Optimization of two-component composite armor against ballistic impact

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Abstract

Using Florence's model we determined an optimal design of a two-component ceramic-faced lightweight armor against normal ballistic impact. The solution is found in a closed form that allowed us to determine the thicknesses of the plates in the optimal armor as functions of the specified areal density of the armor, parameters determining the material properties of the armor's components and characteristics of the impactor.

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1. Introduction

Florence's model [1] yields a relatively simple expression for the ballistic limit velocity (BLV), and it is actually the only model suitable for an analytic optimization of two-component ceramic armors. Florence proposed to use his model for determining an armor with the minimum areal density and gave an example of a numerical calculation for a ceramic/aluminum shield [1]. Similar calculations for ceramic/GFRP shield were performed by Hetherington and Rajagopalan [2]. Later, Hetherington [3] considered the problem of determining the structure of two-component armor with a given areal density that provides the maximum BLV. He suggested an approximate expression for the optimum value of the front plate to back plate thicknesses ratio. Wang and Lu [4] investigated a similar problem when the total thickness of the armor is specified rather than the areal density. The problem of designing an armor with the minimum areal density for a given BLV was investigated by Ben-Dor et al. [5]. It was shown that the solution of the optimization problem can be presented in terms of the dimensionless variables whereby all the characteristics of the impactor and the armor are expressed as functions of two independent dimensionless parameters. The latter solution allows to find the solution for the optimization problem for an arbitrary two-component composite armor in a closed analytical form. In this study we comprehensively investigated a problem considered in [3] for an arbitrary two-component armor and found the solution in a closed analytical form.

2. Formulation of the problem

Consider a normal impact by a rigid projectile on a two-layer composite armor consisting of a ceramic front plate and a ductile back plate. In this study we employ the following model:

\[ v_{bl}^2 = \frac{a_2 \sigma_2 b_2 z (\gamma_1 b_1 + \gamma_2 b_2 z + m)}{0.91 m^2} \]

\[ z = \pi (R + 2b_1)^2, \]

where \( v_{bl} \) is the ballistic limit velocity, \( m \) is a projectile's mass, \( R \) is a projectile's radius, \( b_i \) are the thicknesses of the plates, \( \sigma_i \) are the ultimate tensile strengths, \( \gamma_i \) are the breaking strain, \( \gamma_1 \) are the densities of the materials of the plates, subscripts 1 and 2 denote a ceramic plate and a back plate, respectively. For \( z = 1 \) Eq. (1) recovers the model suggested by Florence [1] as it was re-worked in [2]. More recently in [5] this model was generalized slightly by introducing a coefficient \( \alpha \) that can be

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determined using the available experimental data in order to increase the accuracy of the predictions.

The objective of the present study is to find the thicknesses of the plates, \(b_1, b_2\), that provide the maximum ballistic limit velocity \(v_{bl}\) for a given areal density of the armor

\[
A = \gamma_1 b_1 + \gamma_2 b_2. \tag{2}
\]

Introduce (for details see [5]) the dimensionless variables \(\tilde{b}_1, \tilde{b}_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\omega}, \tilde{A}\) using the following formulas:

\[
b_i = \tilde{b}_i R, \quad \gamma_i = \frac{m}{\pi R^2} \tilde{\gamma}_i, \quad i = 1, 2,
\]

\[
v_{bl} = \tilde{\omega} \sqrt{\frac{2\sigma_2}{0.91 \gamma_2}}, \quad A = \frac{m}{\pi R^2} \tilde{A}.
\]

Using Eq. (3), the Eqs. (1) and (2) can be rewritten as follows:

\[
\tilde{\omega}^2 = \tilde{\gamma}_2 \tilde{b}_2 \tilde{\omega} \left( \left( \tilde{\gamma}_1 \tilde{b}_1 + \tilde{\gamma}_2 \tilde{b}_2 \right)^2 + 1 \right), \tag{4}
\]

\[
\tilde{A} = \tilde{\gamma}_1 \tilde{b}_1 + \tilde{\gamma}_2 \tilde{b}_2, \tag{5}
\]

where

\[
\tilde{z} = \frac{z}{\pi R^2} = (1 + 2\tilde{b}_1)^2. \tag{6}
\]

Substituting \(\tilde{\gamma}_2 \tilde{b}_2\) from Eq. (5) and \(\tilde{z}\) from Eq. (6) into Eq. (4) we obtain

\[
\tilde{\omega}^2 = \tilde{\omega} \psi(\tilde{A}, \beta, x), \tag{7}
\]

where

\[
\psi(\tilde{A}, \beta, x) = (1 - x) (\beta x + 1) [\tilde{A} (\beta x + 1)^2 + 1], \tag{8}
\]

\[
x = \frac{A_1}{\tilde{A}} = \frac{A_1}{A_1 + A_2}, \quad \beta = \frac{2 \tilde{A}}{\tilde{\gamma}_1}, \quad A_1 = \tilde{\gamma}_1 \tilde{b}_1, \quad A_2 = \tilde{\gamma}_2 \tilde{b}_2. \tag{9}
\]

Thus the problem is reduced to finding \(x, 0 \leq x \leq 1\), which provides the minimum \(\psi\) considered as a function of \(x\).

