Modification of the Method of Local Variations for Shape Optimization of Penetrating Impactors Using the Localized Impactor/Shield Interaction Models

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Abstract: Implementation of the localized interaction models for description of impactor-shield interaction implies that the depth of penetration of the impactor (body of revolution) is represented as an integral over a parameter (the velocity of the impactor) with the integrand depending on this parameter and on integral which in its turn depends on this parameter and on the unknown function determining the shape of the impactor. We suggest a modification of the method of local variations of Banichuk and Chernous’ko for determining the shape of the impactor with the maximum depth of penetration. The suggested optimization procedure can be employed for arbitrary localized interaction models. The latter allows a practically unlimited freedom in choosing a particular localized interaction models for shape optimization of penetrators. As an example of application of the method we studied a version of the localized interaction models with friction coefficient depending on sliding velocity.

Keywords: Impactor; Method of local variations; Optimization; Penetration.

INTRODUCTION

Many engineering models for high-speed penetration modeling belong to the class of localized interaction models (LIMs; Ben-Dor et al., 2005,
2 Ben-Dor et al. (2006a) whereby the integral effect of the interaction between a host medium and a moving rigid non-deformable projectile is described as a superposition of the independent local interactions of the projectile’s surface elements with the medium. Every local interaction is determined by the local geometric and kinematic parameters of the surface element (primarily, by the local velocity of the surface element and the angle between the local surface velocity vector and the local normal vector to the projectile surface) as well as by some global parameters that represent the integral characteristics of the medium (e.g., hardness, density, and so on).

These relatively unsophisticated models take into account the influence of the shape of the impactor on penetration and they are very convenient for shape optimization of penetrating impactors. Overview of the publications on this subject can be found in Ben-Dor et al. (2005, 2006a). The problem of determining the shape of the impactor (body of revolution) that attains the maximum depth of penetration (DOP) can be reduced to a classical variational problem for integral functional only for the simplest LIMs, particularly, for two-term model without friction (Ben-Dor et al., 2001). However, even the three-term model without friction yields the variational problem where the criterion of optimization is the function of integral functionals (FIF). Although the latter problem can be investigated analytically (Ben-Dor et al., 2006b) this is rather the exception than the rule for such functionals. Consequently, although the two-term model with constant friction can be also reduced to the maximization of FIF, the analytical approach is not feasible, and numerical methods must be used (Ben-Dor et al., 2003; Jones and Rule, 2000).

Generally, in the case of LIM, the DOP is represented as an integral over a parameter (the velocity of the impactor) with the integrand depending on this parameter and integral which in its turn depends on the parameter and on the unknown function determining the shape of the impactor. In this study we show that the method of local variations (Banichuk et al., 1969; Chernous’ko and Banichuk, 1973) that was employed previously for some special LIMs (Ben-Dor et al., 2002, 2003), can be used on a regular basis for numerical investigation of the problem of determining the impactor (body of revolution) with the maximum DOP provided that impactor-shield interaction is described by a LIM.

We applied this approach to the model of impactor-shield interaction with a non-constant friction coefficient (FC). According to the data reported in the literature (for details see the review in Ben-Dor et al., 2006c) a typical behavior of FC is well documented. FC decreases (and assumes very low values for pairs metal/metal) with the increase of sliding velocity (SV) and pressure starting from a low (as compared to the velocities encountered in high speed penetration) velocities. However, there are only a few studies where the corresponding relationships for
the conditions of high-speed penetration are proposed (Davis, 2003; Jones et al., 2003; Klepaczko, 2002), and, to the best of our knowledge, only in Neely-Horton et al. (2004) the problem of impactor’s shape optimization is considered taking into account a nonconstant FC. The function describing the generatrix of the impactor with the maximum DOP is chosen by Neely-Horton et al. (2004) among functions with three free parameters. In this study, we solve the problem about impactor penetrating to a maximum depth as a variational problem.

