Modeling of penetration by rigid impactors

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1. Introduction

Empirical and semi-empirical models are widely used in impact dynamics, in particular, for modeling of high-speed penetration into geological targets, metals, concrete and composites (see, e.g., references in Adeli et al. (1986), Backman and Goldsmith (1978), Bangash and Bangash (2005), Ben-Dor et al. (2005, 2006), Bulson (1997), Corbett et al. (1996), Li et al. (2005) and low-velocity penetration into granular media (Ambroso et al., 2005a,b; Goldman and Umbanhowar, 2008; Hou et al., 2005; Katsuragi and Durian, 2007; Lothe et al., 2004; Seguin et al., 2008; Tsimring and Volfson, 2005; Uehara et al., 2003). Many of these models employ the dependences between the impact (initial) velocity and the depth of penetration (DOP), which are often obtained by statistical processing of experimental results and are not based on the theoretical models of penetration. Consequently, it seems of interest to consider an inverse problem of determining such penetration models, i.e., the dependencies of forces acting on a penetrator vs. the instantaneous depth of penetration and velocity, which yield the known phenomenological correlations. To the best of our knowledge, there are only a few publications on this subject. This approach was considered and applied for solving particular problems of penetration mechanics by Ben-Dor et al. (2008), Eisl어 et al. (1998) and Pilyugin (2004) under the assumption that the resistance force depends only on the impactor’s velocity and does not depend on its location inside the shield. Ambroso et al. (2005a) analyzed low-speed penetration into granular media (for more details, see Section 4 in this study). In this note, we suggest a general approach to the problem and demonstrate its application for two classes of the dependencies of the resistance force vs. velocity and the instantaneous location of the penetrator.

2. Formulation of the problem

Consider a normal penetration of a rigid projectile into a semi-infinite shield along the axis $H$ where the coordinate $H$ determining the instantaneous location of the penetration is defined as the distance between some characteristic point of
the impactor (e.g., the center of mass or the leading edge of the impactor) and the front surface of the shield; the effects associated with the stage of the incomplete immersion of the impactor’s nose in the shield are neglected.

Assume that the empirical dependence, \( H_{\text{max}} = \Psi(V_{\text{imp}}) \), between the impact (initial) velocity, \( V_{\text{imp}} \), and the DOP, \( H_{\text{max}} \), is known. The problem is to determine a function \( R \) which denotes the total force acting at the impactor, that implies this empirical dependence for \( H \geq H_{\text{max}} \), where \( H = \Psi(0) \). Note that usually \( H = 0 \) in the case of high-speed impact. In this study, we assume that the total force, \( R \), depends on the impactor’s velocity, \( V \), and the instantaneous coordinate of the impactor, \( H \).

Newton’s second law describing the motion of the impactor with mass \( m \) can be written as follows:

\[
mV/dV/dh = -R(H, V).
\] (1)

The DOP, \( H_{\text{max}} \), which is defined as a depth whereby the penetrator’s velocity vanishes, is determined by the equation \( \zeta(V_{\text{imp}}, H_{\text{max}}) = 0 \), where \( \zeta = \zeta(V_{\text{imp}}, H) \) is the solution of Eq. (1) with the initial condition \( V(H_\text{t}) = V_{\text{imp}} \).

Using the dimensionless variables Eq. (1) can be rewritten as follows:

\[
dw/dh = -D(h, w),
\] (2)

where

\[
h = \frac{H}{H_\text{o}}, \quad h_\text{t} = \frac{H_{\text{t}}}{H_\text{o}}, \quad v = \frac{V}{V_\text{o}}, \quad w = \frac{V^2}{V_\text{o}^2}, \quad D(h, w) = \frac{2H_\text{o}}{mV_\text{o}^2} R(H_\text{o}, V_\text{o}, v).
\] (3)

\( V_\text{o} \) is some reference velocity and \( H_\text{o} \) is a characteristic length. The empirical dependence between the squared dimensionless impact velocity, \( w_{\text{imp}} = (V_{\text{imp}}/V_\text{o})^2 \), and the dimensionless DOP, \( h_{\text{max}} = H_{\text{max}}/H_\text{o} \), can be expressed in the following two forms:

\[
h_{\text{max}} = F_w(w_{\text{imp}}), \quad F_w(0) = h_\text{t} \quad \text{or} \quad w_{\text{imp}} = F_h(h_{\text{max}}), \quad F_h(h_\text{t}) = 0, \] (4)

where \( F_h \) and \( F_w \) are the known, mutually inverse functions.

