Area rules for penetrating bodies

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Abstract

Area rules are determined for the normal penetration of a projectile into a ductile target under conditions of the localized projectile–target interaction. Bodies of revolution and three-dimensional projectiles with a close shape and the same distribution of the cross-sectional area along the longitudinal axis are considered. It is shown that the difference in the ballistic limits of the bodies is of the order of $\varepsilon^2$ where $\varepsilon$ is the order of the difference in their shapes.

1. Introduction

Area rules are known in gas dynamics where they have been determined for different high velocity flight conditions. A brief review of area rules in gas dynamics can be found in Ref. [1]. The essence of the rules is elucidated in the following. Consider an arbitrary three-dimensional projectile and a reference body of revolution of the same length. Assume that both projectiles have the same distribution of the cross-sectional area along the longitudinal axis and their boundary contours are close at every section with an estimated difference of the order of $\varepsilon$. Then the difference in the drag forces acting on these projectiles is of the order of $\varepsilon^2$. Area rules were determined for different ranges of flight conditions using some exact models of the flow over the projectile. The area rules can be proved to be valid and extended to the general localized projectile–host medium interaction model [1]. Since the localized interaction approach used in this study to generalize area rules for penetrating impactors has not yet gained general recognition in the mechanics of penetration, its essence is briefly explained in the following.

The principal element of the localized interaction approach are the localized interaction models [1] whereby the integral effect of the interaction between a host medium and a moving projectile is described as a superposition of the independent local interactions of the projectile’s surface elements with the medium. Every local interaction is determined by the local geometric and kinematic parameters of the surface element (primarily, by the local projectile velocity and the angle between the velocity vector and the local unit normal vector at the projectile surface) as well as by some global parameters which account for the integral characteristics of the host medium.

Various specific localized models have been suggested for modeling penetration and perforation. The most widely used models are discussed in Refs. [2–4]. For the case of penetration into a plastic barrier, a localized model was suggested and experi-
mentally verified in Ref. [5]. Experimental results [6] on the penetration of high velocity impactors with conical heads into metal targets showed that, with high accuracy, the variable component of the drag force could be determined using the localized Newtonian formula [1] which is widely used in hypersonic gas dynamics. Some penetration models employed in Ref. [7] for some bodies of revolution can be classified as the localized interaction models.

Because the localized interaction models allow the description of the local effect of the projectile-medium interaction and to simulate the motion of an impactor inside a target, they can be applied to investigate the effect of the impactor shape on the efficiency of the perforation and to determine the optimal shapes of the impactors. The study reported in Ref. [8] is an example of such an approach for bodies of revolution. Analytical investigations [1, 9–11] led to conclusions about the efficiency of impactors with unconventional three-dimensional shapes. These theoretical results agreed with experiments on penetration into metal and soil targets [9, 11]. The goal of this study is to analyze the effect of the shape of a three-dimensional impactor on the penetration using some ideas from gas dynamics. Certainly, the results obtained in gas dynamics cannot be extended directly to the penetration phenomenon because for the analysis of penetration the equation of the nonstationary projectile motion in the target must be studied.

Many phenomenological localized interaction models were suggested for the description of the effect of the host medium on a penetrating body. However, a general assumption about the localized interaction between the impactor and the target is less restrictive than the assumptions adopted in various phenomenological models. Therefore, there are reasons to believe that such a more general approach will yield more reliable and well-based results.

\[ d\tilde{F} = \left( \Omega_\rho \tilde{n}^0 + \Omega_\tau \tilde{\tau}^0 \right) dS, \]

where

\[ \tilde{\tau}^0 = -\frac{\tilde{v}^0 + u\tilde{n}^0}{\sqrt{1 - u^2}}, \quad u = -\tilde{v}^0 \cdot \tilde{n}^0 \]

and \( d\tilde{F} \) is the force acting on the surface element \( dS \) of the impactor; \( \tilde{v} \) is the local velocity of the impactor, \( \tilde{n} \) and \( \tilde{\tau} \) are the local normal and tangent vectors at the impactor's surface, respectively, \( \Omega_\alpha, \Omega_\rho, \Omega_\tau \) are functions determining the model (\( \Omega_\rho = \Omega_\tau = 0 \) if \( u \leq 0 \)), superscript zero denotes a unit vector. Hereafter the parameters \( a \), which characterize properties of the target material are not listed as arguments of the functions \( \Omega_\rho, \Omega_\tau \).

