Numerical solution for shape optimization of an impactor penetrating into a semi-infinite target

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Abstract

The shape of the impactor with the maximum depth of penetration (DOP) for a given impact velocity is found using a numerical procedure for solving a corresponding non-classical variational problem. It is shown that the optimum shape in a general case is close to a blunt cone. The variation of the optimal shape of the impactor and the dependence of the DOP vs. the initial (impact) velocity and friction coefficient is studied. The analysis is performed also for optimal conical impactors.

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1. Introduction

There are only a few publications on impactor’s shape optimization. In the first studies on this problem, the indirect criteria for optimization were used, namely, the resistance during motion of the penetrator inside a target with a constant velocity [1–4] or with a varying speed determined by the equation of motion of the impactor [5] and the "dynamical work" [6] determined by the model described in [7] (the adjusted solution is given in [8]). More recently, the direct criteria for optimization, the depth of penetration (DOP) in the case of a semi-infinite target and the ballistic limit velocity for a target with a finite thickness, were employed. Three-dimensional impactors [9–13] and bodies of revolution [14,15] were considered applying the criterion of the maximum DOP (MDP) into a semi-infinite target while the resistance of the target at the initial stage of penetration was neglected. The latter factor was taken into account in [16–21] where the impactor’s shape optimization problems for homogeneous and non-homogeneous targets were investigated.

2. Description of the model and formulation of the problem

Consider a normal penetration of a rigid penetrator (a body of revolution) into a ductile semi-infinite target. The notations are shown in Fig. 1. The coordinate $h$, the current DOP, is defined as the distance between the nose of the impactor and the front surface of the target. The cylindrical coordinates $x$, $\rho$, $\theta$ are associated with the impactor. The part of lateral surface of the impactor between the cross-sections $x = 0$ and $x = \chi$ interacts with the target where (see Fig. 2)

\[
\chi(h) = \begin{cases} 
  h & \text{if } 0 \leq h \leq L \\
  L & \text{if } h > L 
\end{cases}
\]

We assume that the impactor–target interaction at a given location at the impactor’s surface which is in
The total force $\vec{F}$ is determined by integrating the local force over the impactor–target contact surface. Using formulae from differential geometry applied to a body of revolution

$$dS_{lat} = \Phi \sqrt{\Phi_x^2 + 1} \, dx \, d\theta, \quad \vec{x} \cdot \vec{n} = \Phi \Phi_x / \sqrt{\Phi_x^2 + 1}$$  \hspace{1cm} (6)

the following expression for the drag force $D$ can be derived:

$$D = \vec{x}^0 \left( \int_{S_{nose}} d\vec{F}_{nose} + \int_{S_{lat}} d\vec{F}_{lat} \right)$$

$$= \pi (a_1 v^2 + a_0) r^2 + 2 \pi \int_0^{\chi(h)} \left( a_1 \frac{\Phi_x^2}{\Phi_x^2 + 1} v^2 + a_0 \right)$$

$$\times (\Phi_x + \mu) \Phi \, dx$$ \hspace{1cm} (7)

where the function $\rho = \Phi(x)$ determines the projectile's shape and $r = \Phi(0), R = \Phi(L)$ (see Fig. 1).

For certain $h_0 \gg L$ and function $f(x)$ the averaging procedure can be used (see Fig. 2):

$$\int_0^{\chi(h)} f(x) \, dx \approx \frac{1}{h_0} \int_0^{h_0} \int_0^{\chi(h)} f(x) \, dx \, dh$$

$$= \frac{1}{h_0} \int_0^{h_0} \int_x^{\chi(h)} f(x) \, dx \, dh$$

$$= \int_0^{L} \left( 1 - \frac{x}{h_0} \right) f(x) \, dx \approx \int_0^{L} f(x) \, dx$$ \hspace{1cm} (8)

Apply this procedure to the integral in Eq. (7). Then integrating from 0 to $\chi(h)$ can be changed to integrating from 0 to $L$.

Taking into account that:

$$\frac{d^2 h}{dt^2} = \frac{1}{2} \frac{dw}{dh}, \quad w(h) = v^2,$$ \hspace{1cm} (9)

the equation of motion of the impactor with the mass $m$ can be written as follows:

$$\frac{m}{2} \frac{dw}{dh} + a_1 K_1 w + a_0 K_0 = 0$$ \hspace{1cm} (10)

where

$$K_0 = r^2 + 2 \int_0^L (\Phi_x + \mu) \Phi \, dx$$

$$= r^2 + \Phi_x^2 \frac{L}{h = 0, 2 \mu L_0 = R^2 + 2 \mu L_0}$$ \hspace{1cm} (11)

$$K_1 = r^2 + 2 \int_0^L \frac{\Phi_x^2 (\Phi_x + \mu) \Phi \, dx}{\Phi_x^2 + 1} = K_0 - 2 l_1$$ \hspace{1cm} (12)

$$I_0 = \int_0^L \Phi \, dx, \quad I_1 = \int_0^L \frac{(\Phi_x + \mu) \Phi \, dx}{\Phi_x^2 + 1}$$ \hspace{1cm} (13)

Consider the motion of the impactor between the initial $(h = 0, w = v_0^2)$ and the final $(h = h_0, w = 0)$ locations. Eq. (10) yields:
Although the above expression for \( h_* \) does not contain the parameter \( \tau \) in explicit form, the model, in contrast to [15], allows one to account for the flat bluntness of the impactor. The latter is of importance because a high speed optimal impactor has a bluntness as it is shown below.

