (5) describe the difference between the local fluid velocity and particle velocity arising due to the small but finite inertia of the particle. For the turbulent flow with low Mach numbers:  $\nabla \cdot \mathbf{u} \approx -\rho^{-1}(\mathbf{u} \cdot \nabla)\rho \neq 0$ . The equation for  $\nabla \cdot \mathbf{v}$  can be easily obtained from Eq. (5) and the Navier-Stokes equation

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{u} - \tau_s \nabla \cdot \left( \frac{d\mathbf{u}}{dt} \right) + O(\tau_s^2) = -\frac{1}{\rho}(\mathbf{u} \cdot \nabla)\rho + \frac{\tau_s}{\rho} \nabla^2 p + O(\tau_s^2), \quad (7)$$

where  $p$  are the fluid pressure fluctuations. Due to inertia, particles inside the turbulent eddies drift out to the boundary regions between the eddies. These are regions with small velocity fluctuations and maximum pressure fluctuations. Consequently, particles are accumulated in the regions with the maximum pressure fluctuations of the turbulent fluid, i.e.,  $\nabla \cdot \mathbf{v} \propto (\tau_s/\rho) \nabla^2 p \neq 0$  even for the incompressible fluid. For large Peclet numbers, when the molecular diffusion of particles in Eq. (1) can be neglected, we can estimate  $\nabla \cdot \mathbf{v} \propto -dn_p/dt$ . Therefore, inertial particles are accumulated (i.e.,  $dn_p/dt \propto -(\tau_s/\rho) \nabla^2 p > 0$ ) in regions with maximum pressure of turbulent fluid, where  $\nabla^2 p < 0$ . Similarly, there is an outflow of inertial particles from the regions with the minimum pressure of fluid. In case of homogeneous and isotropic turbulence a drift from regions with increased (decreased) concentration of particles by a turbulent flow of fluid is equiprobable in all directions.

On the contrary, in a temperature stratified turbulence, the turbulent heat flux  $\langle \mathbf{u} \theta \rangle$  does not vanish. This implies that the fluctuations of fluid temperature,  $\theta$ , and velocity are correlated, and, therefore, fluctuations of pressure are correlated with the fluctuations of velocity due to a non-zero turbulent heat flux,  $\langle \mathbf{u} \theta \rangle \neq 0$ . The increased pressure of the surrounding fluid is accompanied by the particles accumulation, and the direction of the mean flux of particles coincides with the direction of the heat flux towards the minimum of the mean temperature.<sup>34,44</sup>

Equation for the mean number density  $N$  of particles reads

$$\frac{\partial N}{\partial t} + \nabla \cdot [N(\mathbf{V} + \mathbf{W}_g) + \mathbf{F}^{(n)}] = D_m \Delta N, \quad (8)$$

where  $\mathbf{V}_p = \mathbf{V} + \mathbf{W}_g$  is the mean particle velocity,  $\mathbf{V}$  is the mean fluid velocity,  $\mathbf{W}_g = \tau_s \mathbf{g}$  is the terminal fall velocity of particles,  $D_m = k_B T / 6\pi \rho \nu a_p$  is the coefficient of molecular (Brownian) diffusion,  $k_B$  is the Boltzman constant, and  $T$  and  $\rho$  are the fluid mean temperature and density, respectively. Hereafter for simplicity we consider the case of a zero mean fluid velocity,  $\mathbf{V} = 0$ . The turbulent flux of particles,  $\mathbf{F}^{(n)} = \langle n \mathbf{v} \rangle$ , includes contributions of turbulent thermal diffusion and turbulent diffusion, i.e.,

$$\mathbf{F}^{(n)} = \mathbf{V}^{\text{eff}} N - D_T \nabla N. \quad (9)$$

Here  $D_T \approx \ell_0 u_0$  is the coefficient of turbulent diffusion,  $\mathbf{V}^{\text{eff}}$  is the effective pumping velocity caused by the turbulent thermal diffusion

$$\mathbf{V}^{\text{eff}} = -\tau \langle \mathbf{v} (\nabla \cdot \mathbf{v}) \rangle, \quad (10)$$

where  $\tau$  is the turbulent correlation time. Equation (10) for the effective velocity has been derived using different methods in Refs. 34 and 36–40. The expression (10) can be obtained in a simple way using the dimensional consideration. Estimating the left-hand side of Eq. (3) as

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v} - \langle n \mathbf{v} \rangle) - D_m \nabla^2 n \approx \frac{n}{\tau}, \quad (11)$$

we obtain an expression for the turbulent component  $n$  of the particle number density

$$n \approx -\tau \nabla \cdot (N \mathbf{v}) = -\tau [N(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)N]. \quad (12)$$

Therefore, the turbulent flux of particles  $F_i^{(n)} = \langle v_i n \rangle$  is given by the following expression:

$$F_i^{(n)} = -N \tau \langle v_i (\nabla \cdot \mathbf{v}) \rangle - \tau \langle v_i v_j \rangle \nabla_j N, \quad (13)$$

where the first term in the right-hand side of Eq. (13) determines the turbulent flux of particles due to the turbulent thermal diffusion:  $-N \tau \langle v_i (\nabla \cdot \mathbf{v}) \rangle = V_i^{\text{eff}} N$ , and the second term in the right-hand side of Eq. (13) describes the contribution of turbulent diffusion:  $\tau \langle v_i v_j \rangle \nabla_j N = D_T \nabla_i N$ .

A detailed analysis,<sup>34,44</sup> using equation of state for an ideal gas and applying the identity  $\tau_s = \rho W_g L_P / P$ , yields the effective velocity in the following form:

$$\mathbf{V}^{\text{eff}} = -\alpha D_T \frac{\nabla T}{T}, \quad (14)$$

where  $\alpha$  is

$$\alpha = 1 + \frac{\text{St} \ln(\text{Re})}{\sqrt{\text{Re}} \text{Ma}^2}. \quad (15)$$

Here  $L_P^{-1} = |\nabla P|/P$ ,  $\text{Ma} = u_0/c_s$  is the Mach number, and  $c_s$  is the sound speed. For gases and non-inertial particles  $\alpha = 1$ . A steady-state solution of Eq. (8) is given by the following formula:

$$\frac{N(z)}{N_0} = \left[ \frac{T(z)}{T_0} \right]^{-\frac{\alpha D_T}{D_m + D_T}} \exp \left[ - \int_{z_0}^z \frac{W_g}{D_m + D_T} dz' \right], \quad (16)$$

where  $N_0 = N(z = z_0)$  and  $T_0 = T(z = z_0)$  are the mean number density of particles and the mean fluid temperature, respectively, calculated at the boundary  $z = z_0$ . If there is a gradient of temperature in a vertical  $z$  direction, Eq. (16) implies that small particles are accumulated in the vicinity of the mean temperature minimum. This causes formation of large-scale inhomogeneous distributions of the mean particle number density.

### III. GOVERNING EQUATIONS FOR ANALYSIS OF INSTABILITY

Let us study fluctuations of the particle number density. The methodology and approach used for investigation of the tangling clustering instability are similar to the methodology and approach used for study of the inertial clustering instability.<sup>15,17</sup> We apply the path-integral approach for random compressible flow with the small yet finite correlation time for the derivation of the equation for the correlation function of the particle number density. This approach is described comprehensively in Refs. 36 and 37. The equation for the two-point second-order correlation function of the particle number density fluctuations,

$$\Phi(t, \mathbf{R}) = \langle n(t, \mathbf{x}) n(t, \mathbf{x} + \mathbf{R}) \rangle,$$

is given by

$$\frac{\partial \Phi}{\partial t} = [B(\mathbf{R}) + 2\mathbf{U}^{(A)}(\mathbf{R}) \cdot \nabla + \hat{D}_{ij}(\mathbf{R}) \nabla_i \nabla_j] \Phi(t, \mathbf{R}) + I(\mathbf{R}), \quad (17)$$

where  $\mathbf{U}^{(A)}(\mathbf{R}) = (1/2)[\mathbf{U}(\mathbf{R}) - \mathbf{U}(-\mathbf{R})]$ ,

$$\hat{D}_{ij} = 2D_m \delta_{ij} + D_{ij}^T(0) - D_{ij}^T(\mathbf{R}), \quad (18)$$

$$D_{ij}^T(\mathbf{R}) \approx 2 \int_0^\infty \langle v_i[0, \boldsymbol{\xi}(t, \mathbf{x}|0)] v_j[\tau, \boldsymbol{\xi}(t, \mathbf{x} + \mathbf{R}|\tau)] \rangle d\tau, \quad (19)$$

$$B(\mathbf{R}) \approx 2 \int_0^\infty \langle b[0, \boldsymbol{\xi}(t, \mathbf{x}|0)] b[\tau, \boldsymbol{\xi}(t, \mathbf{x} + \mathbf{R}|\tau)] \rangle d\tau, \quad (20)$$

$$U_i(\mathbf{R}) \approx -2 \int_0^\infty \langle v_i[0, \boldsymbol{\xi}(t, \mathbf{x}|0)] b[\tau, \boldsymbol{\xi}(t, \mathbf{x} + \mathbf{R}|\tau)] \rangle d\tau \quad (21)$$