The solution of this problem depends only on two parameters, \(\tilde{A}\) and \(\tilde{\gamma}_1\). If

\[
x^{\text{opt}} = \phi_0(\tilde{A}, \tilde{\gamma}_1), \tag{11}
\]

provides the minimum \(\psi\) (hereafter a superscript \(\text{opt}\) denotes the optimal parameters), then the principal dimensionless parameters associated with the optimal solution (the thickness of the ceramic plate, \(b_1^{\text{opt}}\), the areal densities of the plates, \(A_1^{\text{opt}}\) and \(A_2^{\text{opt}}\), and their ratio, \(\omega^{\text{opt}}\), the ballistic limit velocity, \(\omega^{\text{opt}}\)) are also some functions of \(\tilde{A}\) and \(\tilde{\gamma}_1\)

\[
b_1^{\text{opt}} = \frac{\tilde{\omega}^{\text{opt}}}{\tilde{\gamma}_1} \phi_0(\tilde{A}, \tilde{\gamma}_1) = \phi_1(\tilde{A}, \tilde{\gamma}_1), \tag{12}
\]

\[
A_1^{\text{opt}} = \tilde{A} \phi_0(\tilde{A}, \tilde{\gamma}_1), \tag{13}
\]

\[
A_2^{\text{opt}} = \frac{\tilde{\gamma}_2 \tilde{b}_2^{\text{opt}}}{\tilde{A}} = \tilde{A} [1 - \phi_0(\tilde{A}, \tilde{\gamma}_1)] = \phi_2(\tilde{A}, \tilde{\gamma}_1). \tag{14}
\]

\[
x^{\text{opt}} = \frac{A_1^{\text{opt}}}{A_1^{\text{opt}}} = \frac{A_2^{\text{opt}}}{A_2^{\text{opt}}} = \frac{1}{\phi_0(\tilde{A}, \tilde{\gamma}_1)} - 1 = \phi_3(\tilde{A}, \tilde{\gamma}_1), \tag{15}
\]

\[
\tilde{\omega}^{\text{opt}} = \sqrt{\tilde{A} \psi \left( \frac{2 \tilde{A}}{\tilde{\gamma}_1}, \phi_0(\tilde{A}, \tilde{\gamma}_1) \right)} = \phi_4(\tilde{A}, \tilde{\gamma}_1). \tag{16}
\]

In order to elucidate the analysis based on the dimensionless variables we will refer (where it is possible) to the special kind of the armor that we will call a “basic armor” (BA). As a BA we selected the ceramic/GFRP armor, and used the experimental data on perforation of the armors with different thicknesses of the plates by a 0.50 inch projectile reported in [3]. For BA a transition from the dimensionless to the dimensional parameters, i.e., for the areal density \(A\) (kg/m\(^2\)), the widths of the plates and the ballistic limit velocity \(v_{bl}\) (m/s) is performed as follows: \(A = 370\tilde{A}, b_i = 6.35\tilde{b}_i, v_{bl} = 133\tilde{\omega} (\tilde{\gamma}_1 = 0.060\) corresponds to \(\tilde{\gamma}_1 = 3499\) kg/m\(^3\)).

3. Investigation of the function \(\psi(\tilde{A}, \beta, x)\)

Let us calculate the derivative

\[
\psi(x, \beta, x) = \frac{\partial \psi}{\partial x} = (\beta x + 1) f(\tilde{A}, \beta, x), \tag{17}
\]

where

\[
f(\tilde{A}, \beta, x) = c_2 x^2 + c_1 x + c_0, \tag{18}
\]

\[
c_0 = \tilde{A} (4\beta - 1) + 2\beta - 1, \quad c_1 = \beta [8\beta - 7] - 3 \tag{19}
\]

\[
c_2 = \tilde{A} \beta^2 (4\beta - 11), \quad c_3 = -5\beta^3.
\]