STATEMENT OF PROBLEM FOR THE GENERAL LIM

The unified description of the LIM is as follows:

\[
\begin{align*}
\mathbf{dF} &= \begin{cases} 
\left[ \Omega_n(u, v)\mathbf{n}^0 + \Omega_t(u, v)\mathbf{\tau}^0 \right] ds & \text{if } 0 < u < 1 \\
\Omega_n(1, v)\mathbf{n}^0 ds & \text{if } u = 1 \\
0 & \text{if } u \leq 0
\end{cases}, \\
\mathbf{\tau}^0 &= -\mathbf{\bar{v}}^0 + u \cdot \mathbf{n}^0 \sqrt{1 - u^2}, \\
u &= -\mathbf{\bar{v}}^0 \cdot \mathbf{n}^0 = \cos v,
\end{align*}
\]

where (see Fig. 1) \(d\mathbf{F}\) is the force acting at the surface element \(dS\) of the projectile that is in contact with the host medium, \(\mathbf{n}^0\) and \(\mathbf{\tau}^0\) are the inner normal and tangent unit vectors at a given location on the projectile’s surface, respectively, \(\mathbf{\bar{v}}^0\) is an unit vector of the surface element velocity of the projectile, \(\mathbf{\bar{v}}, v\) is the angle between the vector \(\mathbf{n}^0\) and the vector \((-\mathbf{\bar{v}}^0)\). The nonnegative functions \(\Omega_n\) and \(\Omega_t\) determine normal stress and shear stress, respectively; they describe the model of the projectile-medium interaction and depend on the parameters that characterize, primarily, the properties of the host medium. The unit tangent vector \(\mathbf{\tau}^0\) lies in the plane of the vectors \(\mathbf{\bar{v}}^0\) and \(\mathbf{n}^0\) and is normal to the vector \(\mathbf{n}^0\); its direction is chosen such that \(\mathbf{\bar{v}}^0 \cdot \mathbf{\tau}^0 < 0\), i.e. the friction force is directed in the positive direction of the vector \(\mathbf{\tau}^0\).

The first and the second equations (see Eq. (1)) determine the interaction force between the impactor and the shield upon their contact while the third equation corresponds to the case when there is no contact. The case with \(u = 1\) is described separately by the second equation because the choice of the direction of the tangent vector \(\mathbf{\tau}^0\) in this situation is undetermined.

Consider a high speed normal penetration of a rigid sharp striker (a body of revolution) into a semi-infinite shield. The basic notations
are shown in Fig. 2. The coordinate $h$, the instantaneous depth of penetration, is defined as the distance between the nose of the impactor and the surface of the shield. In cylindrical coordinates, $x, \rho, \vartheta$, associated with the impactor its nose surface is described by the
following equation:

\[
\rho = \Phi(x), \quad 0 \leq x \leq L, \quad 0 \leq \vartheta \leq 2\pi, \quad \Phi(0) = r, \quad \Phi(L) = R, \quad \Phi' \geq 0,
\tag{4}
\]

where \( L \) is the length of the impactor’s nose that interacts with the shield and the increasing function \( \Phi(x) \) defines the generatrix of the impactor’s nose. If the DOP, \( H \), is much larger than \( L \), the stage of the incomplete immersion of the impactor’s nose in the shield can be neglected, and the drag force \( D \) is calculated by integrating \((-\vec{v}_0 \cdot \vec{d}F)\) over the impactor’s nose surface:

\[
\tilde{D}(v) = \frac{D(v)}{\pi} = r^2 \Omega_n(1, v) + 2 \int_0^L \omega(\Phi', v) \Phi \, dx,
\tag{5}
\]

where

\[
\omega(\Phi', v) = \Phi' \Omega_n(u, v) + \Omega_u(u, v), \quad u = \Phi'/\sqrt{\Phi'^2 + 1}, \quad \Phi' = d\Phi/dx.
\tag{6}
\]

The equation of motion of the impactor, \( m(d^2h/dt^2) = -D \), can be rewritten as follows:

\[
mv(dv/dh) = -\pi\tilde{D}(v),
\tag{7}
\]

where the velocity of the impactor, \( v \), is considered to be a function of \( h \), and \( m \) is the mass of the impactor. The DOP for a given impact velocity, \( v_{imp} \), is defined as the depth where the velocity of the impactor vanishes. The formula for the DOP, \( H \), can be obtained from Eq. (7):

\[
\tilde{H} = \frac{\pi}{m}H = \int_0^{v_{imp}} \frac{vdv}{D(v)},
\tag{8}
\]

We consider the problem of maximizing the DOP for a given impact velocity. The model describing impactor/shield interaction, parameters determining the mechanical properties of the shield, the mass of the impactor, the length and the shank radius of the nose of the projectile are assumed to be given. The problem is reduced to the optimization of the functional \( \tilde{H} \) in Eq. (8), whereas the solution must satisfy two last conditions in Eq. (4).