(for details of derivations, see Ref. 15). Here  $b = \text{div } \mathbf{v}$ ,  $D_{ij}^T(\mathbf{R})$  is the scale-dependent turbulent diffusion tensor,  $\delta_{ij}$  is the Kronecker tensor,  $I(\mathbf{R})$  is the source of particle number density fluctuations, and  $\langle \dots \rangle$  denotes averaging over the statistics of turbulent velocity field and the Wiener process

$\mathbf{w}(t)$ . The Wiener trajectory  $\xi(t, \mathbf{x}|s)$  in the expressions for the turbulent diffusion tensor  $D_{ij}^T(\mathbf{R})$  and other transport coefficients is defined as follows:

$$\xi(t, \mathbf{x}|s) = \mathbf{x} - \int_s^t \mathbf{v}[\tau, \xi(t, \mathbf{x}|\tau)] d\tau + \sqrt{2D_m} \mathbf{w}(t-s), \quad (22)$$

where  $\mathbf{w}(t)$  is the Wiener random process which describes the Brownian motion (molecular diffusion) and has the following properties:  $\langle \mathbf{w}(t) \rangle_w = 0$ ,  $\langle w_i(t+\tau)w_j(t) \rangle_w = \tau \delta_{ij}$ , and  $\langle \dots \rangle_w$  denotes the mathematical expectation over the statistics of the Wiener process. The velocity  $v_i[\tau, \xi(t, \mathbf{x}|\tau)]$  describes the Eulerian velocity calculated at the Wiener trajectory.

To simplify the averaging procedure in derivation of Eq. (17) we used a model of random velocity field which fully loses memory at random instants. The velocity fields before and after renewal are assumed to be statistically independent. Between the renewals the velocity field can be random with its intrinsic statistics. To obtain a statistically stationary random velocity field we assumed that the velocity fields between renewals have the same statistics. The random renewal instants destroy stationarity of the velocity field. On the other hand, between the random renewal instants the velocity field is stationary in statistical sense. To perform calculations in the closed form we assumed that the random renewal times are determined by a Poisson process. We also considered a model of a random velocity field where Lagrangian trajectories, i.e., the integrals  $\int v(\mu, \xi) d\mu$  and  $\int b(\mu, \xi) d\mu$  have Gaussian statistics.

This model employs three random processes: (i) the Wiener random process which describes Brownian motions, i.e., the molecular diffusion; (ii) Poisson process for random renewal times; and (iii) the random velocity field between the renewals. This model reproduces important features of some real turbulent flows. For example, the interstellar turbulence which is driven by supernovae explosions, loses memory in the instants of explosions (see, e.g., Ref. 47). Such flows also can be reproduced in direct numerical simulations.

Equation (17) with  $I(\mathbf{R}) = 0$  and for a delta-correlated in time random incompressible ( $b = 0$ ) velocity field was derived by Kraichnan.<sup>48</sup> In this case:  $B(\mathbf{R}) = \nabla_i \nabla_j \hat{D}_{ij}(\mathbf{R})$  and  $U_i^{(A)}(\mathbf{R}) = \nabla_j \hat{D}_{ij}(\mathbf{R})$ , and Eq. (17) is reduced to  $\partial \Phi / \partial t = \nabla_i \nabla_j [\hat{D}_{ij}(\mathbf{R}) \Phi(t, \mathbf{R})]$ . For a turbulent compressible flow with a finite correlation time Eq. (17) was derived using a stochastic calculus.<sup>15</sup> In particular, Wiener path integral representation of the solution of the Cauchy problem for Eq. (1), the Feynman-Kac formula, and Cameron-Martin-Girsanov theorem were used for the derivation of Eq. (17).<sup>36,37,49</sup>

The source function  $I(\mathbf{R})$  in Eq. (17) is related to the two source terms  $-(\mathbf{v} \cdot \nabla)N - N \nabla \cdot \mathbf{v}$  in the right-hand side of Eq. (3), and the explicit expression for  $I(\mathbf{R})$  is as follows (see Ref. 44):

$$I(\mathbf{R}) = B(\mathbf{R})N^2 + \mathbf{U}^{(S)}(\mathbf{R}) \cdot \nabla N^2 + \frac{3}{4} D_{ij}^T(\mathbf{R}) (\nabla_i N) (\nabla_j N), \quad (23)$$

where  $\mathbf{U}^{(S)}(\mathbf{R}) = (1/2)[\mathbf{U}(\mathbf{R}) + \mathbf{U}(-\mathbf{R})]$  and  $\nabla_i^{(x)} \nabla_j^{(y)} N(t, \mathbf{x}) N(t, \mathbf{y}) = (3/4) (\nabla_i N) (\nabla_j N)$ , and  $\nabla_i \equiv \nabla_i^{(\mathbf{R})}$ .

The meaning of the turbulent transport coefficients  $B(\mathbf{R})$  and  $\mathbf{U}(\mathbf{R})$  is as follows. The function  $B(\mathbf{R})$  is determined by the compressibility of the particle velocity field. The vector  $\mathbf{U}(\mathbf{R})$  determines a scale-dependent drift velocity which describes transport of fluctuations of particle number density from smaller scales to larger scales, i.e., in the regions with larger turbulent diffusion. The scale-dependent tensor of turbulent diffusion  $D_{ij}^T(\mathbf{R})$  is equal to the tensor of the molecular (Brownian) diffusion in very small scales, while in the vicinity of the maximum scale of turbulent motions it coincides with the tensor of turbulent diffusion. It should be noticed also that if  $\nabla N \neq 0$  a nonzero source term  $I(\mathbf{R})$  causes the production of the particle number density fluctuations due to the tangling of the mean particle number density by the turbulent velocity field.

## A. Degree of compressibility

If the turbulent velocity field is not delta-correlated in time (e.g., the correlation time is small yet finite), the tensor of turbulent diffusion,  $D_{ij}^T(\mathbf{R})$ , is compressible, i.e.,  $(\partial/\partial R_i) D_{ij}^T(\mathbf{R}) \neq 0$ . The

parameter  $\sigma_T$  that characterizes the degree of compressibility of the tensor of turbulent diffusion, is defined as follows:

$$\sigma_T \equiv \frac{\nabla_i \nabla_j D_{ij}^T(\mathbf{R})}{\nabla_i \nabla_j D_{mn}^T(\mathbf{R}) \epsilon_{imp} \epsilon_{jnp}} \approx \frac{\langle (\nabla \cdot \tilde{\xi})^2 \rangle}{\langle (\nabla \times \tilde{\xi})^2 \rangle}, \quad (24)$$

where  $\epsilon_{ijk}$  is the fully antisymmetric Levi-Civita unit tensor,  $\tilde{\xi} = \xi - \mathbf{x}$  with  $|t - s| \gg \tau_0$ . If the turbulent velocity field is a delta-correlated in time random process, then  $\nabla \cdot \tilde{\xi} = -(\nabla \cdot \mathbf{v})(t - s)$  and  $\nabla \times \tilde{\xi} = -(\nabla \times \mathbf{v})(t - s)$ , and, hence, the expression for  $\sigma_T$  reads

$$\sigma_T = \frac{\langle (\nabla \cdot \mathbf{v})^2 \rangle}{\langle (\nabla \times \mathbf{v})^2 \rangle} \equiv \sigma_v, \quad (25)$$

where  $\sigma_v$  is the degree of compressibility of the particle velocity field. The parameter  $\sigma_v$  depends on the Stokes number<sup>17</sup>

$$\sigma_v = \frac{(8/3)St^2}{1 + \beta St^2}, \quad (26)$$

where the coefficient  $\beta \sim 1$ .

For small finite correlation time of turbulent velocity field, i.e., for small Strouhal numbers,  $Sr = \tau_c \sqrt{\langle \mathbf{u}^2 \rangle} / \ell \ll 1$ , the parameter  $\sigma_T$  can be estimated using Eq. (C12) in Ref. 15 (see also Ref. 44) as follows:

$$\sigma_T = \sigma_v + \frac{2Sr^2}{3} \left( 1 + \frac{913\sigma_v^2}{12(1 + \sigma_v)} \right) + O(Sr^4), \quad (27)$$

where  $\tau_c$  is the correlation time of random velocity field. Here the condition  $Sr \ll 1$  is supposed to be valid in the viscous range of scales.

