Shape Optimization of Impactors Against a Finite Width Shield Using a Modified Method of Local Variations

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Abstract: We suggest a modification of the method of local variations of Banichuk and Chernous’ko that expands further the scope of applications of this method beyond the applications considered in our previous study. This modification allows us to solve numerically the variational problem of determining the shape of the impactor having a minimum ballistic limit velocity (BLV) when the criterion of optimization is a double integral with an integrand including a double integral over a variable area. The integrands in both integrals depend on the unknown function determining the generatrix of the impactor. We found that impactors with flat or conical nose have a minimum BLV and derived an analytical criterion that determines the choice between these two shapes depending on the parameters describing the size of the impactor’s nose and properties of the material of the shield. The proposed approach can be useful in solving other optimization problems.

Keywords: Ballistic limit velocity; Impact; Method of local variations penetration; Optimization; Perforation.

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1. INTRODUCTION

Overview of the studies on shape optimization of penetrating impactors using engineering models of impactor-shield interaction can be found in recent review and monograph of Ben-Dor et al. (2005, 2006). It must be emphasized these studies considered variational problems arising for semi-infinite shields whereas shape optimization of impactors against shields with a finite thickness was not considered at all (except for Ben-Dor et al., 2002). The latter study is an exception in this respect although it employed very simple model, which allowed application of the classical calculus of variations. The absence of studies on shape optimization of impactors against shields with a finite thickness can be explained by the following reasons. Analysis of perforation of thick shields with finite thickness must take into account plug formation and the incomplete immersion of the impactor at the initial and final stages of penetration. However, incorporation of these factors even in the simple two-term model of impactor-shield interaction (Ben-Dor et al., 2005, 2006) yields a formula for the ballistic limit velocity (BLV) of the impactor (body of revolution) in the form of a nonintegral and cumbersome functional depending on the generatrix of the impactor. Variational problem for the functionals of this type cannot be solved analytically, and to the best of our knowledge, commonly used numerical methods cannot be directly used for the solution of the problem.

The method of local variations (Banichuk et al., 1969; Chernous’ko and Banichuk, 1973) was successfully employed in its original version for shape optimization of bodies of revolution penetrating into semi-infinite shields when the special two-term impactor-shield interaction model is used (Ben-Dor et al., 2003). Modification of the method of local variations (Ben-Dor et al., 2007) allowed solving similar problem for a wide class of impactor-shield interaction models. In this study, we suggest a further modification of the method of local variations (MLV) for more complicated functionals and apply the suggested modification for numerical optimization of the shape of the impactor against a finite width shield.

2. FORMULATION OF THE PROBLEM AND MATHEMATICAL MODEL

Consider a high-speed normal penetration of a rigid body of revolution into a ductile shield with thickness $b$. The notations are shown in Figs. 1(a)–(b). The coordinate $h$, the depth of the penetration, is defined as the distance between the nose of the impactor and the front surface of the shield. The coordinates $x$ and $\rho$ are associated with the impactor.
Figure 1(a)–(b). Coordinates and notations. Two-stage penetration model.

The part of the lateral surface of the impactor between the cross-sections \( x = \theta \) and \( x = \Theta \) interacts with the shield where (see Figs. 2(a)–(b))

\[
\vartheta(h; b, L) = \begin{cases} 
0 & \text{if } 0 \leq h \leq b \\
(h - b) & \text{if } b \leq h \leq b + L
\end{cases},
\]

\[
\Theta(h; b, L) = \begin{cases} 
h & \text{if } 0 \leq h \leq L \\
L & \text{if } L \leq h \leq b + L
\end{cases}
\]

(1)

It is assumed that the impactor generally consists of a non-cylindrical nose with the length \( L \) and of a cylindrical part. Only the nose interacts with the shield, and interaction between the impactor and the
shield occurs when $0 \leq h \leq b + L$. In coordinates $x$, $\rho$ associated with the impactor the surface of the nose is described by the following equation:

$$\rho = \Phi(x), \quad \Phi(0) = r, \quad \Phi(L) = R, \quad \Phi' \geq 0, \quad (2)$$

where the increasing function $\Phi(x)$ determines the generatrix of the impactor’s nose.

