ANALYTICAL SOLUTION FOR PENETRATION BY RIGID CONICAL IMPACTORS USING CAVITY EXPANSION MODELS

G. Ben-Dor, A. Dubinsky and T. Elperin
Department of Mechanical Engineering, The Pearlstone Center for Aeronautical Engineering, Ben-Gurion University of the Negev, P. O. Box 653, Beer-Sheva 84105, Israel

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Introduction

The cylindrical cavity expansion approximation allows to determine the explicit analytical expressions for the forces acting on a penetrating impactor at each location of a target-impactor contact surface [1-3]. However in order to determine penetration characteristics (the ballistic limit velocity, the residual velocity, the depth of the penetration) it is necessary to solve an equation of motion of an impactor with appropriate initial conditions. At this stage of the solution additional simplifying assumptions (e.g., simplified description of the incomplete immersion of the impactor into a target and/or assumption about a constant velocity in calculating the target resistance) are introduced in order to obtain analytical formulae. In this study the analytical formulae for penetration characteristics are derived without using these simplifying assumptions.

Description of the model

Consider a normal penetration of a rigid conical-nosed impactor of mass M into a target. All the notations are shown in Fig. 1. The coordinate h is defined as the distance of the nose of the impactor from the front surface of the target. The coordinate x is associated with the target. Only the nose of the impactor with the length L interacts with the target, and the impactor may have also the cylindrical part with the length \( \ell \), \( r_0 \) is the impactor’s base radii and \( \theta \) is the half angle of the apex of the cone. We take into account that for incomplete immersion of the impactor into a target only a part of the impactor interacts with the target. This part of the lateral surface of the nose of the impactor is located between the cross-sections \( x = x_1 \) and \( x = x_2 \) (see Fig. 1) where (for details see [4])

\[
\begin{align*}
    x_1(h) &= \begin{cases} 
    0 & \text{if } 0 \leq h \leq b \\
    h & \text{if } b \leq h \leq b + 1 
    \end{cases}, \\
    x_2(h) &= \begin{cases} 
    h & \text{if } 0 \leq h \leq 1 \\
    1 & \text{if } 1 \leq h \leq b + 1 
    \end{cases}
\end{align*}
\]  

(1)
for a target with a finite thickness, and

\[ \bar{x}_1(h) = 0, \quad \bar{x}_2(h) = \begin{cases} \bar{h} & \text{if } 0 \leq \bar{h} \leq 1 \\ 1 & \text{if } \bar{h} \geq 1 \end{cases} \] (2)

for a semi-infinite target, \( \bar{x}_1 = x_1/L \), \( \bar{x}_2 = x_2/L \), \( \bar{h} = h/L \), \( \bar{b} = b/L \).

Fig. 1. Coordinated and notations.

The impactor-target interaction is described by the following solution of the problem of hole's expansion from the time \( t=0 \) [1-3]:

\[ p = a_2 \dot{R}^2 + a_1 \ddot{R} + a_0 \] (3)

where \( R \) is the radius of the hole, \( p \) is the pressure applied in the normal direction at the part of the impactor's surface, coefficients \( a_j \) \((j = 0, 1, 2)\) depend on the properties of the material of the target, and formulas for these coefficients can be found in the above cited studies. Since (see Fig. 1) \( R = \beta x \), \( \dot{R} = \beta \dot{x} \), \( \ddot{R} = \beta \ddot{x} \), \( x = h - \xi \), \( \beta = \tan \theta \), Eq. (3) can be written as follows:
\[ p = a_2 \beta^2 h^2 + a_1 \beta^2 x h + a_0 \]  

(4)

The force \( d\vec{F} \) acting at the surface element \( dS \) of the impactor at a given location at the surface of the impactor which is in contact with the target can be described as \( d\vec{F} = (p n^0 + k p t^0) dS \), where \( n^0 \) and \( t^0 \) are the inner normal and tangent unit vectors at a given location at the surface of the impactor, respectively, \( k \) is the friction coefficient. The total force \( \vec{F} \) is determined by integrating the local force over the impactor-target contact surface \( S \). The drag force \( D \) is directed in the opposite direction to the projection of the total force on the direction of the motion of the impactor \( h^0 \):

\[ D = -\vec{h}^0 \int_{S} d\vec{F} = 2\pi \beta (\beta + k) \int_{x_1(h)}^{x_2(h)} px dx \]  

(5)

Using the variable \( h^2 = v^2 = w(h) \) the equation of motion of the impactor \( M d^2 h/dt^2 = -D \) can be written as follows:

\[ [A_0 + A_1 J_2(h)] \frac{dw}{dh} + [A_2 w + 1] J_1(h) = 0 \]  