The mechanism of coupling related to the tangling of the gradient of the mean temperature gradient is quite robust. The properties of the tangling are not very sensitive to the exponent of the energy spectrum of the background turbulence. Anisotropy effects do not introduce new physics in the clustering process because the main contribution to the tangling clustering instability is at the Kolmogorov (viscous) scale of turbulent motions, where turbulence can be considered as nearly isotropic, while anisotropy effects can be essential in the vicinity of the maximum scales of the turbulent motions. Using these arguments, we consider the tensor  $D_{ij}^T(\mathbf{R})$  for isotropic and homogeneous turbulent flow in the following form:

$$D_{ij}^T(\mathbf{R}) = D_T \left[ [F(R) + F_c(R)] \delta_{ij} + RF'_c \frac{R_i R_j}{R^2} + \frac{RF'}{2} \left( \delta_{ij} - \frac{R_i R_j}{R^2} \right) \right], \quad (28)$$

where  $F(0) = 1 - F_c(0)$  and  $F' = dF/dR$ . The function  $F_c(R)$  describes the compressible (potential) component, whereas  $F(R)$  corresponds to vortical (incompressible) part of the turbulent diffusion tensor.

## B. Derivation of expression for the function $B(R)$

Taking into account the equation of state of an ideal gas we obtain  $p/P = \varrho/\rho + \theta/T + O(\varrho\theta/\rho T)$ . For small Stokes numbers,  $\nabla \cdot \mathbf{v} \approx (\tau_s/\rho) \nabla^2 p$ . This allows us to estimate  $B(\mathbf{R})$  as

$$B(\mathbf{R}) \approx \frac{2\tau_s^2}{\rho^2} \langle \tau [\nabla^2 p(\mathbf{x})] [\nabla^2 p(\mathbf{y})] \rangle \approx \frac{2\tau_s^2}{\rho^2} \frac{P^2}{T^2} \langle \tau [\nabla^2 \theta(\mathbf{x})] [\nabla^2 \theta(\mathbf{y})] \rangle, \quad (29)$$

where  $\rho, T, P$  and  $\varrho, \theta, p$  are the mean and fluctuations of the fluid density, temperature, and pressure, respectively, and  $\nabla^2 p(\mathbf{x}) = [\nabla^{(x)}]^2 p(\mathbf{x})$ . Hereafter we omit the argument  $t$  in the correlation function. In  $\mathbf{k}$  space the correlation function  $\langle \tau [\nabla^2 \theta(\mathbf{x})] [\nabla^2 \theta(\mathbf{y})] \rangle$  reads

$$\langle \tau [\nabla^2 \theta(\mathbf{x})] [\nabla^2 \theta(\mathbf{y})] \rangle = \int \bar{\tau}(k) k^4 \langle \theta(\mathbf{k}) \theta(-\mathbf{k}) \rangle \exp(i\mathbf{k} \cdot \mathbf{R}) d\mathbf{k}. \quad (30)$$

Taking into account that  $\langle \theta(\mathbf{k})\theta(-\mathbf{k}) \rangle = \langle \theta^2 \rangle \tilde{E}_\theta(k)/4\pi k^2$ , and integrating in  $\mathbf{k}$  space we arrive to the following expressions for the functions  $B(\mathbf{R})$ :

$$B(\mathbf{R}) \approx \frac{2 \text{St}^2 c_s^4}{3 \tau_\eta u_0^4} \left( \frac{\ell_0 \nabla T}{T} \right)^2 \text{Re}^{1/2}, \quad (31)$$

where  $\tilde{E}_\theta(k) = (2/3) k_0^{-1} (k/k_0)^{-5/3}$  is the spectrum function of the temperature fluctuations for  $k_0 \leq k \leq \ell_\eta^{-1}$ , with  $k_0 = \ell_0^{-1}$  and  $\tilde{\tau}(k) = 2\tau_0 (k/k_0)^{-2/3}$ . To determine  $\langle \theta^2 \rangle$  we used the budget equation for the temperature fluctuations  $E_\theta = \langle \theta^2 \rangle / 2$

$$\frac{DE_\theta}{Dt} + \text{div } \Phi_\theta = -(\mathbf{F} \cdot \nabla)T - \varepsilon_\theta, \quad (32)$$

which for homogeneous turbulence in a steady state yields

$$\langle \theta^2 \rangle = -2 \tau_0 (\mathbf{F} \cdot \nabla)T = \frac{2}{3} (\ell_0 \nabla T)^2, \quad (33)$$

where  $F_i = \langle u_i \theta \rangle = -D_T^{(\theta)} \nabla_i T$  is the turbulent heat flux,  $D_T^{(\theta)} = u_0 \ell_0 / 3$  is the coefficient of the turbulent diffusion of the temperature fluctuations, and the dissipation rate of  $E_\theta$  is  $\varepsilon_\theta = \langle \theta^2 \rangle / 2 \tau_0$  (see, e.g., Ref. 50).

On the other hand, the function  $B(\mathbf{R})$  at very small scales has a universal form

$$B(\mathbf{R}) = \frac{20 \sigma_v}{\tau_\eta (1 + \sigma_v)} \approx \frac{20 \sigma_v}{\tau_\eta} \approx \frac{160 \text{St}_{\text{eff}}^2}{3 \tau_\eta} \quad (34)$$

because at these scales the velocity field is smooth and nearly isotropic. Here we introduced the effective Stokes number using Eqs. (31) and (33)

$$\text{St}_{\text{eff}} = \text{St } \Gamma, \quad \Gamma(\text{Ma}, \text{Re}, \ell_0/L_T) = \left[ 1 + \frac{\text{Re}^{1/2}}{81 \text{Ma}^4} \left( \frac{\ell_0 \nabla T}{T} \right)^2 \right]^{1/2}, \quad (35)$$

where  $L_T$  is the characteristic scale of the mean temperature variations. The case of  $\Gamma = 1$  corresponds to the inertial clustering instability. To derive Eq. (34) we took into account that for a Gaussian velocity field<sup>15</sup>

$$\langle (\nabla \cdot \mathbf{v})^2 \rangle = \frac{80}{3 \tau_\eta^2} \text{St}_{\text{eff}}^2, \quad \langle (\nabla \times \mathbf{v})^2 \rangle = \frac{10}{\tau_\eta^2}. \quad (36)$$

Equation (36) yields

$$\sigma_v \equiv \frac{\langle (\nabla \cdot \mathbf{v})^2 \rangle}{\langle (\nabla \times \mathbf{v})^2 \rangle} = \frac{8}{3} \text{St}_{\text{eff}}^2 \ll 1. \quad (37)$$

Taking typical parameters for atmospheric turbulence  $\text{Re} = 10^7$ ,  $u_0 = 1$  m/s, and  $\ell_0 = 100$  m within the temperature inversion, such that the mean temperature gradient,  $|\nabla T|$ , is of the order of 1 K per 100 m, we obtain  $\Gamma \approx 2.5 \times 10^3$ , or for  $\text{Re} = 10^6$ ,  $u_0 = 0.3$  m/s, and  $\ell_0 = 30$  m the parameter  $\Gamma$  is  $\Gamma \approx 5 \times 10^3$ .

#### IV. SOLUTION FOR TANGLING CLUSTERING INSTABILITY WITH ZERO SOURCE TERM ( $I = 0$ )

It is convenient to rewrite Eq. (17) for the two-point second-order correlation function  $\Phi(R)$  in a non-dimensional form with coordinate  $R$  measured in units of the Kolmogorov scale,  $\ell_\eta$ , time in units of the Kolmogorov time,  $\tau_\eta$ , and the function  $\Phi$  in units of  $N^2$

$$\frac{\partial \Phi}{\partial t} = \frac{1}{M(R)} \left[ \Phi'' + 2 \left( \frac{1}{R} + \chi(R) \right) \Phi' \right] + 2\tilde{U}R \Phi' + B(R) \Phi + I(R), \quad (38)$$

where

$$\frac{1}{M(R)} = \frac{2}{\text{Sc}} + \frac{2}{3}[1 - F - (RF_c)'], \quad \chi(R) = -\frac{M(R)}{3}(F - 2F_c)', \quad (39)$$

$$I(R) = \frac{4(\alpha - 1)^2}{3} \left( \frac{\ell_0 \nabla T}{T} \right)^2 \left( 1 - \frac{3}{2M(R)} - \frac{R \chi(R)}{M(R)} \right) + B(R), \quad (40)$$

$\text{Sc} = \nu/D_m$  is the Schmidt number and  $\mathbf{U} = \tilde{U}(R) \mathbf{R}$ . Typically, in many applications, e.g., in the atmospheric turbulence,  $\text{Sc} \gg 1$  for small inertial particles. The two-point correlation function  $\Phi(R)$  satisfies the following boundary conditions:  $\Phi'(R = 0) = 0$  and  $\Phi(R \rightarrow \infty) = 0$ . This function has a global maximum at  $R = 0$  and therefore it satisfies the conditions

$$\Phi''(R = 0) < 0, \quad \Phi(R = 0) > |\Phi(R > 0)|.$$

For a steady-state regime and in the absence of the tangling clustering instability the solution of Eq. (38) was obtained in Ref. 44 for  $I(R) \neq 0$ . In this section we will consider solution of Eq. (38) for the case of the tangling clustering instability and  $I(R) = 0$ . Since the Schmidt number,  $\text{Sc} = \nu/D_m \gg 1$ , the molecular diffusion scale is much less than the viscous Kolmogorov scale. A general form of the turbulent diffusion tensor in the viscous range of scales is obtained taking into account that  $F(R) = (1 - R^2)/(1 + \sigma_T)$  and  $F_c(R) = \sigma_T(1 - R^2)/(1 + \sigma_T)$ , which yields

$$D_{ij}^T(\mathbf{R}) = C_1 R^2 \delta_{ij} + C_2 R_i R_j, \quad (41)$$

$$C_1 = \frac{2(2 + \sigma_T)}{3(1 + \sigma_T)}, \quad C_2 = \frac{2(2\sigma_T - 1)}{3(1 + \sigma_T)}, \quad (42)$$

and the other functions in this range of scales are  $\tilde{U} = 20 \sigma_v / 3(1 + \sigma_v)$  and  $B = 20 \sigma_v / (1 + \sigma_v)$ .

