On the Generalized Degrees-of-Freedom of the Phase Fading Z-Interference Channel with a Relay

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Abstract

We study the generalized degrees of freedom (GDoF) of the phase fading (PF) Z-interference channel with a relay (Z-ICR) in the weak interference regime. We consider the scenario in which the relay node receives transmissions only from one of the sources but its transmissions are received at both destinations. We first derive two upper bounds on the achievable GDoF: the first bound is based on the genie-aided approach, and the second bound is based on the cut-set theorem. We then derive an achievable GDoF by employing the decode-and-forward (DF) strategy at the relay and by treating interference as noise at each receiver. Finally, we derive conditions on the gains of the links in the channel, under which this simple scheme achieves the GDoF upper bound. Therefore, we obtain conditions under which treating interference as noise is GDoF-optimal for the PF Z-ICR. It is observed that relaying strictly increases the GDoF of the channel compared to the scenario without a relay. Our results support the application of relaying in scenarios in which interference is weak.

I. INTRODUCTION

The interference channel (IC) models a communication scenario in which two source-destination pairs communicate over a shared medium. The capacity region of the IC is generally unknown but it has been characterized for some special scenarios. The cases of very strong and of strong interference between the communicating pairs were studied in [1] and [2], respectively, and it was shown that in these scenarios the optimal strategy at each receiver is to decode both the interfering message as well as the desired message. On the other extreme, the case of weak interference (WI) was studied in [3]-[5], and it was shown that when interference is weak enough, treating interference as noise is sum-rate optimal. As treating interference as noise is done via a low complexity, simple, point-to-point (PtP) decoding strategy, it motivates identifying additional scenarios in which treating interference as noise carries optimality.

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In this work we study the IC with an additional relay node. In the IC with a relay (ICR), the relay node assists the communications from both sources to their corresponding destinations. The optimal transmission strategy for the relay node in this channel is not known in general. One of the main difficulties is that when the relay assists one pair, it may degrade the performance of the other pair. The sum-rate capacity of the Gaussian IC with a potent relay in the WI regime was characterized in [6], which showed that in such a scenario, compress-and-forward (CF) at the relay and treating interference as noise at the destinations is sum-rate optimal. The sum-rate capacity of the ICR in the WI regime when the relay has a finite power remains unknown to date. Other figures-of-merit for characterizing the performance of the ICR in the WI regime are the degrees-of-freedom (DoF) and the generalized DoF (GDoF). In [7] it was shown that for ICs with time-varying/frequency-selective channel coefficients, when global channel knowledge is available at all nodes, and when all links have the same scaling behaviour vs. the signal-to-noise ratio (SNR), adding a relay does not increase the DoF region, and the achievable DoF for each pair in the ICR remains upper bounded by 1. On the other hand, in [8] it was shown that relaying can increase the GDoF of the symmetric Gaussian ICR. In [8], several GDoF upper bounds were derived using the cut-set theorem and genie aided methods for the case in which the source-destination, source-relay, and relay-destination links scale differently. The work [8] proposed a communication scheme which achieves the GDoF upper bound under certain conditions. Specifically, it was shown that in the WI regime, when the source-relay links are weaker than the interfering links, the Han-Kobayashi (HK) scheme is GDoF-optimal. The optimal GDoF of the Gaussian ICR for the case in which the interfering links are weaker than the source-relay links, however, has not been characterized to date.

The Z-ICR is a special case of the ICR in which one of the interfering links is missing, e.g., as a result of shadowing in the channel. The sum-rate capacity of the phase fading Z-ICR for the scenario in which the relay node receives transmissions only from one of the sources was first studied in [10]. In [10], it was shown that when interference is weak enough, the sum-rate capacity is achieved by employing the DF strategy at the relay and by treating interference as noise at each receiver.