We assume that the impactor-shield interaction is described by the following widely used model (for details see, e.g., Ben-Dor et al., 2005, 2006; Recht, 1990):

$$d\vec{F} = [a_1(-\vec{v}^0 \cdot \vec{n}^0)^2 v^2 + a_0] \vec{n}^0 dS, \quad (3)$$

where $d\vec{F}$ is the force acting at the lateral surface element $dS$ of the projectile that is in contact with the shield, $\vec{n}^0$ is the inner normal unit vectors at a given location on the projectile’s surface, $\vec{v}^0$ is a unit vector of the surface element velocity of the projectile, $\vec{v}$ (translational
velocity of the impactor), the parameters \( a_0 \) and \( a_1 \) characterize the properties of the material of the shield. It is usually assumed that \( a_1 \) equals to the density of the material of the shield, \( \gamma_{sh} \).

The drag force acting at the lateral surface of the impactor, \( D_{imp} \), is found by integrating \((-\vec{v}) \cdot d\vec{F} \) over this surface. Using formulae of differential geometry we obtain:

\[
D_{imp}(h, v) = 2\pi (a_0 I_0 + a_1 I_1 v^2),
\]

where

\[
I_\mu(h) = \int_{\Theta(h; b, L)}^{\Theta(h; b, L)} G_\mu(\Phi, \Phi') dx = \int_0^L \eta(x, h; b, L) G_\mu(\Phi, \Phi') dx,
\]

\[
\Phi' = d\Phi/dx
\]

\[
G_\mu(\Phi, \Phi') = \Phi \Phi' [(\Phi^2 + 1)]^\mu, \quad \mu = 0, 1,
\]

\[
\eta(x, h; b, L) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \Theta(h; b, L) \\
1 & \text{if } \Theta(h; b, L) \leq x \leq \Theta(h; b, L) \\
0 & \text{if } \Theta(h; b, L) \leq x \leq L
\end{cases}
\]

For a blunt impactor, we use a simplified model of plugging (see, e.g., Goldsmith and Finnegan, 1971; Sagomonyan, 1988) whereby it is assumed that a cylindrical plug with the radius \( r \), height \( b \), and mass

\[
m_{plug} = \pi \gamma_{sh} r^2 b
\]

is formed at the beginning of penetration, and this plug moves together with the impactor while \( h \leq b \) (the first stage of penetration, see Fig. 1(a)). Formula for the resistance force associated with the plug during this stage of the penetration reads:

\[
D_{plug}(h) = 2\pi \tau_{sh} r (b - h),
\]

where \( \tau_{sh} \) is the shear strength of the material of the shield. If \( h = b \), the impactor and the plug break apart, and the impactor with mass \( m \) moves under the action of the force \( D_{imp} \) (the second stage of penetration, see Fig. 1(b)). Consequently, equations describing the motion of the impactor read:

\[
(m + m_{plug}) \ddot{h} + D_{imp}(h, \dot{h}) + D_{plug}(h) = 0, \quad 0 \leq h < b,
\]

\[
m \ddot{h} + D_{imp}(h, \dot{h}) = 0, \quad b \leq h \leq b + L
\]
Equations (10) and (11) can be replaced by one equation:

\[
[m + m_{plug} \delta(h; b)] \ddot{h} + D_{imp}(h, \dot{h}) + \delta(h; b)D_{plug}(h) = 0,
\]

where

\[
\delta(h; b) = \begin{cases} 
1 & \text{if } 0 \leq h \leq b \\
0 & \text{if } h > b 
\end{cases}
\]

Let us introduce a new function \( w = v^2 = \dot{h}^2 \) of the independent variable \( h \) and take into account that \( 2\ddot{h} = dw/dh \). Then Eq. (12) can be written as

\[
\frac{1}{2} [m + m_{plug} \delta(h; b)] \frac{dw}{dh} + D_{imp}(h, \sqrt{w}) + \delta(h; b)D_{plug}(h) = 0
\]

Substituting \( D_{imp}, m_{plug}, \) and \( D_{plug} \) from Eqs. (4), (8), and (9), respectively, into Eq. (14) we obtain:

\[
[m + \pi\gamma_{sh}r^2 b \delta(h; b)] \frac{dw}{dh} + 4\pi[a_1 I_1(h)w + a_0 I_0(h) + \tau_{sh}(b - h) \delta(h; b)] = 0
\]

The ballistic limit velocity (BLV) is defined as the initial velocity of the impactor that is required in order to emerge from the shield with zero velocity. Let \( w = w(h) \) be the solution of Eq. (15) with the initial condition \( w(b + L) = 0 \). Then the BLV, \( v_{bl} \), is determined by the formula

\[
v_{bl}^2 = w_{bl} = w(0).
\]