In the *molecular diffusion range of scales*,  $a_p / \ell_\eta \leq R \leq 1 / \sqrt{\text{Sc}}$ , all terms  $\propto R^2$  are small and can be neglected. Note that we consider the case when the particle radius,  $a_p$ , is the minimum scale of the problem, so that  $a_p \leq \ell_D$ , where  $\ell_D = \ell_0 / \text{Pe}^{3/4} = \ell_\eta / \sqrt{\text{Sc}}$  is the molecular diffusion scale, and  $\text{Pe} = u_0 \ell_0 / D_m$  is the Peclet number. The solution of Eq. (38) in this range reads

$$\Phi(R) = \left( 1 - \frac{\text{Sc}(B - \gamma \tau_\eta)}{12} R^2 \right) \exp(\gamma t), \quad (43)$$

where  $B > \gamma \tau_\eta$  and  $\gamma$  is the growth rate of the tangling clustering instability.

In the *turbulent diffusion region of scales*,  $1 / \sqrt{\text{Sc}} \ll R \ll 1$ , the molecular diffusion term  $\propto 1/\text{Sc}$  is negligible, and we seek for the solution of Eq. (38) in this region in the following form:

$$\Phi(R) \propto R^{-\beta} \exp(\gamma t). \quad (44)$$

Using the Corrsin integral,  $\int_0^\infty R^2 \Phi(R) dR = 0$ , we obtain that  $\beta = \lambda \pm i\kappa$  is a complex number,  $\kappa^2 > 0$ , where

$$\kappa^2 = \frac{4(B - \gamma \tau_\eta)(C_1 + C_2) - (C_1 - C_2 + 2\tilde{U})^2}{4(C_1 + C_2)^2}, \quad (45)$$

$$\lambda = \frac{C_1 - C_2 + 2\tilde{U}}{2(C_1 + C_2)} = \frac{3 - \sigma_T}{2(1 + 3\sigma_T)} + \frac{10\sigma_v}{1 + \sigma_v} \left( \frac{1 + \sigma_T}{1 + 3\sigma_T} \right). \quad (46)$$

Hence the real part of solution (44) is reduced to

$$\Phi(R) = C R^{-\lambda} \cos(\kappa \ln R + \varphi) \exp(\gamma t), \quad (47)$$

where  $C$  is the constant.

Since the correlation function  $\Phi(R)$  has a global maximum in  $R = a_p / \ell_\eta \ll 1$ , the parameter  $\sigma_T \leq 3$ . The function  $\Phi(R)$  sharply decreases with the increase of  $R$ , for  $R \gg 1$ . The growth rate of the second moment of particles number density and the constant  $C$  can be obtained by matching the correlation function  $\Phi(R)$  and its first derivative  $\Phi'(R)$  at the boundaries of the above ranges,

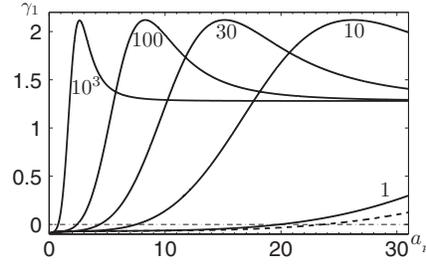


FIG. 1. The growth rate  $\gamma_1$  of the tangling clustering instability (in units of  $1/\tau_\eta$ ) of the first mode ( $m = 1$ ) versus the particle radius  $a_p$  for different values of parameter  $\Gamma$ , and  $\sigma_T = 1$ ,  $Sc = 10^6 a_p$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ . The dashed line corresponds to the inertial clustering instability ( $\Gamma = 1$ ).

i.e., in the points  $R = 1/\sqrt{Sc}$  and  $R = 1$ . The matching yields  $\kappa/2(C_1 + C_2) \approx \pi m/\ln Sc$  (where  $m = 2k + 1$ ), and the growth rate for the  $m$ th mode of the tangling clustering instability is given by the following formula:

$$\gamma_m = \frac{1}{3(1 + 3\sigma_T)} \left[ \frac{200\sigma_v(\sigma_T - \sigma_v)}{(1 + \sigma_v)^2} - \frac{(3 - \sigma_T)^2}{2(1 + \sigma_T)} - \frac{2\pi^2 m^2 (1 + 3\sigma_T)^2}{(1 + \sigma_T) \ln^2 Sc} \right], \quad (48)$$

where  $\sigma_v$  is given by Eq. (37),  $\sigma_T \approx 1$ , and  $m = 1, 2, 3, \dots$ . The first mode ( $m = 1$ ) has the minimum threshold for the excitation of the tangling clustering instability.

The tangling clustering instability depends on the ratio  $\sigma_T/\sigma_v$ . For the  $\delta$ -correlated in time random Gaussian compressible velocity field  $\sigma_T = \sigma_v$  (for details, see Refs. 15). In this case the second moment  $\Phi(t, \mathbf{R})$  can only decay, in spite of the compressibility of the velocity field. On the contrary, for the finite correlation time of the turbulent velocity field  $\sigma_T \neq \sigma_v$ , and the correlation function  $\Phi(R)$  grows exponentially in time, i.e., the tangling clustering instability is excited.

Figures 1 and 2 show the growth rate [see Eq. (48)] of the tangling clustering instability of the first mode ( $m = 1$ ) versus the particle radius  $a_p$  for different values of parameter  $\Gamma$ . One can see from these figures that the characteristic time of the tangling clustering instability is of the order of the Kolmogorov time scale (the growth rate  $\gamma_1$  in Figs. 1 and 2 is measured in units of the inverse Kolmogorov time). Remarkably, for every parameter  $\Gamma$  (Re, Ma) there is a rather sharp maximum of the function  $\gamma_1(a_p)$ . This implies that depending on the parameters of the turbulence there is a preferential particle size for which the particle clustering due to the excitation of the tangling clustering instability is much faster than for other values of the particle size. Moreover, the growth rate of the tangling clustering instability is much larger than that of the inertial clustering instability (the growth rate for the inertial clustering instability in turbulence with a zero mean temperature gradient is shown in Fig. 1 by the dashed line).

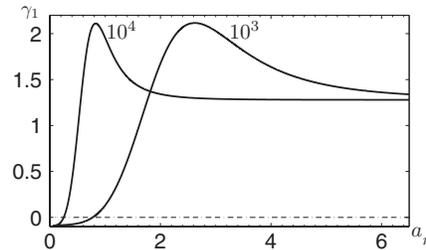


FIG. 2. The growth rate  $\gamma_1$  of the tangling clustering instability of the first mode ( $m = 1$ ) versus the particle radius  $a_p$  for larger values of parameter  $\Gamma$ , and  $\sigma_T = 1$ ,  $Sc = 10^6 a_p$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ .

## V. SOLUTION FOR TANGLING CLUSTERING INSTABILITY WITH NON-ZERO SOURCE TERM ( $I \neq 0$ )

In this section we obtain solution of Eq. (38) which includes both the tangling clustering instability and the source term for the tangling clustering. This implies that we consider solution of Eq. (38) in the vicinity of the thresholds of the excitation of the tangling clustering instability.

Let us consider the *turbulent diffusion range of scales*,  $1/\sqrt{\text{Sc}} \ll R \ll 1$  and introduce the following function  $\Psi(t, R) = \Phi(t, R)R^{1+\mu}$ , where

$$\mu = \frac{1 + \sigma_T}{1 + 3\sigma_T} \left( 1 - 2\sigma_T + \frac{10\sigma_v}{1 + \sigma_v} \right). \quad (49)$$

Equation (38) is reduced to the Schrödinger type equation

$$\frac{\partial \Psi(t, R)}{\partial t} = \frac{1}{M(R)} \Psi'' - \mathcal{U} \Psi + I(R), \quad (50)$$

with the potential  $\mathcal{U}$  and  $1/M(R)$  in the form

$$\mathcal{U} = -\frac{2}{3} \left[ \frac{40\sigma_v}{1 + \sigma_v} \left( 1 + \frac{5\sigma_v}{(1 + \sigma_v)(1 + 3\sigma_T)} \right) + (2\sigma_T - 1) \left( 1 - (2\sigma_T - 1) \frac{1 + \sigma_T}{1 + 3\sigma_T} \right) \right], \quad (51)$$

$$\frac{1}{M(R)} = \frac{2(1 + 3\sigma_T)}{3(1 + \sigma_T)} R^2, \quad I(R) = B R^{1+\mu}, \quad (52)$$

and  $B = 20\sigma_v/(1 + \sigma_v)$ . Here we took into account that the main contribution to the source term  $I(R)$  for large Reynolds numbers is due to the first term  $B(\mathbf{R})N^2$  in Eq. (23). Other contributions are negligible  $\sim 10\text{Re}^{-3/2} \ln^2 \text{Re}$  [see Eqs. (15) and (35)–(37)].