Hereafter, the parameters \(\tilde{A}\) and \(\beta\) are not listed as arguments of the corresponding functions. Consider a behavior of the function \(\psi(\tilde{A}, \beta, x)\) at the interval determined by Eq. (10). To this end let us calculate the values of the functions \(\psi(x)\) and \(f(x)\) at the end points of this interval

\[
\psi(0) = \tilde{A} + 1 > 0, \tag{20}
\]

\[
\psi(1) = 0, \tag{21}
\]

\[
f(0) = c_0 = \frac{\tilde{A} + 1}{\tilde{\gamma}_1} g(\tilde{A}, \tilde{\gamma}_1), \tag{22}
\]

\[
f(1) = -[\tilde{A} \beta^2 + 3\beta^2 + (3\tilde{A} + 1) \beta + \tilde{A} + 1] < 0, \tag{23}
\]

where

\[
g(\tilde{A}, \tilde{\gamma}_1) = \frac{4\tilde{A} (2\tilde{A} + 1)}{\tilde{A} + 1} - \tilde{\gamma}_1. \tag{24}
\]

The curve \(g(\tilde{A}, \tilde{\gamma}_1) = 0\) divides the domain \(\tilde{A} \geq 0, \tilde{\gamma}_1 \geq 0\) into two sub-domains determined by the conditions \(g < 0\) and \(g > 0\), correspondently (see Fig. 1). Consider now these two cases in more details taking into account that the third degree polynomial \(f(x)\) can have 1 or 3 real roots.
Assume that
\[ g(\bar{A}, \bar{\gamma}_1) < 0. \] (25)

Since \( f(x) \rightarrow +\infty \) when \( x \rightarrow -\infty \) and \( f(0) < 0 \), the equation \( f(x) = 0 \) has a root at the interval \( -\infty < x < 0 \) and, consequently, 0 or 2 roots at the interval \( 0 < x < 1 \).

Let us assume now that
\[ g(\bar{A}, \bar{\gamma}_1) > 0. \] (26)

Taking into account Eq. (23) one can conclude that the equation \( f(x) = 0 \) has 1 or 3 roots at the interval \( 0 < x < 1 \). Since \( \psi_1(0) > 0 \), then there exists an arbitrary small \( \varepsilon \) such that \( \psi_1(\varepsilon) > \psi(0) > \psi_1(1) \) and, consequently, a maximum \( \psi(\bar{x}) \) is attained not at the end points of the interval \([0,1]\) but at some interior point where \( \psi_1(\bar{x}) = f(\bar{x}) = 0 \).

Numerical simulation shows that equation \( f(x) = 0 \) does not have roots at the interval \([0,1]\) if Eq. (25) is valid, and it has one root in the opposite case. Thus, the maximum \( \bar{w} \) is attained at the point \( \bar{x} = 0 \) (the first case) and at the point where \( f(x) = 0 \) (the second case). The behavior of the function \( \bar{w}(x) \) is shown in Fig. 2(a) and (b), whereas Fig. 2(a) illustrates the transition from the case 1 (\( \bar{A} < \bar{A}' \)) to the case 2 (\( \bar{A} > \bar{A}' \)), where \( \bar{A} = \bar{A}' \) is the solution of the equation \( g(\bar{A}, \bar{\gamma}_1) = 0 \).

Let us consider now the case determined by Eq. (25). Formally, \( \psi(x) \) attains its maximum value for \( \bar{b}_1 = 0 \) when the employed physical model is not valid. Let us show that the case determined by Eq. (25) is of no practical significance.

The inequality given by Eq. (25) can be solved for positive \( \bar{A} \)
\[ \bar{A} < \Theta(\bar{\gamma}_1), \] (27)

where
\[ \Theta(\bar{\gamma}_1) = (1/16)[\bar{\gamma}_1 - 4 + \sqrt{\bar{\gamma}_1^4 + 24\bar{\gamma}_1 + 16}]. \] (28)

On the other hand, decreasing \( \psi(x) \) implies the inequality
\[ \bar{w} \leq \bar{w}(0) = \sqrt{\bar{A}} \Psi(0) = \sqrt{\bar{A}(\bar{A} + 1)}. \] (29)

Combining these inequalities we obtain
\[ \bar{w} < \theta(\bar{\gamma}_1), \] (30)

where
\[ \theta(\bar{\gamma}_1) = \sqrt{\Theta(\bar{\gamma}_1)[\Theta(\bar{\gamma}_1) + 1]}. \] (31)