**Main Contributions**

In this work we study for the first time the GDoF of the Z-ICR for the scenario in which the relay node receives transmissions only from one of the sources. We consider the case in which the channel is subject to i.i.d. phase fading (PF). The phase fading model is encountered in many practical scenarios, such as in wideband orthogonal frequency-division multiplexing (OFDM) communications, and it is a result of lack of perfect frequency synchronization between the transmitter and the receiver. Note that the GDoF analysis was previously applied only to time-invariant AWGN channels, which follows since if the channel is not constant (e.g., fading channels), then the GDoF may become a random variable. For the phase fading model, however, as the squared magnitude of each channel coefficient is a constant, then the received SNR at each link is constant and hence, GDoF analysis is relevant for this channel model as well. We study the case in which the average attenuation of each link scales differently as a function of SNR, and we let each receiving node have causal channel state information only on its
incoming links (Rx-CSI). The channel considered in this paper has four fundamental differences from the channel model studied in [8]: First, we consider a fading scenario while [8] studied the Gaussian channel. Second, unlike [8], the channel structure studied in this work is not symmetric. The third difference is that [8] presented optimality only for the scenario where the source-relay links are weaker than the interfering links, while the optimality of our work corresponds to the opposite scenario. Lastly, here the relay receives transmissions only from one of the sources. The transmissions of the relay, however, are received at both destinations. Thus, the relay cannot forward desired information to the destination of the source that is not received at the relay. It follows that, the GDoF of the phase fading Z-ICR, studied in this work, cannot be derived as a special case of Gaussian ICR studied in [8].

Our main contributions are as follows:

1) We derive two upper bounds on the achievable GDoF of the PF Z-ICR: one is based on the genie-aided approach, and the other is based on the cut-set theorem.
2) We derive a lower bound on the achievable GDoF of the PF Z-ICR by using DF at the relay, and by treating interference as noise at each receiver.
3) We identify conditions on the scaling of the links under which our lower bound achieves the GDoF upper bound, thereby characterizing the optimal GDoF of the PF Z-ICR in the WI regime.
4) We show that adding a relay to the PF Z-IC strictly increases the GDoF of the channel in the WI regime.

The rest of this paper is organized as follows: In Section II, we provide the notations and the system model considered in this work. In Sections III and IV we provide the two upper bounds and the lower bound on the GDoF, respectively. In Section V, we provide sufficient conditions for the optimality of our achievability scheme, and discuss the implications of the results. Lastly, Section VI summarizes the work.

II. NOTATIONS AND SYSTEM MODEL

We denote random variables (RVs) with upper-case letters, e.g., $X,Y$, and their realizations with lower-case letters, e.g., $x,y$. We denote the probability density function (p.d.f.) of a continuous RV $X$, defined over the set of complex numbers $\mathbb{C}$, with $f_X(x)$. Capital double-stroke letters are used for denoting matrices, e.g., $A$, with the exception that $E\{X\}$ denotes the stochastic expectation of $X$. Bold-face letters, e.g., $x$ denote column vectors, the $i$’th element of a vector $x$ is denoted with $x_i$, and $x^j$ denotes the vector $(x_1,x_2,...,x_j)^T$ where superscript “$T$” denotes transposition. $X^*$ denotes the conjugate of $X$, $A^H$ denotes the Hermitian transpose of $A$, and $|A|$ denotes the determinant of $A$. We denote the circularly symmetric, complex Normal distribution with mean $\mu$ and variance $\sigma^2$ with $CN(\mu,\sigma^2)$. We use the subscript “G”, e.g., $X_G$, to denote an RV which is distributed according to a circularly symmetric, complex Normal distribution with the same mean and variance as the indicated RV, e.g., $X_G \sim CN(E\{X\},\text{var}\{X\})$. All logarithms are with a natural basis. We also define: $(x)^+ \triangleq \max\{x,0\}$. We denote $f(\text{SNR}) \triangleq \text{SNR}^c$ if $\lim_{\text{SNR} \to \infty} \log \frac{f(\text{SNR})}{\log \text{SNR}} = c$. Lastly, given $f(\text{SNR}) \triangleq \text{SNR}^c$ and $g(\text{SNR}) \triangleq \text{SNR}^d$, we write $f(\text{SNR}) \lesssim g(\text{SNR})$ when $c \leq d$.