We choose the initial conditions which correspond to a turbulence without particle clusters:  $\Psi(t = 0, R) = 0$ . Now we seek a solution of Eq. (50) in the following form:  $\Psi(t, R) = \sum_{m=1}^{\infty} f_m(t) \Psi_m(R)$  and  $I(R) = B \sum_{m=1}^{\infty} A_m \Psi_m(R)$ , where  $\Psi_m(R)$  are the eigenfunctions determined by the following equation with  $I(R) = 0$ :

$$\frac{1}{M(R)} \Psi_m'' - (\mathcal{U} + \gamma_m) \Psi_m = 0. \quad (53)$$

Consequently, the function  $f_m(t)$  is determined by the following equation:

$$\frac{\partial f_m}{\partial t} = \gamma_m f_m + B A_m. \quad (54)$$

Taking into account the orthogonality of the eigenfunctions,

$$\frac{\int_0^{\infty} M(R) \Psi_m(R) \Psi_n(R) dR}{\int_0^{\infty} M(R) \Psi_m^2(R) dR} = \delta_{mn}, \quad (55)$$

and solving Eq. (54) we obtain the expressions for the function  $f_m(t)$  and  $A_m$

$$f_m(t) = \frac{20\sigma_v A_m}{(1 + \sigma_v) \gamma_m} [\exp(\gamma_m t) - 1], \quad (56)$$

$$A_m = \frac{\int_0^{\infty} M(R) R^{1+\mu} \Psi_m(R) dR}{\int_0^{\infty} M(R) \Psi_m^2(R) dR}. \quad (57)$$

Taking into account that  $\Phi_m(R = a_p) = [\Psi_m(t, R) R^{-(1+\mu)}]_{R=a_p} = 1$  we obtain the correlation function  $\Phi(R = a_p, t)$  in the form

$$\Phi(R = a_p, t) = \frac{20\sigma_v}{1 + \sigma_v} \sum_{m=1}^{\infty} \frac{A_m}{\gamma_m} [\exp(\gamma_m t) - 1]. \quad (58)$$

When the tangling clustering instability is excited, it causes formation of a cluster with the particle number density inside the cluster, which is much larger than the mean particle number density.

The solution for  $\Phi(t, R)$  [see Eq. (47)], which satisfies the above conditions, has the following dimensional form:

$$\Phi(t, R) = N^2 \frac{20\sigma_v}{1 + \sigma_v} \sum_{m=1}^{\infty} \frac{A_m}{\gamma_m} \left(\frac{R}{\ell_D}\right)^{-\lambda} \cos\left[\kappa \ln\left(\frac{R}{\ell_D}\right)\right] [\exp(\gamma_m t) - 1]. \quad (59)$$

Here the correlation function  $\Phi(t, R)$  has the global maximum at  $R = \ell_D/\ell_\eta$ , i.e., we assumed that in the molecular diffusion region the correlation function  $\Phi(t, R)$  is nearly constant,  $\Phi(t, R) \approx 1$  [see Eq. (43)]. The first minimum of the correlation function  $\Phi(t, R)$  for the mode  $m = 1$  is located in  $R = R_{\min} = \exp(1/\lambda)$ , and it is given by the following expression:

$$\frac{\Phi_{\min}}{N^2} = -\frac{B A_1 \kappa}{e \gamma_1 \lambda} \text{Sc}^{-\lambda/2} [\exp(\gamma_1 t) - 1], \quad (60)$$

where we took into account that  $\cot[\kappa \ln(R_{\min}/\ell_D)] = -\kappa/\lambda$ . On the other hand, the maximum value of the correlation function  $\Phi(t, R)$  is

$$\frac{\Phi_{\max}}{N^2} = \frac{B A_1}{\gamma_1} [\exp(\gamma_1 t) - 1], \quad (61)$$

where  $\Phi_{\max} = \Phi(R = a_p, t)$  [see Eq. (58)]. Equations (60) and (61) yield

$$\frac{\Phi_{\min}}{\Phi_{\max}} = -\frac{\pi}{e \lambda} \left(\frac{\text{Sc}^{-\lambda/2}}{\ln \text{Sc}}\right). \quad (62)$$

Since  $n_p = N + n \geq 0$ , the function  $\Phi(t, \mathbf{R}) \equiv \langle n(t, \mathbf{x})n(t, \mathbf{y}) \rangle = \langle n_p(t, \mathbf{x})n_p(t, \mathbf{y}) \rangle - N^2 \geq -N^2$ . Therefore, the minimal possible value of the function  $\Phi(t, \mathbf{R})$  is  $\Phi_{\min} = -N^2$ . This condition together with Eq. (62) allow us to estimate the maximum number density of particles attained inside the cluster during the tangling clustering instability

$$\frac{n_p^{\max}}{N} = \left(1 + \frac{e \lambda}{\pi} \text{Sc}^{\lambda/2} \ln \text{Sc}\right)^{1/2}. \quad (63)$$

The maximum value of the particle number density inside the cluster,  $n_{\max}/N$ , versus the particle radius is shown in Fig. 3 for different values of parameter  $\Gamma$ . The discontinuity of the first derivative of  $n_{\max}/N$  which is seen in Fig. 3 is related to the transition from one mechanism of particle tangling clustering due to the source term to another mechanism caused by the tangling clustering instability. The exponential growth at the linear stage of the instability is saturated by the nonlinear effects. The values of  $n_{\max}/N$  in Fig. 3 are calculated using Eq. (63) that takes into account possible saturation of the tangling clustering instability caused by the exhaustion of the particles in the region surrounding the cluster. Inspection of Fig. 3 shows that the particle number density inside the cluster can increase by a factor of  $10^4$  in comparison with the mean particle number density.

There are also other mechanisms leading to the nonlinear saturation of the tangling clustering instability discussed in detail in Ref. 15. However, as follows from our analysis the main significant

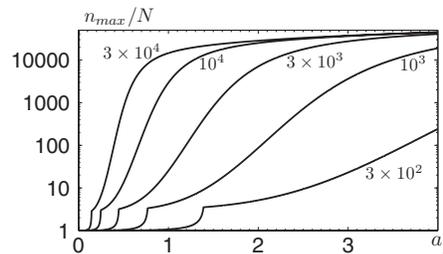


FIG. 3. The particle number density inside the cluster  $n_{\max}/N$  versus the particle radius  $a_p$  for different values of parameter  $\Gamma$ , and  $\sigma_T = 1$ ,  $\text{Sc} = 10^6 a_p$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ .

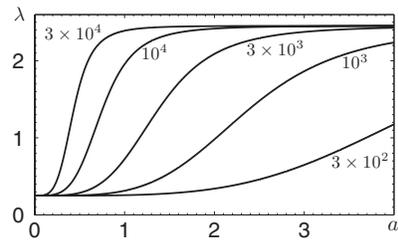


FIG. 4. The exponent  $\lambda$  versus the particle radius  $a_p$  for different values of parameter  $\Gamma$  and  $\sigma_T = 1$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ .

mechanism of saturation of the growth of the tangling clustering instability for small particles is exhaustion of the particles in the surrounding area. Indeed, the tangling clustering instability causes strong redistribution of particles so that inside the clusters the particle number density strongly increases at some instant, while in the surrounding regions it decreases. With the decrease of the number density of particles the hydrodynamic description becomes inapplicable. It should be noted that we consider situation when there is only the redistribution of the particles without their creation or annihilation. This implies that particles from the cluster vicinity are concentrated in the central part of it, which can be expressed using the Corrsin integral of the correlation function of the particle number density fluctuations:  $\int_0^\infty \Phi(t, R) R^2 dR = 0$ . This condition implies that the tail of the correlation function  $\Phi(t, R)$  must be negative [i.e., there is the anti-correlation tail of the function  $\Phi(t, R)$ ]. The transition from central positive part  $\Phi(t, R) > 0$  to the negative tail of  $\Phi(t, R)$  occurs at the distance that is of the order of several Kolmogorov scales. Note, that in contrast to the inertial clustering, the tangling clustering instability accumulates particles into the cluster from the scales which are much larger than the Kolmogorov length scale. The reason is that the tangling mechanism generates fluctuations of the particle number density in all scales of inertial range in turbulence with imposed mean temperature gradient. Consequently, the concentration of particles inside the cluster increases due to the tangling clustering instability by several orders of magnitude.