Let us now estimate \( \bar{\gamma}_1 \). Substituting the mass of the cylindrical impactor in term of its density \( \gamma_{impr} \), length \( L \) and radius of the base \( R \),
\[ m = \pi R^2 L, \] (32)

into the second equation in Eq. (3) for \( \bar{\gamma}_1 \) we obtain
\[ \bar{\gamma}_1 = \frac{R}{L} \frac{\bar{\gamma}_1}{\gamma_{impr}}, \] (33)
i.e., indeed \( \gamma_1 \) is much less than 1. Then Taylor’s series expansion for small \( \gamma_1 \) yields

\[
\Theta(\gamma_1) = 0.25\gamma_1 - (0.25\gamma_1)^2 + O(\gamma_1^3),
\]

\( \Theta(\gamma_1) = 0.5\sqrt{\gamma_1} + O(\gamma_1^{5/2}). \)

Clearly, inequality given by Eq. (30) taking into account Eq. (35) is valid only for very small \( \gamma_1 \) that does not correspond to a ballistic impact conditions. Thus, e.g., for “basic armor” \( \gamma_1 = 0.06 \) and Eq. (30) implies that \( v_{bl} < 17 \text{ m/s} \). Note that equation \( A - 0.25\gamma_1 = 0 \) is a good approximation of the curve shown in Fig. 1.

### 4. Optimal armor

In a case when Eq. (26) is valid, the equation \( f(x) = 0 \) has one real root that is the location of the maximum of a function \( \psi(x) \). This root can be determined using Cardano’s formulae [6]

\[
x^\text{opt} = \sqrt[3]{\sqrt{D} - 0.5q - \sqrt[3]{D + 0.5q - (1/3)c_2/c_3}},
\]

where

\[
D = (p/3)^3 + (q/2)^2,
\]

\[
p = \frac{3c_1c_3 - c_2^2}{3c_3^2}, \quad q = \frac{2c_2^3}{27c_3^2} - \frac{c_1c_2}{3c_3} + \frac{c_0}{c_3}.
\]

The solution of the optimization problem in a graphical form is shown in Figs. 3–6. Fig. 4 shows that the dependence \( x^\text{opt} \) vs. \( \bar{A} \) is close to a linear function for every \( \gamma_1 \). Even Eq. (36) represents the solution of the considered problem in closed form. The family of the curves plotted in Fig. 4 can be more simply approximated with the average accuracy of 3% in the range 0.04 \( \leq \gamma_1 \leq 0.1 \), 0.05 \( \leq \bar{A} \leq 0.35 \) as follows:

Fig. 3. Values \( x \) corresponding to the optimal armor vs. given areal density of the armor.

Fig. 4. Optimal thickness of ceramic plate vs. given areal density of the armor.

Fig. 5. Optimal ratio of the areal density of the back plate, \( \bar{A}_2 \), to the areal density of the ceramic plate, \( \bar{A}_1 \), vs. given areal density of the armor.

Fig. 6. Maximum ballistic limit velocity vs. given areal density of the armor.
Given areal density, \( \bar{\gamma}_1 \) vs. given areal density, \( \bar{n} \) showed in Figs. 7–9 for BLV. The typical results that support this conclusion are density of the armor without considerable loss in the maximum, i.e., the thicknesses of the plates may be changed in the vicinity of the optimal values for a given areal density of the armor without considerable loss in the BLV. The typical results that support this conclusion are showed in Figs. 7–9 for \( \bar{\gamma}_1 = 0.06 \). Let us consider the dependence of the BLV and \( \bar{w} \) on some parameter, \( \eta \) (for instance, the thickness of the ceramic plate), that varies within the limits specified in the formulation of the problem. The lower, \( \eta^{\text{inf}} \), and the upper, \( \eta^{\text{sup}} \), values of the interval for \( \eta \) that imply variation of \( \bar{w} \) in the interval \( [(1 - \varepsilon)\bar{w}^{\text{inf}}, \bar{w}^{\text{sup}}] \) for several given values of \( \varepsilon \) are shown in Figs. 7–9 for \( \eta = \zeta = A_2/A_1 = A_2/A_1 \) (Fig. 7), \( \eta = x = A_1/A_2 = A_1/A_1 \) (Fig. 8) and \( \eta = \bar{b}_1 \) (Fig. 9). The results showed in Figs. 7–9 support the above property of the optimal solution. Thus, there exists a broad range of possible designs for the optimal lightweight two-component armor among the designs with almost identical ballistic performance.

5. Concluding remarks

A closed-form solution for the two-component armor optimization problem is found among the designs with a given areal density when the ballistic limit velocity (BLV) of the impactor is a target function. In addition to the exact solution, the simplified approximate solution is proposed as well. The behavior of the BLV near the optimum is investigated. It is shown that the thicknesses of the plates can be changed in a quite broad range in the neighborhood of the optimal design of the armor without decline in its defense properties.

References