**ADAPTATION OF THE METHOD OF LOCAL VARIATIONS**

The Newton–Cotes formulas for approximate calculation of the integral in Eq. (8) can be written as follows (Korn and Korn, 1968):

\[
\tilde{H} \approx \Delta v \sum_{i=0}^N \alpha_i \frac{v_i}{\tilde{D}(v_i)},
\tag{9}
\]
where $N + 1$ is the number of nodes in which the values of the integrand are calculated,

$$
\Delta v = v_{imp}/N, \quad v_i = i\Delta v, \quad i = 0, 1, \ldots, N,
$$

(10)

The coefficients $\alpha_i$ depend on the version of the Newton-Cotes integration formula. In particular, $\alpha_0 = \alpha_1 = \cdots = \alpha_{N-1} = 1, \alpha_N = 0$ for the rectangle formula, and $\alpha_0 = \alpha_N = 0.5, \alpha_1 = \alpha_2 = \cdots = \alpha_{N-1} = 1$ for the trapezoid formula. The appropriate choice of $N$ allows attaining the required accuracy of the approximation. If $\tilde{D}(0) \neq 0$ then the term with $i = 0$ in the sum in Eq. (9) can be omitted.

Equations (5), (6), and (9) allow representing $\tilde{H}$ as a function of the integral functionals:

$$
\tilde{H} = f(J_0, J_1, \ldots, J_N) = \Delta v \sum_{i=0}^{N} \alpha_i \frac{v_i}{J_i},
$$

(11)

where

$$
J_i = \tilde{D}(v_i) = r^2 \Omega_n(1, v_i) + 2 \int_0^L \omega(\Phi', v_i) \Phi \, dx.
$$

(12)

An efficient numerical method for solving such problems is the method of local variations (Banichuk et al., 1969; Chernous’ko and Banichuk, 1973). A brief description of this method may be found in Ben-Dor et al. (2006a). For the sake of completeness, in the following section, we summarize the basic concepts of this method. The unknown function $\Phi(x)$ is represented approximately as a piecewise linear function determined by its values at the nodes of interpolation. Then every integral $J_i$ can be represented as a sum whereby every term depends only on the values of the function $\Phi(x)$ in no more than in two adjacent nodes of interpolation. The essence of the method of local variations is that it determines the rule of sequential variation of the values of the function $\Phi(x)$ in the nodes that maximizes the optimization criterion. The important advantage of the method is that it requires recalculating no more than two terms in the sum when the function is varied in any node. Decreasing the function variation increment and increasing the number of nodes of interpolation allow attaining the desired accuracy. As most of the existing numerical methods, the method of local variations allows to find only the local extremum. In order to overcome the latter shortcoming calculations must be performed with different initial functions, and the obtained results must be compared.

Equations (11) and (12) imply that the variation of the function in an interior interpolation node requires recalculation of $2(N + 1)$ terms in the sums for the integrals $J_0, J_1, \ldots, J_N$ and calculation of the new value of the optimization criterion.
Consider the particular case when the model of impactor-shield interaction is such that the function \( \omega(\Phi', v) \) can be represented as

\[
\omega(\Phi', v) = \sum_{\kappa=1}^{M} \varphi_\kappa(v) \psi_\kappa(\Phi'),
\]

(13)

where \( \varphi_\kappa, \psi_\kappa (\kappa = 1, 2, \ldots, M) \) are some functions. Then Eq. (12) can be rewritten as follows:

\[
J_i = \tilde{D}(v_i) = r^2 \Omega_n(1, v_i) + 2 \sum_{\kappa=1}^{M} \varphi_\kappa(v_i) G_\kappa, \quad G_\kappa = \int_{0}^{L} \psi_\kappa(\Phi') \Phi \, dx. \quad (14)
\]

Consequently, variation of the function in an interior interpolation node requires recalculating \( 2M \) terms in the sums for the integrals \( G_1, G_2, \ldots, G_M \) and recalculating the optimization criterion. In this particular case, since usually \( M \ll N \), the procedure requires less computational time.

**TWO-TERM MODEL WITH FRICTION COEFFICIENT DEPENDENT ON SLIDING VELOCITY**

Consider the most widely used model of impactor-shield interaction with friction (for details see Ben-Dor et al., 2005, 2006a; Recht, 1990):

\[
\Omega_n(u, v) = a_2 u^2 v^2 + a_0, \quad \Omega_t(u, v) = \mu \Omega_n(u, v),
\]

(15)

where coefficients \( a_0 \) and \( a_2 \) depend on the mechanical properties of the material of the shield, and let us assume that the FC \( \mu \) depends on the SV \( w \), i.e.

\[
\mu = \mu(w), \quad w = v(-\bar{v}^0 \cdot \bar{v}^0) = v \sin v = v \sqrt{1 - u^2}.
\]