Using the dimensionless variables

\[
\tilde{h} = \frac{h}{L}, \quad \tilde{w} = \frac{w}{w_s}, \quad w_s = \frac{a_0}{\gamma_{sh}}, \quad \text{(16)}
\]

Equation (15) can be rewritten as follows:

\[
\frac{d\tilde{w}}{d\tilde{h}} + f(\tilde{h})\tilde{w} + g(\tilde{h}) = 0,
\]

where

\[
f(\tilde{h}) = \frac{k_s I_1(\tilde{h})}{\varphi(\tilde{h})}, \quad g(\tilde{h}) = \frac{\tau_s r(b - \tilde{h}) \delta(\tilde{h}; \tilde{b}) + I_0(\tilde{h})}{\varphi(\tilde{h})},
\]

\[
\varphi(\tilde{h}) = 0.25[m_s + \tilde{b}\tilde{r}^2 \delta(\tilde{h}; \tilde{b})], \quad \tilde{r} = \tau - \tilde{J},
\]

\[
I_\mu(\tilde{h}) = \int_0^1 \eta(\tilde{x}, \tilde{h}; \tilde{b}, 1) G_\mu(\tilde{\Phi}, \tilde{\Phi}) d\tilde{x}, \quad \tilde{J} = \int_0^1 \tilde{\Phi} d\tilde{x}, \quad \text{(18)}
\]
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\[ \Phi = \frac{\Phi}{L} \], \[ \Phi' = \frac{d\Phi}{d\bar{x}} \], \[ \bar{x} = \frac{x}{L} \], \[ \bar{b} = \frac{b}{L} \], \[ \tau = \frac{R}{L} \],

\[ k_* = \frac{a_1}{\gamma_{sh}} \], \[ \tau_* = \frac{\tau_{sh}}{a_0} \], \[ m_* = \frac{m}{\pi \gamma_{sh} L^3} \].

Equation (17) yields the following general expression for the dimensionless BLV:

\[ \bar{v}_{bl}^2 = \bar{v}_{bl} = \int_0^{\bar{b}+1} g(\bar{h}) \exp\left( \int_0^{\bar{h}} f(\bar{H}) d\bar{H} \right) d\bar{h}, \quad \bar{v}_{bl} = \frac{\bar{v}_{bl}}{\sqrt{\bar{w}_*}} \].

Equation (19) yields formulas for the particular cases of the sharp cone nosed impactor

\[ \bar{u}_{bl}^{cone} = \frac{\tau^2 + 1}{k_* \tau^2} \left[ \exp\left( \frac{2k_* \tau^4 \bar{b}}{(\tau^2 + 1) m_*} \right) - 1 \right], \quad \Phi(\bar{x}) = \tau \bar{x}, \]

and a cylindrical impactor with a flat bluntness

\[ \bar{w}_{bl}^{cyl} = \frac{2\tau_* b^2}{m_* + \bar{b}^2 \tau_*}, \quad \Phi(\bar{x}) = \tau. \]

Let us consider the problem of minimizing the BLV. It is assumed that the model describing impactor-shield interaction, parameters determining the mechanical properties of the shield, the mass of the impactor, and the length and the shank radius of the nose of the projectile, is given. In dimensionless variables, the problem is reduced to the determining the non-decreasing function \( \Phi(\bar{x}) \) that satisfies the constraints

\[ \Phi(1) = \tau, \quad \tau \bar{x} \leq \Phi(\bar{x}) \leq \tau, \quad \Phi'(\bar{x}) \geq \frac{\tau - \Phi(\bar{x})}{1 - \bar{x}} \geq 0, \quad 0 \leq \bar{x} < 1, \]

and minimizes the functional in Eq. (19). The parameters \( \bar{b}, \tau, k_*, \tau_* \), and \( m_* \) are assumed to be known. The second constraint in Eq. (22), stipulates that the optimum shape is sought between the sharp cone and the cylinder. The third inequality represents the necessary (but not sufficient) condition of convexity of the generatrix: the tangent in every point of the generatrix must be located not lower than the straight line segment between this point and the final point of the generatrix. Hereafter we assume that the parameter \( a_2 \) has the meaning of the density of the shield material, i.e., \( k_* = 1 \).