The value  $n_{\text{max}}$  depends strongly on the Schmidt number  $Sc$  and on the exponent  $\lambda$  [see Eq. (63)]. The exponent  $\lambda$  depends on the degree of compressibility of the particle velocity field,  $\sigma_v \propto St^2 \Gamma(\text{Ma}, \text{Re}, \ell_0/L_T)$  [see Eq. (35)]. The calculated values of exponent  $\lambda$  versus the particle radius for different values of parameter  $\Gamma$  are shown in Fig. 4, that explains strong dependence of the maximum particle number density inside the cluster  $n_{\text{max}}$  on the particle radius and the parameter  $\Gamma$ .

In general, other nonlinear mechanisms may limit the growth of the tangling clustering instability in the nonlinear stage of its evolution, and therefore they limit the maximum attainable value of the particle number density,  $n_p^{\text{max}}$ , inside the cluster. For example, a momentum coupling of particles and turbulent fluid becomes essential when the mass loading parameter  $\zeta = m_p n_{\text{max}} / \rho$  is of the order of unity.<sup>1</sup> Introducing a mean particle mass density  $\bar{\rho}_p = m_p N$  (e.g., in cloud physics it is a liquid water content measured in  $\text{g}/\text{cm}^3$ ), we obtain the following constraint:  $n_p^{\text{max}} / N \leq \zeta \rho / \bar{\rho}_p$ . Using  $\rho = 1.3 \times 10^{-3} \text{ g}/\text{cm}^3$  and  $\bar{\rho}_p = 10^{-6} \text{ g}/\text{cm}^3$  we arrive at  $n_p^{\text{max}} / N \leq 1300$  for  $\zeta = 1$ . This limiting effect should be taken into account together with the saturation mechanism caused by the exhaustion of the particles in the vicinity of the cluster.

Figure 5 shows the temporal evolution of the number density of particles inside the cluster during the excitation of the tangling clustering instability for particles of different radii. Figure 5 reveals several interesting features pertinent to the instability which deserve to be mentioned. The tangling clustering instability is less effective for very small particles,  $a_p \leq 0.5 \mu\text{m}$ . The reason is that this instability is saturated by the exhaustion of the particles in the vicinity of the cluster for very low value of  $n_{\text{max}}/N \approx 13.2$  [see Eq. (63) and Fig. 4]. For larger particles ( $a_p > 0.6 \mu\text{m}$ ), the concentration inside the cluster can increase due to the tangling clustering instability by several orders of magnitude. On the other hand, as follows from Fig. 2 the growth rate of the tangling clustering instability is the same for particles with  $a_p = 0.565 \mu\text{m}$  and for all particles with

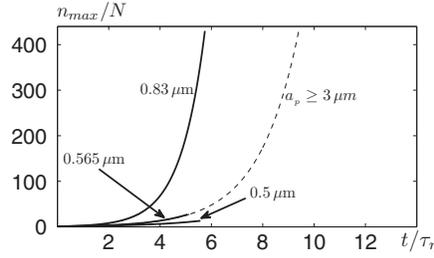


FIG. 5. The particle number density inside the cluster  $n_{\max}/N$  as a function of time for different values of the particle radius  $a_p$  and  $\Gamma = 10^4$ ,  $\sigma_T = 1$ ,  $Sc = 10^6 a_p$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ . The dashed line is for  $a_p \geq 3 \mu\text{m}$ .

$a_p \geq 3 \mu\text{m}$ . However, contrary to the case  $a_p \geq 3 \mu\text{m}$ , the saturated value of the particle number density enhancement due to the instability for  $a_p = 0.565 \mu\text{m}$  is very low,  $n_{\max}/N \approx 27.2$ .

## VI. EFFECT OF DROPLET EVAPORATION ON TANGLING CLUSTERING INSTABILITY

Let us study the effect of droplet evaporation on tangling clustering instability in stably stratified turbulence. The equation for the instantaneous number density  $n_p(t, \mathbf{x})$  of droplets of the radius  $a_p$  reads

$$\frac{\partial n_p}{\partial t} + \nabla \cdot (n_p \mathbf{v}) = D_m \Delta n_p - \frac{n_p}{\tau_{\text{ev}}} + I_0, \quad (64)$$

where the second term in the right-hand part of Eq. (64) takes into account the droplet evaporation with the characteristic time  $\tau_{\text{ev}}$ , and the last term,  $I_0$ , describes source of droplets due to condensation, which for simplicity is assumed to be constant. The equation for fluctuations of the droplet number density,  $n(t, \mathbf{x}) = n_p(t, \mathbf{x}) - N(t, \mathbf{x})$ , reads

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v} - \langle n \mathbf{v} \rangle) = D_m \Delta n - (\mathbf{v} \cdot \nabla) n - N \nabla \cdot \mathbf{v} - \frac{n}{\tau_{\text{ev}}}. \quad (65)$$

The last term in Eq. (65) describes the droplet evaporation. Using Eq. (65) we derive equation for the evolution of the two-point second-order correlation function of the droplet number density,  $\Phi(t, \mathbf{R})$ , see Eq. (17), in which

$$D_{ij}^T(\mathbf{R}) \approx 2 \int_0^\infty G(\tau) \langle v_i[0, \boldsymbol{\xi}(t, \mathbf{x}|0)] v_j[\tau, \boldsymbol{\xi}(t, \mathbf{x} + \mathbf{R}|\tau)] \rangle d\tau, \quad (66)$$

$$B(\mathbf{R}) \approx 2 \int_0^\infty G(\tau) \langle b[0, \boldsymbol{\xi}(t, \mathbf{x}|0)] b[\tau, \boldsymbol{\xi}(t, \mathbf{x} + \mathbf{R}|\tau)] \rangle d\tau, \quad (67)$$

$$U_i(\mathbf{R}) \approx -2 \int_0^\infty G(\tau) \langle v_i[0, \boldsymbol{\xi}(t, \mathbf{x}|0)] b[\tau, \boldsymbol{\xi}(t, \mathbf{x} + \mathbf{R}|\tau)] \rangle d\tau, \quad (68)$$

$G(\tau) = \exp(-\tau/\tau_{\text{ev}})$ , and other terms do not change in case of the droplet evaporation.

Equation for  $\Phi(t, \mathbf{R})$  can be rewritten in the dimensionless form as follows:

$$\frac{\partial \Phi}{\partial t} = \frac{1}{M(R)} \left[ \Phi'' + 2 \left( \frac{1}{R} + \chi(R) \right) \Phi' \right] + \left[ B(R) - \frac{2\tau_D}{\tau_{\text{ev}}} \right] \Phi + I(R), \quad (69)$$

where distance  $R$  is measured in units of Kolmogorov scale  $\ell_\eta$  and time  $t$  is measured in units of  $\tau_\eta^2/\tau^{\text{eff}} \equiv \tau_D/3$ . Here the effective time  $\tau^{\text{eff}}$  is determined by the following expression:

$$\tau^{\text{eff}} = \frac{\tau_\eta \tau_{\text{ev}}}{\tau_\eta + \tau_{\text{ev}}}, \quad (70)$$

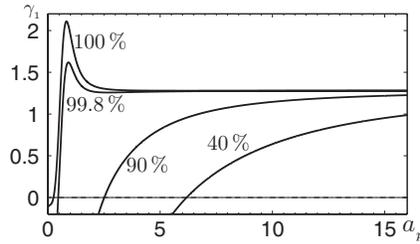


FIG. 6. The growth rate  $\gamma_1$  of the tangling clustering instability (in units of  $1/t_\eta$ ) of the first mode ( $m = 1$ ) versus the particle radius  $a_p$  for different values of the relative humidity  $\phi$ , and  $\Gamma = 10^4$ ,  $\sigma_T = 1$ ,  $\text{Sc} = 10^6 a_p$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ .

and a modified turbulent diffusion time is determined as  $\tau_D = \ell_\eta^2 / D_T^{\text{eff}}$ , the effective turbulent diffusion coefficient  $D_T^{\text{eff}}$  in the Kolmogorov scale reads<sup>51</sup>

$$D_T^{\text{eff}} = \frac{\tau^{\text{eff}} u_\eta^2}{3}, \quad (71)$$

and a modified function  $1/M(R)$  is

$$\frac{1}{M(R)} = \frac{2}{\text{Sc}^{\text{eff}}} + \frac{2}{3}[1 - F - (RF_c)'], \quad (72)$$

where  $\text{Sc}^{\text{eff}} = \text{Sc} \tau^{\text{eff}} / \tau_\eta$ . Equation (69) shows that the evaporation decreases the term  $[B(R) - 2\tau_D / \tau_{\text{ev}}]$  which is responsible for the generation of fluctuations of the droplet number density. Equation (71) has a simple physical meaning. In the case when the droplet evaporation time is much smaller than the turbulent correlation time, the turbulent diffusion coefficient is renormalized as given by Eq. (71) (for details, see Ref. 51).