(16)

Then

\[
\omega(\Phi', v) = (a_2 u^2 v^2 + a_0)[\mu(w) + \Phi'],
\]

(17)

and Eqs. (10)–(12) yield the following expression for the DOP written using the dimensionless variables:

\[
H = \frac{\bar{m}}{\pi} \bar{H} = t_i \bar{H}, \quad \bar{H} = F(\bar{J}_0, \bar{J}_1, \ldots, \bar{J}_N) = \Delta \bar{v} \sum_{i=1}^{N} \zeta_i \frac{\bar{v}_i}{\bar{J}_i} \quad (18)
\]
where
\[
\overline{J}_i = r^2 (\overline{v}_i^2 + 1/t_0) + 2 \int_0^1 \left( \overline{v}_i^2 \frac{\overline{\Phi}'}{\overline{\Phi}^2 + 1} + \frac{1}{t_0} \right) \left[ \overline{\Phi}' + \mu \left( \frac{v_{imp} \overline{v}_i}{\sqrt{\overline{\Phi}^2 + 1}} \right) \right] \overline{\Phi} \, d\overline{x},
\]  
(19)

\[
t_0 = \frac{a_2}{a_0} v_{imp}^2, \quad t_1 = \frac{m}{\pi L^2 a_2}, \quad \overline{v}_i = \frac{v_{imp}}{v_{imp}} = \frac{i}{N},
\]
(20)

\[
\Delta \overline{v} = \frac{\Delta v}{v_{imp}} = \frac{1}{N}, \quad \overline{x} = \frac{x}{L}, \quad \overline{\Phi} = \frac{\Phi}{L}, \quad \overline{\Phi}' = \frac{d\overline{\Phi}}{d\overline{x}}.
\]

Since the theory does not specify the type of the function $\mu(w)$, we use the simple empirical relationship that was proposed for high-speed penetration modeling by Jones et al. (2003) and Davis (2003) (see Fig. 3):

\[
\mu(w) = \begin{cases} 
\mu_0 - (\mu_0 - \mu_{\infty}) \left( \frac{w}{w_0} \right) & \text{if } 0 \leq w \leq w_0, \\
\mu_{\infty} & \text{if } w \geq w_0
\end{cases}
\]
(21)

where $\mu_0$ and $\mu_{\infty}$ are the values of FC for very small and large SV, respectively, and $w_0$ is the value of SV when the transition from SV dependent FC to a constant FC occurs.

Then the optimization problem in a dimensionless form can be formulated as maximization of the functional $\overline{H}$ subjected to the following conditions:

\[
\overline{\Phi}(1) = \tau, \quad \overline{\Phi}'(\overline{x}) \geq 0, \quad \tau \overline{x} \leq \overline{\Phi}(\overline{x}) \leq \tau, \quad 0 \leq \overline{x} \leq 1,
\]
(22)

\[\text{Figure 3. Friction coefficient as a function of sliding velocity (Davis, 2003).}\]
where \( t_0, t_2 = v_{imp}/w_0, \mu_0, \mu_\infty \) and \( \tau = R/L \) are known parameters. The third condition stipulates that the optimum shape is sought between the sharp cone and the cylinder.

**NUMERICAL RESULTS AND CONCLUSIONS**

Davis (2003) calculated and verified the models determined by Eqs. (15) and (21) for several steels on the basis of experimental data for ogive-nosed impactor with \( L = 21.06 \text{mm} \), \( R = 6.35 \text{mm} \) \((\tau = 0.302)\) and mass \( m = 65 \text{g} \). The latter parameter was calculated indirectly using the plots in Davis (2003). We conducted the first series of optimization calculations (Figs. 4(a), (b) and 5(a), (b)) for the same maximum sizes \((L \text{ and } R)\) and mass of the impactor and used the models of Davis (2003) for Experimental Steel 2 with the parameters \( a_0 = 473 \text{ MPa}, a_2 = 1125 \text{ kg/m}^3, \mu_0 = 0.5770, \mu_\infty = 0.0079 \), \( w_0 = 201 \text{ m/s} \) and for AF1410 Steel with the parameters \( a_0 = 479 \text{ MPa}, a_2 = 1125 \text{ kg/m}^3, \mu_0 = 0.5770, \mu_\infty = 0.0551, w_0 = 479 \text{ m/s} \).