In particular, for a cone-shape impactor with an equation of generator

\[
\mathcal{F}(\bar{x}) = r_{cone} + \lambda \bar{x}, \quad \lambda = \tau - r_{cone}, \quad r_{cone} = r_{cone}/L
\]

Eq. (20) yields:

\[
T_1 = \left( \frac{\lambda + \mu}{\lambda^2 + 1} \right)^\tau \left( \tau - \frac{1}{2} \lambda \right)
\]

where \( r_{cone} \) is the radius of the flat bluntness of the cone.

We consider the problem of maximizing the DOP \( h_* \), for the semi-infinite target with the known parameters determining its mechanical properties and the friction coefficient. The mass of the impactor, the length and base radius of the nose of the projectile are assumed to be given. The problem is reduced to the optimization of the functional \( h_* \) in Eq. (18) with given parameters \( \mu, t, \tau \) whereas the solution \( \Phi(\bar{x}) \) must satisfy the conditions:

\[
\Phi(1) = \tau \quad \text{and} \quad \Phi(\bar{x}) \geq 0, \quad \tau \bar{x} \leq \Phi(\bar{x}) \leq \tau, \quad 0 \leq \bar{x} \leq 1
\]

The condition \( \tau \bar{x} \leq \Phi \) is introduced in order to eliminate from consideration sharp awl-shaped bodies that are of no practical significance.

Since the solution for \( \mu \neq 0 \) cannot be obtained in a closed analytical form, a numerical solution of the problem is found.

3. Method of solution

To solve the above maximization problem we developed a computer code based on the method of local variations [24]. \( \Phi(\bar{x}) \) is represented approximately as a piecewise linear function determined by the values \( \Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n-1)}, \Phi^{(n)} = \tau \) at the \( n + 1 \) points of interpolation, respectively, \( \bar{x}^{(0)} = 0, \bar{x}^{(1)}, \ldots, \bar{x}^{(n-1)}, \bar{x}^{(n)} = 1 \) where \( \Phi^{(j)} = \Phi(\bar{x}^{(j)}), \bar{x}^{(j)} = j \Delta \bar{x}, \Delta \bar{x} = 1/n, j = 0, 1, \ldots, n \). Then the integrals in Eq. (20) can be represented as:

\[
T_1(\Phi^{(0)}, \ldots, \Phi^{(n)}) = \sum_{j=1}^{n} T_1(\Phi^{(j)}, \Phi^{(j-1)})
\]

where

\[
T_1(\Phi^{(j)}, \Phi^{(j-1)}) = \Delta \bar{x} F_1 \left( \frac{\Phi^{(j)} + \Phi^{(j-1)}}{2}, \frac{\Phi^{(j)} - \Phi^{(j-1)}}{\Delta \bar{x}} \right)
\]

Eqs. (23) and (24) can be rewritten as follows:

\[
\Phi^{(j)} = \tau \quad \text{and} \quad \Phi^{(j)} \geq \Phi^{(j-1)}, \quad j = 1, \ldots, n
\]

\[
\tau \bar{x}^{(j)} \leq \Phi^{(j)}, \quad j = 0, \ldots, n - 1
\]

The maximization problem can be now formulated as to find \( n, \Phi^{(0)}, \ldots, \Phi^{(n-1)} \) that satisfy the conditions of Eqs. (27)–(29) and provide a maximum value \( h_* \). Eq. (18) with a required accuracy. The brief description of the numerical procedure that comprises three steps is presented below.

Step 1: Select the initial approximation (IA), i.e., the initial values \( n \) and \( \Phi^{(0)}, \ldots, \Phi^{(n-1)} \) taking into account Eqs. (27)–(29) and calculate \( T_1 \) and the current value of the criterion \( h_* \). Since the method assures determining only a local maximum, several calculations must be performed with various IAs. The IAs we use are the power-law functions (\( \Phi^{(j)} = \tau \bar{x}^{(j)}, 0 < \bar{x}^{(j)} < 1 \)) and the power-law functions (\( \Phi^{(j)} = \tau \bar{x}^{(j)}, 0 < \bar{x}^{(j)} < 1 \)).