The analysis similar to that performed in Sec. IV yields the growth rate for the mode  $m$  of the tangling clustering instability

$$\gamma_m = \frac{1}{3(1 + 3\sigma_T)} \left[ \frac{200\sigma_v(\sigma_T - \sigma_v)}{(1 + \sigma_v)^2} - \frac{(3 - \sigma_T)^2}{2(1 + \sigma_T)} - \frac{2\pi^2 m^2 (1 + 3\sigma_T)^2}{(1 + \sigma_T) \ln^2 \text{Sc}^{\text{eff}}} \right] - \frac{2\tau_\eta^2}{\tau^{\text{eff}} \tau_{\text{ev}}}, \quad (73)$$

where  $m = 1, 2, 3, \dots$ . This growth rate  $\gamma_m$  of the second moment of particles number density was obtained by matching the correlation function  $\Phi(R)$  and its first derivative  $\Phi'(R)$  at the points  $R = 1/\sqrt{\text{Sc}^{\text{eff}}}$  and  $R = 1$ , which also yields:  $\kappa/2(C_1 + C_2) \approx \pi m / \ln \text{Sc}^{\text{eff}}$ . Therefore, the evaporation of droplets causes decrease of the instability growth rate. Figure 6 shows the growth rate  $\gamma_1$  of the tangling clustering instability for different values of the relative humidity  $\phi$  versus the particle radius. Here we used the following expression for the evaporation time of droplets  $\tau^{\text{ev}} = 2.1 \times 10^{-3} a_p^2 / (1 - \phi)$ , where the droplet radius is in microns and time is in seconds (see, e.g., Ref. 52). Inspection of Fig. 6 shows that the evaporation of droplets strongly affects the tangling clustering instability for small droplets, i.e., it increases the instability threshold in the droplet radius

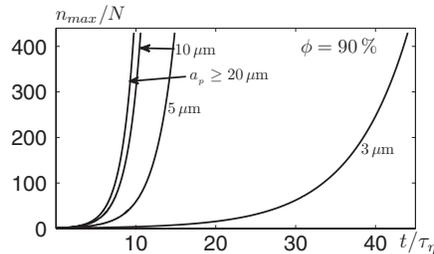


FIG. 7. The particle number density inside the cluster  $n_{\text{max}}/N$  as a function of time for different values of the particle radius  $a_p$ , and the relative humidity  $\phi = 90\%$ ,  $\Gamma = 10^4$ ,  $\sigma_T = 1$ ,  $\text{Sc} = 10^6 a_p$ . The particle radius  $a_p$  is given in  $\mu\text{m}$ .

depending on the relative humidity  $\phi$ . In addition, there is sharp maximum of the growth rate for 1  $\mu\text{m}$  droplets if the relative humidity is close to the supersaturated values: 99.8% and 100%. For low relative humidity the growth rate of the tangling clustering instability is less in comparison with the supersaturated case.

Figure 7 shows the temporal evolution of the number density of particles inside the cluster during the excitation of the tangling clustering instability for the relative humidity  $\phi = 90\%$  and different droplet radius. This figure also demonstrates that the evaporation of droplets strongly affects the tangling clustering instability.

## VII. DISCUSSION AND CONCLUSIONS

The present study has been inspired by the previous work,<sup>44</sup> where it was shown that the tangling clustering of inertial particles in the temperature stratified turbulence holds the potential to promote a strong clustering with the considerably enhanced particle concentration inside the cluster. In this study based on the thorough theoretical analysis, it is demonstrated that the temperature fluctuations strongly contribute to the tangling clustering instability. Temperature fluctuations caused by tangling of the mean temperature gradient by the velocity fluctuations, produce pressure fluctuations and considerably enhance particle clustering. The growth rate of the tangling clustering instability is by a factor of  $\sqrt{\text{Re}} (\ell_0/L_T)^2 / (3\text{Ma})^4$  larger than the growth rate of the inertial clustering instability. The main reason why the tangling clustering instability for a low-Mach-number turbulent fluid flow is much more effective than the inertial clustering instability is the following. In both cases the particle clustering is determined by the two-point correlation function of the Laplacian of fluid pressure fluctuations. In non-stratified turbulence the pressure fluctuations are of the order of  $\rho \mathbf{u}^2$ . In the temperature stratified turbulence there are also additional pressure fluctuations caused by temperature fluctuations:  $p \propto P(\theta/T) \propto \rho c_s^2 (\theta/T)$ . Therefore, the ratio of the two-point correlation functions of the Laplacian of fluid pressure fluctuations in stratified and non-stratified flows is proportional to  $\text{Ma}^{-4} (\theta/T)^2$ , i.e., it is a large parameter when Ma is small.

The growth rate of the tangling clustering instability and the particle number density inside the cluster after saturation of the instability on the nonlinear stage depends on the parameter  $\Gamma \approx (\text{Re}^{1/4}/9\text{Ma}^2) \ell_0 |\nabla T|/T$ . We also found that depending on the parameters of turbulence and the mean temperature gradient there is the preferential clustering of particles of a particular size (the growth rate of the tangling clustering instability has a sharp maximum at this size). The growth of the particle number density inside the cluster caused by the tangling clustering instability is significantly larger (by several orders of magnitudes) than the increase of the particle number density inside the cluster caused by the source tangling clustering.<sup>44</sup>

We demonstrated the strong effect of the droplet evaporation on this instability. The tangling clustering instability in the temperature stratified turbulence may enhance significantly the collision rate of small particles, which is of interest for atmospheric physics and many other practical applications. In particular this effect can substantially accelerate the coalescence of small droplets in atmospheric turbulence with temperature gradients.

## ACKNOWLEDGMENTS

This research was supported in part by the Israel Science Foundation governed by the Israeli Academy of Sciences (Grant No. 259/07), by EU COST Actions MP0806 and ES1004, by the EC FP7 project ERC PBL-PMES (Grant No. 227915), by the Russian Government Mega Grant (Grant No. 11.G34.31.0048), and by the Grant of Russian Ministry of Science and Education (Contract No. 8648).

<sup>1</sup>C. T. Crowe, J. D. Schwarzkopf, M. Sommerfeld, and Y. Tsuji, *Multiphase Flows with Droplets and Particles*, 2nd ed. (CRC Press LLC, NY, 2011).

<sup>2</sup>Z. Warhaft, "Passive scalars in turbulent flows," *Annu. Rev. Fluid Mech.* **32**, 203 (2000).

<sup>3</sup>Z. Warhaft, "Laboratory studies of droplets in turbulence: Towards understanding the formation of clouds," *Fluid Dyn. Res.* **41**, 011201 (2009).

<sup>4</sup>S. Balachandar and J. K. Eaton, "Turbulent dispersed multiphase flow," *Annu. Rev. Fluid Mech.* **42**, 111 (2010).