The total force \( \tilde{F} \) is determined by integrating the local force over the contact surface \( \sigma \). The expression for the drag force \( D \) acting on the impactor reads:

\[ D = \tilde{F} \cdot (-\tilde{v}^0) \]

\[ = \int \int_{\sigma} \left[ u\Omega_\rho(u, v) + \sqrt{1 - u^2} \Omega_\tau(u, v) \right] dS \]

Consider a normal impact of a convex, sharp, rigid 3D impactor moving in the target with the velocity directed along the \( x \)-axis i.e. \( \tilde{V}^0 = -\tilde{X}^0 \). The notations are shown in Fig. 1. The shape of the impactor in cylindrical coordinates \( x, \rho, \theta \) (see Fig. 1), can be represented as \( \rho = \Phi(x, \theta) \). Assume that its

2. Description of the model

Consider a rigid impactor penetrating into a ductile target with a high speed. A wide range of the models describing impactor-target interaction at a point of an impactor's surface which is in contact

Fig. 1. Coordinates and notations.
shape is such that the resultant force is directed along the \( x \)-axis. Examples of such impactors are projectiles with axial symmetry. The part of the lateral surface of the impactor between the cross-sections \( x = x_1 \) and \( x = x_2 \) interacts with the target where

\[
x_1(h) = \begin{cases} 
0 & \text{if } 0 \leq h \leq b \\
b - h & \text{if } b \leq h \leq b + L 
\end{cases}
\]

\[
x_2(h) = \begin{cases} 
h & \text{if } 0 \leq h \leq L \\
L & \text{if } b \leq L \leq b + L 
\end{cases}
\]  

(4)

Using formulae of differential geometry

\[
\vec{n}^0 \cdot \vec{x}^0 = u = u_1/u_0, \quad u_0 = \sqrt{\Phi^2(\Phi_x^2 + 1) + \Phi_y^2},
\]

(5)

\[
u_1 = \Phi \Phi_x, \quad dS = u_0 \, dx \, d\theta
\]

where subscripts \( x \) and \( \theta \) denote derivatives with respect to the corresponding argument. Eq. (3) can be rewritten as

\[
D[\Phi(x, \theta); h, v] = \int_{x(h)}^{x_2(h)} \int_0^{2\pi} \Omega(u, v) u_1 \, dx \, d\theta
\]

(6)

where \( \Omega(u, v) = \Omega_p(u, v) + \sqrt{1 - u^2} \Omega(u, v)/u \).

The equation of motion of the impactor with mass \( m \) reads:

\[
m \frac{d^2 h}{dt^2} = -D[\Phi(x, \theta); h, \frac{dh}{dt}]
\]

(7)

After a change of variables, \( (dh/dt)^2 = W(h) \), Eq. (7) can be rewritten as

\[
m \frac{dW}{2 \, dh} = -\tilde{D}[\Phi(x, \theta); h, W]
\]

(8)

where

\[
\tilde{D}[\Phi(x, \theta); h, W] = D[\Phi(x, \theta); h, \sqrt{W}]
\]

(9)

Hereafter we define the ballistic limit \( v_\ast \) as an initial velocity of the impactor when it emerges from the target with zero velocity i.e.

\[
W(b + L) = 0
\]

(10)