(6)

where \( v \) is the velocity of the impactor, \( \bar{x} = x/L \),

\[ J_\mu(h) = \frac{\int_{x_1(h)}^{x_2(h)} \bar{x}^\mu d\bar{x}}{\int_{x_1(h)}^{x_2(h)} d\bar{x}} = \frac{[\bar{x}^\mu_2(h)]^{\mu+1} - [\bar{x}^\mu_1(h)]^{\mu+1}}{\mu + 1} , \quad \mu = 1, 2 \]  

(7)

\[ A_0 = \frac{M}{4\pi a_0 \beta (\beta + k) L^3} > 0, \quad A_1 = \frac{a_1 \beta^2}{2a_0} > 0, \quad A_2 = \frac{a_2 \beta^2}{a_0} > 0, \]  

(8)

**Target with a finite thickness**

If an initial (impact) velocity \( v_{in} = v(0) \) is considered as an initial condition, the solution of Eq. (6) can be written as follows:

\[ \Psi(v_{in}, v) = \psi(h) \]  

(9)

where
\[
\Psi(v_{in}, v) = \frac{\psi_0^2}{v^2} \frac{dw}{A_2 w + 1}, \quad \psi(h) = \int_0^h \frac{J_1(z)dz}{A_0 + A_1 J_2(z)}
\]  

(10)

If \(v = v_{res}\) for \(h = b + 1\), we arrive at the relation between the initial velocity \(v_{in}\) and the residual velocity \(v_{res}\), \(\Psi(v_{in}, v_{res}) = \psi(b + 1)\). The ballistic limit velocity \(v_*\), i.e., the initial velocity which corresponds to zero residual velocity, is determined from the equation \(\Psi(v_*, 0) = \psi(b + 1)\).

Omitting the mathematical details, we present formulae which can be derived from the above equations:

\[
\frac{\psi_0^2}{v_*^2} - B \frac{\psi_{res}^2}{v_*^2} = 1, \quad v_* = \sqrt{\frac{B - 1}{A_2}},
\]  

(11)

where

\[
B = \exp[A_2 \psi(b + 1)], \quad \psi(b + 1) = 3Gg^2/(2A_1)
\]  

(12)

\[
G = \begin{cases} 
\chi(g) - \chi\left[(1 - b)g\right] & \text{if } b \leq 1 \\
\chi(g) + \left((3A_0/A_1) + 1\right)^{-1/3} & \text{if } b \geq 1
\end{cases}
\]

(13)

\[
\chi(z) = \int_0^z \frac{dz}{1 - \xi^2} = \frac{1}{\sqrt{3}} \arctan\left(\frac{2z + 1}{\sqrt{3}}\right) - \frac{1}{6} \ln \frac{(z - 1)^2}{z + z + 1} - \frac{\pi}{6\sqrt{3}}
\]  

(14)

Semi-infinite target

Equation (9) allows to find the depth of penetration of an impactor into a semi-infinite target \(h_*\) for a given initial (impact) velocity \(v_{in}\). Substituting \(v = 0\) and \(h = h_*\) in Eq. (9) yields the following equation:

\[
\Psi(v_{in}, 0) = \psi(h_*)
\]  

(15)

where \(\Psi\) and \(\psi\) are determined by Eq. (10) taking into account Eq. (2). Since the function \(\psi(h_*)\) increases from 0 to \(\infty\), when \(h_*\) increases from 0 to \(\infty\), Eq. (15) has a unique solution for every \(v_{in} > 0\). Omitting the algebraic manipulations we present only the final formulas:
\[ h_\ast = \begin{cases} 
\left( \frac{1}{2} \frac{v_\text{in}^2}{v_1^2} + 1 \right)^{2A_1/A_2} - 1 \right)^{1/3} & \text{if } v_\text{in} \leq v_1 \\
1 + \frac{2A_1(e+1)}{3} \left[ \frac{1}{A_2} \ln \left( A_2 v_\text{in}^2 + 1 \right) - \frac{1}{A_1} \ln \left( \frac{e+1}{e} \right) \right] & \text{if } v_\text{in} \geq v_1
\end{cases}
\] (16)

where
\[ v_1 = \sqrt{\left( \frac{e+1}{e} \right)^{A_2/(2A_1)} - 1} / A_2, \quad e = \frac{3A_0}{A_1} \] (17)

The above formulae take into account two stages of penetration. In the first, initial stage, the impactor's nose only partially penetrates into the target, and the penetration resistance depends on the depth of penetration. If the impact velocity is relatively small, the penetration terminates at this stage. If the impact velocity is sufficiently large for penetration to a depth more than \( L \) (the second stage), the penetration resistance becomes constant when \( h > L \).

Concluding remark

Some known phenomenological localized interaction models [5-7] can be described by Eq. (3) with \( a_1 = 0 \). The obtained above formulas can be adopted for this case by passing to a limit \( A_1 \to 0 \).

References