The Z-ICR consists of two transmitters $\{Tx_k\}_{k=1}^2$, two receivers $\{Rx_k\}_{k=1}^2$, and a full-duplex relay node. $Tx_k$ sends messages to $Rx_k$, $k \in \{1,2\}$. The relay node receives transmissions only from $Tx_1$. This channel model is
schematically depicted in Fig. 1. The received signals at Rx1, Rx2 and the relay at time \( i \) are denoted by \( Y_{1,i}, \ Y_{2,i}, \) and \( Y_{3,i}, \) respectively. The channel inputs from Tx1, Tx2 and the relay at time \( i \) are denoted by \( X_{1,i}, \ X_{2,i}, \) and \( X_{3,i}, \) respectively. Let \( H_{kl,i} \) denote the channel coefficient from node \( k \) to node \( l \) at time \( i \). The relationship between the channel inputs and its outputs is given by:

\[
Y_{1,i} = H_{11,i}X_{1,i} + H_{21,i}X_{2,i} + H_{31,i}X_{3,i} + Z_{1,i}
\]

\[
Y_{2,i} = H_{22,i}X_{2,i} + H_{32,i}X_{3,i} + Z_{2,i}
\]

\[
Y_{3,i} = H_{13,i}X_{1,i} + Z_{3,i},
\]

\( i \in \{1, 2, ..., n\}. \) Here, \( Z_1, \ Z_2, \) and \( Z_3 \) are mutually independent RVs, each distributed according to \( \mathcal{CN}(0, 1) \), i.i.d. in time and independent of the channel inputs and the channel coefficients. Each channel input has a per-symbol unit power constraint: \( \mathbb{E}\{|X_k|^2\} \leq 1, k \in \{1, 2, 3\}. \)

As we consider phase fading, then the channel coefficients are given by \( H_{kl,i} = a_{kl}e^{j\Theta_{kl,i}}, \) where \( a_{kl} \in \mathbb{R}_+ \) are non-negative constants corresponding to the attenuation of the signal magnitude from node \( k \) to node \( l, \) and \( \Theta_{kl,i} \) are uniformly distributed over \([0, 2\pi)\), i.i.d. in time and independent of each other and of the additive noises \( Z_k, \ k \in \{1, 2, 3\}. \) We assume that the destinations and the relay node have instantaneous causal Rx-CSI: The channel coefficients causally available at Rx1 are represented by \( \tilde{H}_1 = (H_{11,1}, H_{21,1})^T \in \mathbb{C}^3 \triangleq \tilde{\gamma}_1, \) for Rx2 they are represented by \( \tilde{H}_2 = (H_{22,1}, H_{32,1})^T \in \mathbb{C}^2 \triangleq \tilde{\gamma}_2, \) and for the relay they are represented by \( \tilde{H}_3 = H_{13,1} \in \mathbb{C} \triangleq \tilde{\gamma}_3. \) Let \( \tilde{H} = (\tilde{H}_1^T, \tilde{H}_2^T, \tilde{H}_3^T)^T \in \mathbb{C}^6 \) be the vector of all channel coefficients. We now state several definitions:

**Definition 1.** An \( (R_1, R_2, n) \) code for the Z-ICR consists of two message sets \( \mathcal{M}_k = \{1, 2, ..., 2^{nR_k}\}, \ k = 1, 2, \) two encoders at the sources; \( e_k : \mathcal{M}_k \mapsto \mathbb{C}^n, k \in \{1, 2\}; \) and two decoders at the destinations; \( g_k : \tilde{\gamma}_k^\mathbb{n} \times \mathbb{C}^n \mapsto \mathcal{M}_k, \ k = 1, 2. \) Since the relay receives transmission only from Tx1, the transmitted signal at the relay at time \( i \) is obtained...
as $x_{3,i} = t_i (y_3^{i-1}, h_3^{i-1}) \in \mathcal{C}, i = 1, 2, ..., n.$

**Comment 1.** Note that since the message sets at the sources are independent and there is no feedback, then the signals transmitted from $\text{Tx}_1$ and from $\text{Tx}_2$ are independent as well. Additionally, since the relay receives transmissions only from $\text{Tx}_1$, then its transmitted signal is independent of the signal from $\text{Tx}_2$, i.e.,

$$f_{x_1, x_2, x_3}(x_1, x_2, x_3) = f_{x_1, x_3}(x_1, x_3) \cdot f_{x_2}(x_2).$$ (1)

Lastly, we define for time index $i$, $\rho_i \triangleq \mathbb{E}\{X_{1,i}X_{3,i}\}, 0 \leq |\rho_i| \leq 1$.