3. ADAPTATION OF THE METHOD OF LOCAL VARIATIONS

The functional \( \bar{w}_{bl}[\Phi(\bar{x})] \) in Eq. (19) is quite involved and the known methods of minimization cannot be applied directly for its minimization.
We will reduce the problem of minimization of this functional to a variational problem for the functional that is a function of some integral functionals. The latter problem can be solved using an efficient numerical method of local variations (Banichuk et al., 1969; Chernous’ko and Banichuk, 1973).

The Newton–Cotes formulas for approximate calculation of the external integral in Eq. (19) can be written as follows (Korn and Korn, 1968):

\[ \overline{w}_{bl} \approx \Delta \overline{h} \sum_{i=0}^{N} \alpha_{i}^{(N)} g(\overline{h}_{i}) \exp \left( \int_{0}^{\overline{h}_{i}} f(\overline{H}) d\overline{H} \right) , \]  

where \( N + 1 \) is the number of nodes in which the values of the integrand are calculated,

\[ \Delta \overline{h} = (\overline{b} + 1)/N, \quad \overline{h}_{i} = i \Delta \overline{h}, \quad i = 0, 1, \ldots, N \]  

Coefficients \( \alpha_{i}^{(N)} \) depend on the type of the Newton–Cotes formula. In particular, \( \alpha_{0}^{(N)} = \alpha_{1}^{(N)} = \cdots = \alpha_{N-1}^{(N)} = 1, \quad \alpha_{N}^{(N)} = 0 \) for the rectangle rule formula, and \( \alpha_{0}^{(N)} = \alpha_{N}^{(N)} = 0.5, \quad \alpha_{1}^{(N)} = \alpha_{2}^{(N)} = \cdots = \alpha_{N-1}^{(N)} = 1 \) for the trapezoid rule formula.

The integral in Eq. (23) can be approximated in a similar manner and Eq. (23) can be rewritten as follows:

\[ \overline{w}_{bl} \approx \Delta \overline{h} \sum_{i=0}^{N} \alpha_{i}^{(N)} g(\overline{h}_{i}) \exp \left( \Delta \overline{h} \sum_{j=0}^{i} \alpha_{j}^{(i)} f(\overline{h}_{j}) \right) , \]  

where for simplicity we used the same nodes in both sums. The appropriate choice of \( N \) allows attaining the required accuracy of the approximation.

Inspection of Equations (18) and (25) shows that the functional \( \overline{w}_{bl} [\overline{\Phi} (\overline{x})] \) is, generally, a function of \( (2N + 1) \) integrals, \( \overline{T}_{0}(\overline{h}_{1}), \overline{T}_{0}(\overline{h}_{2}), \ldots, \overline{T}_{0}(\overline{h}_{N}); \overline{T}_{1}(\overline{h}_{1}), \overline{T}_{1}(\overline{h}_{2}), \ldots, \overline{T}_{1}(\overline{h}_{N}) \) and \( \overline{J} \), taking into account that \( \overline{T}_{0}(0) = \overline{T}_{1}(0) = 0 \).

4. NUMERICAL RESULTS, DISCUSSION, AND CONCLUSIONS

As most of the existing numerical methods, the method of local variations allows determining only the local extrema. In order to overcome the latter shortcoming, calculations must be performed with different initial functions, and the obtained results must be compared. In our calculations, we use the following initial functions:

\[ \overline{\Phi} (\overline{x}) = \xi + (\tau - \xi) \overline{x}^\varepsilon, \quad \xi \leq \tau \]  

(26)
with different $\xi$ and $\varepsilon$. In the majority of calculations performed using these initial functions, iterations converge to one of the following three functions describing (1) sharp cone, (2) cylinder, and (3) a non-convex generatrix that is very close to the generatrix of a sharp cone and yields practically the same BLV.