- <sup>5</sup>R. Monchaux, M. Bourgoïn, and A. Cartellier, "Analyzing preferential concentration and clustering of inertial particles in turbulence," *Int. J. Multiphase Flow* **40**, 1 (2012).
- <sup>6</sup>R. A. Shaw, "Particle-turbulence interactions in atmospheric clouds," *Annu. Rev. Fluid Mech.* **35**, 183 (2003).
- <sup>7</sup>R. E. Britter and S. R. Hanna, "Flow and dispersion in urban areas," *Annu. Rev. Fluid Mech.* **35**, 469 (2003).
- <sup>8</sup>A. Khain, M. Pinsky, T. Elperin, N. Kleeorin, I. Rogachevskii, and A. Kostinski, "Critical comments to results of investigations of drop collisions in turbulent clouds," *Atmos. Res.* **86**, 1 (2007).
- <sup>9</sup>H. Siebert, S. Gerashchenko, A. Gylfason, K. Lehmann, L. R. Collins, R. A. Shaw, and Z. Warhaft, "Towards understanding the role of turbulence on droplets in clouds: In situ and laboratory measurements," *Atmos. Res.* **97**, 426 (2010).
- <sup>10</sup>S. L. Post and J. Abraham, "Modeling the outcome of drop-drop collisions in Diesel sprays," *Int. J. Multiphase Flow* **28**, 997 (2002).
- <sup>11</sup>M. R. Maxey, "The gravitational settling of aerosol particles in homogeneous turbulence and random flow field," *J. Fluid Mech.* **174**, 441 (1987).
- <sup>12</sup>T. Elperin, N. Kleeorin, and I. Rogachevskii, "Self-excitation of fluctuations of inertial particles concentration in turbulent fluid flow," *Phys. Rev. Lett.* **77**, 5373 (1996).
- <sup>13</sup>T. Elperin, N. Kleeorin, I. Rogachevskii, and D. Sokoloff, "Turbulent transport of atmospheric aerosols and formation of large-scale structures," *Phys. Chem. Earth A* **25**, 797 (2000).
- <sup>14</sup>E. Balkovsky, G. Falkovich, and A. Fouxon, "Intermittent distribution of inertial particles in turbulent flows," *Phys. Rev. Lett.* **86**, 2790 (2001).
- <sup>15</sup>T. Elperin, N. Kleeorin, V. L'vov, I. Rogachevskii, and D. Sokoloff, "Clustering instability of the spatial distribution of inertial particles in turbulent flows," *Phys. Rev. E* **66**, 036302 (2002).
- <sup>16</sup>B. Mehlig, M. Wilkinson, K. P. Duncan, T. Weber, and M. Ljunggren, "On the aggregation of inertial particles in random flows," *Phys. Rev. E* **72**, 051104 (2005).
- <sup>17</sup>T. Elperin, N. Kleeorin, M. A. Liberman, V. S. L'vov, and I. Rogachevskii, "Clustering of aerosols in atmospheric turbulent flow," *Environ. Fluid Mech.* **7**, 173 (2007).
- <sup>18</sup>P. Olla and R. M. Vuolo, "Concentration fluctuations of large Stokes number particles in a one-dimensional random velocity field," *Phys. Rev. E* **76**, 066315 (2007).
- <sup>19</sup>I. Fouxon and P. Horvai, "Separation of heavy particles in turbulence," *Phys. Rev. Lett.* **100**, 040601 (2008).
- <sup>20</sup>P. Olla, "Preferential concentration versus clustering in inertial particle transport by random velocity fields," *Phys. Rev. E* **81**, 016305 (2010).
- <sup>21</sup>G. Boffetta, F. De Lillo, and A. Gamba, "Large scale inhomogeneity of inertial particles in turbulent flows," *Phys. Fluids* **16**, L20 (2004).
- <sup>22</sup>L. R. Collins and A. Keswani, "Reynolds number scaling of particle clustering in turbulent aerosols," *New J. Phys.* **6**, 119 (2004).
- <sup>23</sup>J. Chun, D. L. Koch, S. L. Rani, A. Ahluwalia, and L. R. Collins, "Clustering of aerosol particles in isotropic turbulence," *J. Fluid Mech.* **536**, 219 (2005).
- <sup>24</sup>L. Chen, S. Goto, and J. C. Vassilicos, "Turbulent clustering of stagnation points and inertial particles," *J. Fluid Mech.* **553**, 143 (2006).
- <sup>25</sup>H. Yoshimoto and S. Goto, "Self-similar clustering of inertial particles in homogeneous turbulence," *J. Fluid Mech.* **577**, 275 (2007).
- <sup>26</sup>J. Bec, L. Biferale, M. Cencini, A. Lanotte, S. Musacchio, and F. Toschi, "Heavy particle concentration in turbulence at dissipative and inertial scales," *Phys. Rev. Lett.* **98**, 084502 (2007).
- <sup>27</sup>M. van Aartsijk and H. J. H. Clercx, "Preferential concentration of heavy particles in stably stratified turbulence," *Phys. Rev. Lett.* **100**, 254501 (2008).
- <sup>28</sup>J. Bec, L. Biferale, A. Lanotte, A. Scagliarini, and F. Toschi, "Turbulent pair dispersion of inertial particles," *J. Fluid Mech.* **645**, 497 (2010).
- <sup>29</sup>A. Aliseda, A. Cartellier, F. Hainaux, and J. C. Lasheras, "Effect of preferential concentration on the settling velocity of heavy particles in homogeneous isotropic turbulence," *J. Fluid Mech.* **468**, 77 (2002).
- <sup>30</sup>A. M. Wood, W. Hwang, and J. K. Eaton, "Preferential concentration of particles in homogeneous and isotropic turbulence," *Int. J. Multiphase Flow* **31**, 1220 (2005).
- <sup>31</sup>J. Salazar, J. de Jong, L. Cao, S. Woodward, H. Meng, and L. Collins, "Experimental and numerical investigation of inertial particle clustering in isotropic turbulence," *J. Fluid Mech.* **600**, 245 (2008).
- <sup>32</sup>H. Xu and E. Bodenschatz, "Motion of inertial particles with size larger than the Kolmogorov scale in turbulent flows," *Physica D* **237**, 2095 (2008).
- <sup>33</sup>E. W. Saw, R. A. Shaw, S. Ayyalasomayajula, P. Y. Chuang, and A. Gylfason, "Inertial clustering of particles in high Reynolds-number turbulence," *Phys. Rev. Lett.* **100**, 214501 (2008).
- <sup>34</sup>T. Elperin, N. Kleeorin, and I. Rogachevskii, "Turbulent thermal diffusion of small inertial particles," *Phys. Rev. Lett.* **76**, 224 (1996).
- <sup>35</sup>T. Elperin, N. Kleeorin, and I. Rogachevskii, "Turbulent barodiffusion, turbulent thermal diffusion and large-scale instability in gases," *Phys. Rev. E* **55**, 2713 (1997).
- <sup>36</sup>T. Elperin, N. Kleeorin, I. Rogachevskii, and D. Sokoloff, "Passive scalar transport in a random flow with a finite renewal time: Mean-field equations," *Phys. Rev. E* **61**, 2617 (2000).
- <sup>37</sup>T. Elperin, N. Kleeorin, I. Rogachevskii, and D. Sokoloff, "Mean-field theory for a passive scalar advected by a turbulent velocity field with a random renewal time," *Phys. Rev. E* **64**, 026304 (2001).
- <sup>38</sup>R. V. R. Pandya and F. Mashayek, "Turbulent thermal diffusion and barodiffusion of passive scalar and dispersed phase of particles in turbulent flows," *Phys. Rev. Lett.* **88**, 044501 (2002).
- <sup>39</sup>M. W. Reeks, "On model equations for particle dispersion in inhomogeneous turbulence," *Int. J. Multiphase Flow* **31**, 93 (2005).

- <sup>40</sup> M. Sofiev, V. Sofieva, T. Elperin, N. Kleeorin, I. Rogachevskii, and S. S. Zilitinkevich, "Turbulent diffusion and turbulent thermal diffusion of aerosols in stratified atmospheric flows," *J. Geophys. Res.* **114**, D18209, doi:10.1029/2009JD011765 (2009).
- <sup>41</sup> N. E. L. Haugen, N. Kleeorin, I. Rogachevskii, and A. Brandenburg, "Detection of turbulent thermal diffusion of particles in numerical simulations," *Phys. Fluids* **24**, 075106 (2012).
- <sup>42</sup> J. Buchholz, A. Eidelman, T. Elperin, G. Grünefeld, N. Kleeorin, A. Krein, and I. Rogachevskii, "Experimental study of turbulent thermal diffusion in oscillating grids turbulence," *Exp. Fluids* **36**, 879 (2004).
- <sup>43</sup> A. Eidelman, T. Elperin, N. Kleeorin, I. Rogachevskii, and I. Sapir-Katiraie, "Turbulent thermal diffusion in a multi-fan turbulence generator with the imposed mean temperature gradient," *Exp. Fluids* **40**, 744 (2006).
- <sup>44</sup> A. Eidelman, T. Elperin, N. Kleeorin, B. Melnik, and I. Rogachevskii, "Tangling clustering of inertial particles in stably stratified turbulence," *Phys. Rev. E* **81**, 056313 (2010).
- <sup>45</sup> S. Chandrasekhar, "Stochastic problems in physics and astronomy," *Rev. Mod. Phys.* **15**, 1 (1943).
- <sup>46</sup> A. I. Akhiezer and S. V. Peletminsky, *Methods of Statistical Physics* (Pergamon, Oxford, 1981).
- <sup>47</sup> V. G. Lambart, D. D. Sokoloff, and V. N. Tutubalin, "Turbulent diffusion in the interstellar medium," *Astron. Rep.* **44**, 659 (2000).
- <sup>48</sup> R. H. Kraichnan, "Small-scale structure of a scalar field convected by turbulence," *Phys. Fluids* **11**, 945 (1968).
- <sup>49</sup> Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *The Almighty Chance* (World Scientific Publishing, Singapore, 1990), and references therein.
- <sup>50</sup> S. S. Zilitinkevich, T. Elperin, N. Kleeorin, and I. Rogachevskii, "Energy- and flux-budget (EFB) turbulence closure model for stably stratified flows. Part I: Steady-state, homogeneous regimes," *Boundary-Layer Meteorol.* **125**, 167 (2007).
- <sup>51</sup> T. Elperin, N. Kleeorin, and I. Rogachevskii, "Effect of chemical reactions and phase transitions on turbulent transport of particles and gases," *Phys. Rev. Lett.* **80**, 69 (1998).
- <sup>52</sup> J. H. Seinfeld and S. N. Pandis, *Atmospheric Chemistry and Physics: From Air Pollution to Climate Change*, 2nd ed. (John Wiley and Sons, NY, 2006).