Figure 4(a), (b) shows the shape of the generatrix of the optimum impactor which coincides with that of a sharp cone for \( v_{imp} \leq v^*_\text{imp} \), where \( v^*_\text{imp} \) is some boundary value. For \( v_{imp} > v^*_\text{imp} \), increasing \( v_{imp} \) implies the increase of the convexity of the optimum generatrix. For large \( v_{imp} \), when friction is not taken into account, the solution for the optimum generatrix actually recovers the optimum solution of Newton (for details see, e.g., Ben-Dor et al., 2006a). The sharp cone is also an optimum impactor for all considered impact velocities if the FC is assumed to be equal to the maximum value \( \mu = \mu_0 \).

Figure 5(a), (b) allows comparing the DOP of the optimum impactor with the DOPs of the impactors having other shapes. Inspection of these figures shows that the DOPs of the optimum impactor and of the optimum truncated cone are very close, whereas the DOP of the ogive-nose impactor (the corresponding curve coincides with that from Davis, 2003) is slightly lower than the maximum. Note that the values of the

![Figure 4. Shape of the optimal impactor for various impact velocities.](image-url)
DOP for these shapes are close to the experimental data (light circles) from Davis (2003) for ogive-nosed impactors. In the case of the model without friction, the DOP is close to the optimum “SV dependent FC” value for Experimental Steel 2 (Fig. 4(a)) and is significantly less than for AF1410 Steel. This may be explained, mainly, by the difference in the magnitudes of the parameter $w_0$ determining the range of velocities where the influence of friction is essential.

Although the results of calculation using real physical parameters of the impactor and the shield are illustrative, more information about the optimal solution may be obtained when the problem is investigated using the dimensionless variables. The dimensionless DOP of the optimum impactor (the ratio of the DOP of the optimum impactor to the DOP of a sharp cone with the same maximum size), $\eta$, and the function determining the generatrix of the optimum impactor in dimensionless coordinates, depend on the following dimensionless parameters: $t_2 = v_{imp}/w_0$, $t_3 = t_0/t_2^2 = a_2 w_0^2/a_0$, $\mu_0$, $\mu_\infty$, and $\tau$. The typical results of the numerical calculations are shown in Figs. 6–9 for $\mu_0 = 0.5$, $\mu_\infty = 0.03$, $\tau = 0.5$ where the effect of the parameters $t_2$ and $t_3$ on the shape of the optimum impactor and its DOP is analyzed.

Analysis shows that the optimum impactor is either a sharp cone or a body with a convex generatrix and a flat bluntness. The domains where the first and the second types of the shape are realized are shown in Fig. 6. Inspection of Fig. 6 reveals that, for a given material of the shield (given $t_3$ and $w_0$), these shape types occur for relatively low and relatively high impact velocities, $v_{imp}$, respectively. With the increase of $v_{imp}$ the radius of the bluntness of the optimum impactor increases, and the curvature of its generatrix becomes more pronounced (Figs. 7(a), (b)).

The comparison of the DOP of the optimum impactor with the DOPs of the optimum truncated conical impactor, sharp conical impactor and ogive-nosed impactor is showed in Fig. 8. Inspection
Modification of Local Variations for Shape Optimization

Figure 6. Type of the optimum impactor on $t_2$ and $t_3$ plane.

of Fig. 8 shows that the DOPs of the optimum impactor and of the optimum truncated cone are close, whereas the DOP is considerably smaller for the ogive-nose impactor. The radius of the bluntness of the optimum truncated conical impactor is always not less than the radius of the bluntness of the optimum impactor (Fig. 9).

All numerical calculations for determining the optimum shape of the impactor have been performed with the accuracy that guarantees the error in the magnitude of the optimization criterion less than 0.1%. One calculation requires several CPU minutes of IBM PC. Consequently, the suggested adaptation of the method of local variations is applicable and very convenient for solving the problems of shape optimization of penetrating impactors using engineering models. Moreover, this modification of the method of local variations is valid for functionals

Figure 7. Shape of the optimum impactor as a function of parameter $t_2$. 
Figure 8. Dimensionless DOP (normalized by the DOP of the sharp cone) of the optimum impactor, optimum truncated-conical impactor and ogive-shaped impactor as a function of parameters $t_2$ and $t_3$.

Figure 9. Dimensionless radius (normalized by the radius of the shank) of a flat bluntness of the optimum impactor and the optimum truncated-conical impactor as a function of parameters $t_2$ and $t_3$.

with more general structure, particularly, when the optimization criterion is represented as an integral over a parameter with the integrand depending on this parameter and several integrals which in their turn depend on the parameter and on the unknown function.
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