Therefore, the minimum value of the BLV is attained for the sharp cone if $\tilde{w}^{\text{cone}}_{bl} < \tilde{w}^{\text{cyl}}_{bl}$ or for the cylinder if $\tilde{w}^{\text{cone}}_{bl} > \tilde{w}^{\text{cyl}}_{bl}$. If $\tilde{w}^{\text{cone}}_{bl} = \tilde{w}^{\text{cyl}}_{bl}$, then both shapes yield the same BLV. Using Eqs. (20) and (21) one can define the magnitude of the parameter $\tau_*^{\text{div}}$, $\tau_*^{\text{div}}$ for which the condition $\tilde{w}^{\text{cone}}_{bl} = \tilde{w}^{\text{cyl}}_{bl}$ is satisfied:

$$\tau_*^{\text{div}} = \frac{(\tau^2 + 1)(m_\ast + \tilde{b}\tau^2)}{2\tilde{b}^2\tau^3}\left[\exp\left(\frac{2\tau^4\tilde{b}}{(\tau^2 + 1)m_\ast}\right) - 1\right], \quad k_\ast = 1. \quad (27)$$

Clearly, the sharp cone or the cylinder is an optimum shape if $\tau_*^{\text{div}} > \tau_*^{\text{div}}$ or $\tau_* < \tau_*^{\text{div}}$, respectively. For relatively large $m_\ast$,

$$m_\ast \gg \tilde{b}\tau^2, \quad 2\tau^4\tilde{b} \ll (\tau^2 + 1)m_\ast, \quad (28)$$

Equation (27) can be simplified by keeping only two terms in Taylor series expansion for the exponent:

$$\tau_*^{\text{div}} = \frac{\tau}{\tilde{b}}. \quad (29)$$

Since the solution is determined by simple analytical relationships, the obtained results are exemplified only in two plots. Figures 3(a)–(b) shows the variation of $\tau_*^{\text{div}}$ as a function of $\tilde{b}$ and $m_\ast$ for $\tau = 0.25$ and $\tau = 0.5$. Generally, the shape of the optimum impactor depends on $\tau_\ast$, $\tilde{b}$, $m_\ast$, and $\tau$ but if $\tau_* \leq \tau/\tilde{b}$, then the cylinder is an optimum shape for impactor of any mass. Figures 4(a)–(b) compare the BLV of the impactors with sharp conical, flat (cylinder) and ogive-shaped noses. The values of the BLV in these figures, $v_{bl}$, are normalized by the BLV of the conical-nosed impactor, $v^{\text{cone}}_{bl}$.

The obtained solution has a clear physical meaning in some limiting cases. If $\tau_* \rightarrow 0$, then the BLV of the cylinder $v^{\text{cyl}}_{bl} \rightarrow 0$, whereas the BLV of any non-cylindrical impactor is positive since energy is required for expanding the cavity in the shield during penetration. Clearly, the advantage of the cylinder is retained for small $\tau_*$. On the other hand, for large $\tau_\ast$, perforation with plugging becomes not advantageous, and sharp-nosed projectile is an impactor with an optimum shape under these conditions.

Therefore, we demonstrated that (1) impactor with the optimum sharp nose is the cone and (2) the minimum BLV is attained if the nose of the impactor is conical or the impactor is a cylinder. The choice of the
optimum shape is performed by comparison of the BLVs of these two impactors.

Further investigation of the relationships determined in this study requires analysis of the large volumes of data obtained in experiments and numerical simulations. Each cycle of experiments/simulations must be conducted for the same shields varying the shapes of the impactors (including cylindrical and conical impactors) while keeping $R$, $L$, and $m$ constant. To the best of our knowledge only several series of such calculations and experiments were published in the literature (Børvik et al., 2002a,b; Dey et al., 2004; Gupta et al., 2007). Analysis of these results allows us to draw a conclusion that they do not contradict our findings.
Børvik et al. (2002a,b) studied experimentally and numerically perforation of the thick steel plate by the projectiles of the same mass and diameter with flat, semispherical, and conical noses. It was found that the BLV of the cylinder is considerably less than the BLVs of two other impactors. Dey et al. (2004) published the results of experiments and computer simulations on perforation of shields manufactured from three structural steels by projectiles with flat, conical, and ogival noses. It appears that the cylinder is the impactor with the minimum BLV. Gupta et al. (2007) conducted the experimental and numerical investigations on perforation of aluminum plates by blunt, ogive, and semispherical nosed projectiles. Certainly reference to the latter study in the context of our
investigation is not completely justified since only thin plates were used in Gupta et al. (2007). However it must be noted that for the plate with the maximum width of 2.5 mm, the BLV of the cylinder is considerably smaller than for the impactors with other shapes.

Clearly, the proposed approach can be applied for functionals that differ from the functional in Eq. (19). Consequently, the suggested modification of the method of local variations can be employed for solving optimization problems involving complicated functionals that are encountered in various design problems.

REFERENCES