The solution of Eq. (8) with the boundary condition given by Eq. (10) describes the impactor's motion for perforation with the ballistic limit velocity. The formula for the ballistic limit \( v_\ast \) reads:

\[
v_\ast = \sqrt{W(0)}
\]

(11)

3. Area rules

Along with a three-dimensional impactor \( T \) consider a reference body of revolution \( T_0 \) with general matrix \( \rho = r(x) \) having the same density and the same cross-sectional area \( s(x) \) at every cross-section, \( 0 \leq x \leq L \). Because the formula

\[
V = \int_0^L s(x) \, dx
\]

(12)

is valid for volumes of both impactors, they have the same volume and mass. Assume also that the shapes of both impactors are close in the sense that

\[
\Phi(x, \theta) = r(x) + \varepsilon \xi(x, \theta)
\]

(13)

where \( \xi(x, \theta) \) is some function and \( \varepsilon \) is a small parameter. The assumption of area equivalence and Eq. (13) yield the following estimate \cite{1}:

\[
\delta(x) \equiv \int_0^{2\pi} \xi(x, \theta) \, d\theta = O(\varepsilon^n)
\]

(14)

which will be used subsequently.

Using the general methodology of perturbation theory \cite{12} the solution of Eqs. (8) and (10) can be represented in the form of a series of a small parameter \( \varepsilon \):

\[
W(h) = W_0(h) + W_1(h) \varepsilon + O(\varepsilon^2)
\]

(15)

Let us now determine the first and the second terms of this series. After substitution of \( \Phi(x, \theta) \) from Eq. (13) and \( W(h) \) from Eq. (15) into Eq. (8) the latter equation can be rewritten as

\[
\frac{m}{2} \left[ \frac{dW_0}{dh} + \frac{dW_1}{dh} \varepsilon + O(\varepsilon^2) \right] = -\tilde{D}[r(x) + \varepsilon \xi(x, \theta); W_0 + W_1 \varepsilon + O(\varepsilon^2)]
\]

(16)
Eqs. (10) and (15) yield the boundary conditions for the functions $W_0$ and $W_1$:

$$W_0(L + b) = 0, \quad W_1(L + b) = 0 \quad (17)$$

Using a Taylor series expansion of the right-hand side of Eq. (16) with respect to $\varepsilon$, Eq. (16) can be rewritten as

$$m \left[ \frac{dW_0}{dh} + \frac{dW_1}{dh} \varepsilon + O(\varepsilon^2) \right]$$

$$= -D \left[ r(x); h, W_0 \right] - \left( \partial \hat{D}/\partial \varepsilon \right)_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \quad (18)$$

Equating the coefficients near $\varepsilon^0$ in both sides of Eq. (18), there results the equation for the function $W_0$:

$$m \frac{dW_0}{dh} = -D \left[ r(x); h, W_0 \right] \quad (19)$$

Eq. (19) with the boundary condition given by the first equation in Eq. (17) describes the motion of the impactor $T_0$ for the case of target perforation with a ballistic limit velocity $v_{\varepsilon}$. More sophisticated analysis is required to derive the equation for $W_1$. Using Eqs. (6) and (9) the derivative $\partial \hat{D}/\partial \varepsilon$ can be written as

$$\frac{\partial \hat{D}}{\partial \varepsilon} = \int_{x(h)}^{x(h)} 2\pi \left[ \frac{\partial \hat{\Omega}(u, W)}{\partial u} \frac{\partial u_1}{\partial \varepsilon} + \frac{\partial \hat{U}(u, W)}{\partial W} \frac{\partial W}{\partial \varepsilon} u_1 \right] dx d\theta \quad (20)$$

where

$$\hat{\Omega}(u, W) = \Omega(u, \sqrt{W}) \quad (21)$$

$$u_0 = \sqrt{(r + e \xi_1)^2 (r + e \xi_2)^2 + 1} + e^2 \xi_3^2 ;$$

$$u_1 = (r + e \xi_1)(r + e \xi_2) \quad (22)$$

After some manipulations Eq. (20) yields

$$\left( \partial \hat{D}/\partial \varepsilon \right)_{\varepsilon=0} = 2\pi W_1 I_0 + l \quad (23)$$