The approach suggested in this study is validated and demonstrated for two types of functions which determine the drag force dependence.

**3. The case** \( D(h, w) = D_h(h) D_w(w) \)

Consider the case when the function \( D \) can be written as follows:

\[
D(h, w) = D_h(h) D_w(w), \] (5)

where \( D_h \) and \( D_w \) are some functions. Separating the variables in Eq. (2) and integrating from \( w_{\text{imp}} \) to 0 over \( w \) and from \( h_\text{t} \) to \( h_{\text{max}} \) over \( h \) we obtain:

\[
\int_0^{w_{\text{imp}}} \frac{dw}{D_w(w)} = \int_{h_\text{t}}^{h_{\text{max}}} D_h(h)dh.
\] (6)

Considering Eq. (6) as a definition of the functional dependence \( h_{\text{max}} \) vs. \( w_{\text{imp}} \) and differentiating both parts of Eq. (6) with respect to \( w_{\text{imp}} \) yields:

\[
D_w(w_{\text{imp}})D_h(h_{\text{max}})(dh_{\text{max}}/dw_{\text{imp}}) = 1.
\] (7)

Substituting \( h_{\text{max}} = F_w(w_{\text{imp}}) \) into Eq. (7) reduces Eq. (7) to the identity \( D_w(w_{\text{imp}})D_h(F_w(w_{\text{imp}}))F'_w(w_{\text{imp}}) = 1 \), which must be satisfied for arbitrary \( w_{\text{imp}} \). It is convenient to rewrite this identity using the variable \( w \) instead of \( w_{\text{imp}} \):

\[
D_w(w)D_h(F_w(w))F'_w(w) = 1.
\] (8)

If function \( D_h(h) \) in Eq. (5) is known, Eq. (8) implies the following expression for unknown function, \( D_w(w) \):

\[
D_w(w) = [D_h(F_w(w))F'_w(w)]^{-1}.
\] (9)

Using the dependence between \( w_{\text{imp}} \) and \( h_{\text{max}} \) in the form \( w_{\text{imp}} = F_h(h_{\text{max}}) \) Eq. (9) can be rewritten after the change of variables \( F_w(x) = F_h^{-1}(x) \) as \( D_w(F_h(h))D_h(h) - F'_w(h) = 0 \). If \( D_w(w) \) is known then

\[
D_h(h) = F'_w(h)/D_w(F_h(h)).
\] (10)

Eqs. (9) and (10) imply that the function \( D(h, w) \) is not determined uniquely on the basis of the known correlation between the impact velocity and the DOP.

As an example let us consider the widely used semi-empirical model of high-speed penetration dynamics:

\[
h_{\text{max}} = F_w(w_{\text{imp}}) = k_1 \ln(1 + k_2 w_{\text{imp}}), \quad w_{\text{imp}} = F_h(h_{\text{max}}) = [\exp(h_{\text{max}}/k_1) - 1]/k_2,
\] (11)

where \( k_1 \) and \( k_2 \) are parameters depending on the properties of the shield or/and on the shape of the impactor. In this case Eqs. (9) and (10) imply:

\[
D_w(w) = \frac{1 + k_2 w}{k_1 k_2 D_h(h_\text{t} \ln(1 + k_2 w))}, \quad D_h(h) = \frac{\exp(h/k_1)}{k_1 k_2 D_w([\exp(h/k_1) - 1]/k_2)}. \] (12)
Assume that $D = D(w)$. Substituting $D_h = 1$ into the first of Eqs. (12) yields the following expression for the function $D_w$:

$$D_w(w) = D = k_3 + k_4 w, \quad k_3 = 1/(k_1 k_2), \quad k_4 = 1/k_1. \quad (13)$$

Historically, the model determined by Eqs. (11) and (13) is associated with the names of Dideon, Helie, Petry, Ponsellet and Resel (Adeli et al., 1986; Backman and Goldsmith, 1978; Corbett et al., 1996). Models of this kind with physically different assumptions for calculating $k_1$ and $k_2$ are widely used until now (see references in Ben-Dor et al. (2005) and Li et al. (2005)).