Step 2: The following calculations are performed sequentially for \( j = 0, \ldots, n - 1 \). Let us change \( \Phi^{(j)} \) by the
increments $\Delta \Phi$ and $(-\Delta \Phi)$. If Eqs. (27)–(29) are satisfied then a new value of the criterion $h_+^y$ ($h_-^y$) is calculated otherwise it is set to $h_+^y = -\infty$ ($h_-^y = -\infty$). If $h_+^y > \max(h_+, h_-^y)$ then $\Phi^{(y)} + \Delta \Phi$ is adopted as a new value of the function at the point $x^{(y)}$ and $(h_+ = h_-^y)$ as a new value of the criterion. If $h_-^y > \max(h_+, h_-^y)$ then $\Phi^{(y)} - \Delta \Phi$ and $h_+ = h_-^y$ are adopted as new values of the function and of the criterion, respectively. If $h_+ \geq \max(h_+, h_-^y)$ then $h_+$ and $\Phi^{(y)}$ retain their values. In the calculations we used the increment $\Delta \Phi \propto (\Delta x)^2$. An

![Fig. 3. Shape of the generator of the optimal impactor.](image)

![Fig. 4. DOP of the optimal impactor, $\bar{h}_{\text{max}}$ as a function of the parameter $t$, for different $\tau$ and $\mu$.](image)
important point is that only two integrals at the most, \( I_j^{(j)} \) and \( I_j^{(j+1)} \), must be recalculated for updating the criterion when the value \( \Phi^{(j)} \) is changed. If the value of the function is changed in the cycle of the calculations for \( j = 0, \ldots, n-1 \) then all the cycle is repeated. If the function and the criterion cease to change at Step 2 of the procedure then we proceed to Step 3.

**Step 3:** If the function and the criterion were changed at Step 2 and the change of the criterion is less than the given allowable error then the calculations are stopped. Otherwise, \( n \) is doubled, the values of the function \( \Phi \) in the new nodal points of the grid are calculated using the linear approximation and we return to the Step 2.

### 4. Results and conclusions

Some typical result of numerical calculations are showed at Figs. 3–6.

Fig. 3a and b shows the shape of the generator of the optimum impactor which coincides with that of a sharp cone for \( t \leq t_c \), where \( t_c \) is a boundary value depending of \( \tau \) and \( \mu \) (Fig. 3a). The validity of the latter property for small \( t \) can be proved analytically by expanding \( h_1 \) as a power series in \( tK_1/K_{00} \).

![Fig. 5. Comparison between the DOP for the optimal impactor, \( h_{\text{max}} \) and optimal conical impactor \( h_{\text{max cone}} \).](image)

![Fig. 6. Radius of bluntness of the optimum blunt cone impactor \( r_{\text{opt cone}} \) normalized by the radius of the base of the impactor \( R \), for different \( \tau \) and \( \mu \).](image)
\[ h_{\infty} = \frac{1}{K_1} \sum_{t=1}^{\infty} (-1)^{t-1} \left( \frac{\eta t}{v} \right)^t, \quad \eta = K_1/K_0 \quad (30) \]

Since \( \eta < 1 \) (see Eq. (12)), the series in Eq. (30) converge for \( t \leq 1 \) and for small \( t \):

\[ h_{\infty} \approx \frac{t}{K_0} = \frac{t}{t^2 + 2\mu_0}, \quad T_0 = \int_0^1 \bar{\varphi} \, dx \quad (31) \]

Taking into account Eq. (29) we arrive at the conclusion that the minimum \( T_0 \) (maximum \( h_{\infty} \)) is attained when \( \bar{\varphi} = \bar{\varphi}x \), i.e., in the case of a cone-shape impactor. For \( t > t_c \), the optimal impactor has a plane bluntness.

Fig. 4a–d shows the MDP \( h_{\max} \) as a function of the parameter \( t \) (for convenience using the transformed variables). One can see that the influence of friction coefficient on the \( h_{\max} \) enhances significantly with increasing \( t \) (increasing the impact velocity for a given material of the target).

Fig. 5 shows that not only the shape of the optimum impactor is close to a blunt (in a general case) cone but the MDP of the optimal impactor, \( h_{\max} \), and the MDP of the optimal conical impactor, \( h_{\max} \text{cone} \), are close, and the difference decreases with the increase of the friction coefficient. Thus, the optimal blunt cone, the body with a simple shape, penetrates at the depth that is close to the optimal one. The radius of the bluntness of the optimal cone as a function of \( t \) for different \( t \) and \( \mu \) is showed in Fig. 6a–d.

Theoretical studies of problems of impactor shape optimization (especially those using semi-empirical models) need further comment. The main goal of these investigations, even when they yield quantitative results, is to determine qualitative relations that are useful in conducting experiments for designing optimum impactors or shields. In the latter case the optimum impactor corresponds to the worst case scenario. In particular, the results of the analysis performed in this study show that the blunt cone-shape impactors merit a further comprehensive investigation.

References