where

$$I_0 = \int_{x(h)}^{x(h)} \frac{d\Omega}{dx} \frac{d}{dW}(\hat{u}, W_0) dx \quad (24)$$

$$l = \int_{x(h)}^{x(h)} \left[ \varphi(x, W_0) \delta(x) + \psi(x, W_0) \frac{d\delta(x)}{dx} \right] dx$$

$$\varphi(x, W_0) = r_x \hat{\Omega}(\hat{u}, W_0) \quad (25)$$

$$\psi(x, W_0) = \frac{r_x}{(r_x^2 + 1)^{3/2}} \frac{d\hat{\Omega}}{du}(\hat{u}, W_0)$$

$$+ r \hat{\Omega}(\hat{u}, W_0)$$

$$\hat{u} = r_x/\sqrt{r_x^2 + 1} \quad (26)$$

Integrating by parts the second term in formula for $l$ in Eq. (24), it is found that

$$l = \psi(x, W_0) \delta(x, h, W_0)$$

$$- \psi(x, W_0) \delta(x, h, W_0)$$

$$+ \int_{x(h)}^{x(h)} \left[ \varphi(x, W_0) - \frac{\partial \psi(x, W_0)}{\partial x} \right] \delta(x) dx \quad (27)$$

Using Eqs. (14) and (27), an estimate of the order of

$$l = O(\varepsilon) \quad (28)$$

is obtained. Equating the coefficients near $\varepsilon$ in both sides of Eq. (18) and using Eqs. (23) and (28), the equation for the function $W_1$ is derived:

$$m \frac{dW_1}{dh} = -I_0[h, W_0(h)] W_1 \quad (29)$$

with the boundary condition given by the second equation in Eq. (17). The only solution of this boundary value problem reads

$$W_1(h) = 0 \quad (30)$$
The latter implies the following estimates
\[ W(h) = W_0(h) = O(\varepsilon^2), \]
\[ W(0) = W_0(0) + O(\varepsilon^2) \]
(31)

Therefore, the following estimate for the ballistic limit of the impactor \( T_1 \) is valid
\[ \nu_* = \sqrt{\nu^2_0 + O(\varepsilon^2)} = \nu_0 + O(\varepsilon^2) \]
(32)

where subscript zero denotes the ballistic limit velocity of the impactor \( T_0 \).

Thus the difference of the ballistic limits of the impactors \( T_0 \) and \( T_1 \) is of the order of \( \varepsilon^2 \) i.e. the area rule for the ballistic limit is proved.

Similar laws can be determined for other characteristics of the penetration process. Let the impactor \( T_i \) start its motion in the target with the initial velocity \( \nu_i \) i.e. the following initial condition
\[ W(0) = \nu^2_i \]
(33)

replaces those given by Eq. (10). Then the boundary conditions for \( W_0 \) and \( W_1 \) read:
\[ W_0(0) = \nu^2_i, \quad W_1(0) = 0 \]
(34)

Eq. (30) is the solution of Eq. (29) with the boundary condition given by the second of Eq. (34). Therefore, the following estimate is valid
\[ \nu(h) = \nu_0(h) + O(\varepsilon^2) \]
(35)

where \( \nu \) and \( \nu_0 \) are velocities of impactors \( T_1 \) and \( T_0 \), respectively. Thus if the initial velocities of the impactors \( T_0 \) and \( T_1 \) are the same, the difference in their velocities at any depth \( h \leq L + b \) inside a target is of the order of \( \varepsilon^2 \) if both impactors continue their motion inside the target.