Assume now that $D = D(h)$. Substituting $D_w = 1$ into the second of Eqs. (12) yields:

$$D_h(h) = D = k_5 \exp(k_6 h). \quad (14)$$

Note that the same dependence $h_{\text{max}}$ vs. $w_{\text{imp}}$ as given by Eq. (11) admits different models for the resistance force, $D$. The widely known model given by Eq. (13) is recovered from the set of the admissible models because of the assumption that $D$ depends only on the velocity of the impactor. Clearly, Eq. (12) allows constructing models when $D$ is a function of both arguments, e.g.,

$$D(h, w) = \frac{k_5 w}{1 - \exp(-k_6 h)}, \quad D(h, w) = \frac{k_3 k_4 (1 + k_2 w) h}{\ln(1 + k_2 w)}. \quad (15)$$

4. The case $D(h, w) = D_1(h) w + D_0(h) w^{1 - \beta}$

Let us consider now the case when the function $D$ can be written as follows:

$$D(h, w) = D_1(h) w + D_0(h) w^{1 - \beta}, \quad 0 < \beta \leq 1, \quad (16)$$

where $D_0$ and $D_1$ are some functions and $\beta$ is a parameter. Eq. (2) with the right-hand side given by Eq. (16) is a Bernoulli equation. Its solution with the initial condition $w(h_0) = w_{\text{imp}}$ reads (Korn and Korn, 1968):

$$w(h) = \frac{w_{\text{imp}} - \beta \int_{h_0}^{h} D_0(\tilde{h}) \psi(\tilde{h}) d\tilde{h}}{\psi(h)}, \quad \psi(h) = \exp \left( \beta \int_{h_0}^{h} D_1(\tilde{h}) d\tilde{h} \right). \quad (17)$$

Then the equation for the DOP can be written as follows:

$$w_{\text{imp}} = \beta \int_{h_0}^{h_{\text{max}}} D_0(\tilde{h}) \psi(\tilde{h}) d\tilde{h}. \quad (18)$$

Considering $w_{\text{imp}}$ as a known function of $h_{\text{max}}$, $w_{\text{imp}} = F_0(h_{\text{max}})$, differentiating Eq. (18) with respect to $h_{\text{max}}$ and replacing $h_{\text{max}}$ by $h$ yields:

$$F'_0(h) = \beta D_0(h) \psi(h). \quad (19)$$

Consequently, if the function $D_1$ is known the function $D_0$ can be determined from Eq. (19) as follows:

$$D_0(h) = F'_0(h)/[\beta \psi(h)]. \quad (20)$$

If function $D_0$ is known Eq. (20) can be viewed as an integral equation with respect to the function $D_1$:

$$\int_{h_0}^{h} D_1(\tilde{h}) d\tilde{h} = \chi(h), \quad \chi(h) = \frac{1}{\beta} \ln \left( \frac{F'_0(h)}{\beta D_0(h)} \right). \quad (21)$$

The condition $\chi(h_0) = 0$ yields:

$$D_0(h_0) = F'_0(h_0)/\beta. \quad (22)$$

Then the solution of the Eq. (21) reads (see, e.g., Petrovskii, 1957):

$$D_1(h) = \chi'(h) = \frac{1}{\beta} \left[ \frac{F''_0(h)}{F'_0(h)} - \frac{D_0'(h)}{D_0(h)} \right]. \quad (23)$$

The example that is considered below is concerned with a low-speed penetration into granular media. Uehara et al. (2003) found the following scaling for the DOP of a ball penetrating into a granular bed:

$$H_{\text{max}} = 0.14 \frac{1}{\mu} \left( \frac{\rho_g}{\rho_b} \right)^{1/2} d_b^{2/3} \left( \frac{v_{\text{imp}}^2}{2g} + H_{\text{max}} \right)^{1/3}, \quad (24)$$

where $\rho_g$ and $\rho_b$ are densities of grains and of the impacting ball, respectively, $\mu = \tan \theta$ is the grain-grain friction coefficient, $\theta$ is the repose angle of the grains, $d_b$ is the diameter of the ball, and $g$ is the acceleration of gravity, other notations are specified above. Let us use the dimensionless variables determined by Eq. (3) with
\[ H_0 = x^{3/2}, \quad V_0 = (2g)^{1/2} x^{3/4}, \quad x = 0.14 \frac{1}{\mu} \left( \frac{\rho_h}{\rho_s} \right)^{1/2} d_0^{2/3}. \]  