The latter law holds also if the impactors penetrate into a semi-infinite target. The only difference is that in this case the formulae of functions \( x_i(h) \) and \( x_2(h) \) read
\[ x_i(h) = 0, \quad x_2(h) = \begin{cases} h & \text{if } 0 \leq h \leq L \\ L & \text{if } h \geq L \end{cases} \]
(36)

Estimate for the difference of the maximum depth of penetration does not result from the above analysis. However the following statement is valid if the impactor \( T_0 \) stops its motion in the target while the impactor \( T_1 \) continues to move, the velocity of the second impactor is of the order of \( \varepsilon^2 \).

4. Some useful localized interaction models

The following models are the most widely used [3] among the localized interaction models:
\[ \Omega_p = \left( u, v \right) = A_i(u) v^2 + A_0(u), \]
\[ \Omega_f(u, v) = k \left[ \mu A_i(u) v^2 + A_0(u) \right] \]
(37)

where \( \mu \) is equal to 0 or 1, \( k \) is the friction coefficient and the functions \( A_0 \) and \( A_i \) determine the specific model. In this case the equation of the impactor motion has an analytical solution and a more straightforward proof of the area rules exists. The function \( \hat{D} \) has for the model of Eq. (37) the following form
\[ \hat{D}[\Phi(x, \theta); h, W] = \hat{D}_i[\Phi(x, \theta); h] W + \hat{D}_0[\Phi(x, \theta); h] \]
(38)

where
\[ \hat{D}_i[\Phi(x, \theta); h] = \int_{x(h)}^{x(0)} \int_0^{2\pi} A_i(u) \left[ 1 + k \mu \sqrt{1 - \frac{u^2}{u}} \right] u_1 d x d \theta, \quad \nu=0, 1 \]
(39)

The solution of Eq. (8) with the boundary condition given by Eq. (10) reads
\[ W[\Phi(x, \theta); h] = -\frac{2}{m} \exp\left\{ -\psi[\Phi(x, \theta); h] \right\} \]
\[ \times \int_{L+b}^{h} \hat{D}_0[\Phi(x, \theta); \chi] \]
\[ \times \exp\left\{ \psi[\Phi(x, \theta); \chi] \right\} d \chi \]
(40)

where
\[ \psi[\Phi(x, \theta); \chi] = \frac{2}{m} \int_{L+b}^{\chi} \hat{D}_i[\Phi(x, \theta); \theta] d \theta \]
(41)
Since both \( \partial N_r / \partial \varepsilon \) and the right-hand side of the second of Eq. (24) have a similar structure, an estimate similar to that given by Eq. (28) is valid

\[
\left( \partial N_r / \partial \varepsilon \right)_{\varepsilon=0} = O(\varepsilon) \tag{42}
\]

Eq. (42) implies the estimate for the Taylor-series expansion of \( W[r(x) + \varepsilon \xi(x, \theta); h] \) with respect to \( \varepsilon \):

\[
W[r(x) + \varepsilon \xi(x, \theta); h] = W[r(x); h] + \varepsilon O(\varepsilon^2) \tag{43}
\]

When \( h = 0 \), Eq. (43) represents the area rule for the ballistic limit.

5. Conclusions

The penetration of a three-dimensional rigid projectile into a target is analyzed under the assumption of the localized interaction between the projectile and the target. This general model comprises a wide variety of models used, mainly, to describe penetration into a ductile target. A three-dimensional impactor and a body of revolution with the same distribution of the cross-sectional areas are considered. If these two bodies have close shapes and the difference in their shapes is characterized by a small parameter \( \varepsilon \), the following area rules were determined: the difference in the ballistic limits of both impactors is of the order of \( \varepsilon^2 \); the difference in the velocities of the impactors, at a given depth inside the target, is of the order of \( \varepsilon^2 \) if both impactors have the same initial velocity before penetration.

Special-purpose experiments are needed to determine the range of validity of these theoretically established rules depending on the projectile shape, the target material and the range of the velocities.

References