Eq. (24) can be rewritten as follows:

\[ w_{\text{imp}} = F_h(h_{\text{max}}), \quad F_h(h_{\text{max}}) = h_{\text{max}}^3 - h_{\text{max}}. \quad h_0 = 1. \]  

Solving the cubic algebraic equation \( h_{\text{max}}^3 - h_{\text{max}} = w_{\text{imp}} \) we can determine the function \( F_w(W_{\text{imp}}) \) in the following explicit form:

\[ F_w(W_{\text{imp}}) = \begin{cases} \frac{2}{\sqrt{3}} \cos \left( \frac{1}{2} \arccos W_s \right) & \text{if} \quad 0 \leq W_s \leq 1, \\ \frac{1}{\sqrt{3}} \left( \sqrt{W_s^2 + W^2} \right) & \text{if} \quad W_s > 1, \end{cases} \]  

where \( W_s = 1.5 \sqrt{3} w_{\text{imp}}, W_s = W_s + \sqrt{W^2 - 1} \).

Clearly, in this case \( h_0 \) is the depth at which the penetrating ball stops if it starts penetration into a granular bed without the initial (impact) velocity, i.e., the function \( F(h) \) is defined for \( h \geq 1 \) and information about the motion of a penetrator between the positions \( h = 0 \) and \( h = 1 \) is not available.

Following Ambroso et al. (2005a) let us consider the class of models given by Eq. (16) with \( \beta = 1 \) and \( D_1 = c \), where \( c \) is a constant used for calibrating the model taking into account the experimental data. Substituting these parameters and \( h_0 = 1 \) into Eqs. (17) and (20) allows us to determine \( \psi(h) \), \( D_0(h) \) and formulate the model as follows:

\[ D(h, w) = c w + (3h^2 - 1) \exp(-c(h - h_0)), \quad h \geq h_0. \]  

Since \( h_0 = 0 \) is also the root of the equation \( F_0(h) = 0 \), it is possible to construct a model that is formally valid in the interval \( 0 \leq h < 1 \) as well. Performing the above substitutions with \( h_0 = 0 \) instead of \( h_0 = 1 \) recovers the model proposed previously by Ambroso et al. (2005a) and determined by Eq. (28) with \( h_0 = 0 \). Both models imply the relationship \( h_{\text{max}}^3 - h_{\text{max}} = w_{\text{imp}} \).

5. Concluding remarks

The developed method allows constructing a set of penetration models which imply the given dependence between the impact velocity and the DOP. The most essential features of the proposed method are as follows: (1) The dependence between \( h_{\text{max}} \) and \( w_{\text{imp}} \) does not allow one to determine uniquely the drag force acting on the impactor, \( D(h, w) \). This property considerably expands the applicability of the method by allowing us to analyze different classes of functions which take into account physics of penetration phenomenon. These classes of functions can be constructed by combining several functions, each of them depending either on \( h \), or \( w \). (2) In order to determine the expression for \( D(h, w) \) in a closed analytic form it is necessary to select in the analysis such functions that allow solving the equation of motion in quadratures. Clearly, the choice of these functions is not restricted by those considered in this investigation. (3) Procedure for solving the problem depends upon the choice of the class of functions for the analysis. In this study we considered only several characteristic approaches which illustrate the general idea.

Clearly, the application of the described method does not guarantee the adequacy of the obtained equations which must be validated using the traditional in penetration mechanics methods.

The suggested method can be modified and generalized, e.g., the given dependence between the impact velocity and the DOP can be replaced by the dependence between the ballistic limit velocity and the thickness of the penetrated plate. It is feasible also to extend the class of the considered models of penetration and consider other types of functions \( D(h, w) \) which determine the total force. The proposed method can be also useful in other fields of mechanics.

References