**Definition 2.** The average probability of error is defined as $P_e(n) \triangleq \Pr (g_1(\tilde{H}_1^n, Y_1^n) \neq M_1 \text{ or } g_2(\tilde{H}_2^n, Y_2^n) \neq M_2)$, where each source message is selected independently and uniformly from its message set.

**Definition 3.** A rate pair $(R_1, R_2)$ is called achievable if for any $\epsilon > 0$ and $\delta > 0$ there exists some blocklength $n_0(\epsilon, \delta)$ such that for every $n > n_0(\epsilon, \delta)$ there exists an $(R_1 - \delta, R_2 - \delta, n)$ code with $P_e(n) < \epsilon$.

In this work, the SNR refers to the average signal-to-noise ratio on the direct links, i.e., $\text{SNR} \triangleq a_{11}^2 = a_{22}^2$. We define the following exponential link strengths:

$$\alpha = \frac{\log (a_{21}^2)}{\log (\text{SNR})}, \quad \beta = \frac{\log (a_{31}^2)}{\log (\text{SNR})}, \quad \gamma = \frac{\log (a_{13}^2)}{\log (\text{SNR})}, \quad \lambda = \frac{\log (a_{32}^2)}{\log (\text{SNR})}.$$  

Thus, $\alpha$ and $\lambda$ represent the strength of the interference at $\text{Rx}_1$ and $\text{Rx}_2$, respectively, $\beta$ represents the strength of the cooperation signal, and $\gamma$ represents the strength of the source-relay link. Denote the sum-rate as $R_{\text{sum}} \triangleq R_1 + R_2$.

The GDoF is defined similar to [8, Def. 1]:

$$\text{GDoF} \triangleq \lim_{\text{SNR} \to \infty} \frac{R_{\text{sum}}}{\log \left( \frac{1}{\text{SNR}} \right)}.$$  

**III. AN UPPER BOUND ON THE ACHIEVABLE GDOF**

In this section we provide an upper bound on the achievable GDoF of the Z-ICR. This bound is presented in the following theorem:

**Theorem 1.** Consider the phase fading Z-ICR described in Section II. An upper bound on the achievable GDoF of this channel is given by

$$\text{GDoF}^+ = \min \left\{ \max \{2, 1 + \min \{\beta, \gamma\} \}, \max \{\alpha, 1, \beta - \lambda\} + \max \{\lambda, 1 - \alpha\} \right\}. \quad (2)$$

**Proof:** The upper bound is given as a combination of two bounds: The first bound is obtained using a genie, and the second bound is based on the cut-set theorem.
A. An Upper Bound Using a Genie

Consider the following genie signals:

\[ S_{1,i} = H_{32,i}X_{3,i} + Z_{2,i} \]
\[ S_{2,i} = H_{21,i}X_{2,i} + Z_{1,i}, \]

where \( i \in \{1, 2, 3, \ldots, n\} \). Suppose that a genie provides \( S_1 \) to Rx_1 and \( S_2 \) to Rx_2, i.e., the genie provides to Rx_2 an interference-free, noisy version of its desired signal as it is received at Rx_1, and to Rx_1 it provides a noisy version of the relay signal component observed at Rx_2. Let \( M_k \) denote the message transmitted from Tx_k and let \( \hat{M}_k \) denote its estimation at Rx_k. Additionally, let \( P_{e,k}^{(n)} \) denote the probability of error in the estimation of \( M_k \). Define \( n\epsilon_k \Delta = 1 + P_{e,k}^{(n)} nR_k, k \in \{1, 2\} \), then we obtain:

\[
R_1 = h(M_1)
= h(M_1) - h(M_1|Y_1^n, \hat{H}_1^n) + h(M_1|Y_1^n, \hat{H}_1^n)
\leq (a) I(M_1; Y_1^n, \hat{H}_1^n) + n\epsilon_1n
\leq (b) I(X_1^n; Y_1^n, \hat{H}_1^n) + n\epsilon_1n
= I(X_1^n; \hat{H}_1^n) + I(X_1^n; Y_1^n|\hat{H}_1^n) + n\epsilon_1n
\leq (c) I(X_1^n; \hat{H}_1^n) + n\epsilon_1n
\leq (d) I(X_1^n; Y_1^n|\hat{H}_1^n) + I(X_1^n, X_3^n|\hat{H}_1^n, Y_1^n) + I(X_1^n, X_3^n, S_1^n|\hat{H}_1^n, Y_1^n) + n\epsilon_1n
= I(X_1^n, X_3^n, S_1^n|\hat{H}_1^n) + n\epsilon_1n
= h(S_1^n|\tilde{H}_1^n) - h(S_1^n|X_1^n, X_3^n, \tilde{H}_1^n) + h(Y_1^n|S_1^n, \tilde{H}_1^n) - h(Y_1^n|X_1^n, X_3^n, S_1^n, \tilde{H}_1^n) + n\epsilon_1n
\leq (e) h(S_1^n|\tilde{H}_1^n) - h(Z_2^n) + h(Y_1^n|S_1^n, \tilde{H}_1^n) - h(S_2^n|\tilde{H}_1^n) + n\epsilon_1n, \tag{3}
\]

where (a) follows from Fano’s inequality, (b) follows from data processing inequality, (c) follows since channel inputs are independent of the channel coefficients, (d) follows since mutual information is nonnegative, and (e) follows since

\[
h(Y_1^n|X_1^n, X_3^n, S_1^n, \tilde{H}_1^n) = h(Y_1^n|X_1^n, X_3^n, Z_2^n, \tilde{H}_1^n)
= h((H_{21}X_2 + Z_1)^n|X_1^n, X_3^n, Z_2^n, \tilde{H}_1^n)
= h(H_{21}X_2 + Z_1)^n|\tilde{H}_1^n)
= h(S_2^n|\tilde{H}_1^n).
\]
Similarly, for \( R_2 \) we have

\[
nR_2 \leq I(X_n^2; Y_n^2, S_n^2 | \tilde{H}_n^n) + n\epsilon_{2n}
\]

\[
= I(X_n^2; S_n^2 | \tilde{H}_n^n) + I(X_n^2; Y_n^2 | S_n^2, \tilde{H}_n^n) + n\epsilon_{2n}
\]

\[
= h(S_n^2 | \tilde{H}_n^n) - h(S_n^2 | X_n^2, \tilde{H}_n^n) + h(Y_n^2 | S_n^2, \tilde{H}_n^n) - h(Y_n^2 | X_n^2, S_n^2, \tilde{H}_n^n) + n\epsilon_{2n}
\]

\[
= h(S_n^2 | \tilde{H}_n^n) - h(Z_1^n) + h(Y_n^2 | S_n^2, \tilde{H}_n^n) - h(S_1^n | \tilde{H}_n^n) + n\epsilon_{2n}.
\] (4)
Define $\theta_i \triangleq \text{arg}\{h_{11,i}, h_{31,i}^*, \rho_i\}$. Combining (3) and (4), we obtain

$$n(R_{\text{sum}} - \epsilon_{1n} - \epsilon_{2n})$$

\[
\leq h(S_1^n | \tilde{H}_n^n) - h(Z_1^n) + h(Y_1^n | S_1^n, \tilde{H}_n^n) - h(S_1^n | \tilde{H}_n^n) - h(Y_2^n | S_2^n, \tilde{H}_n^n) - h(S_1^n | \tilde{H}_n^n)
\]

\[
= h(Y_1^n | S_1^n, \tilde{H}_n^n) - h(Z_1^n) + h(Y_2^n | S_2^n, \tilde{H}_n^n) - h(Z_2^n)
\]

\[
= \sum_{i=1}^{n} h(Y_{1,i} | Y_{1,i}^{-1}, S_1^n, \tilde{H}_n^n) - h(Z_1^n) + \sum_{i=1}^{n} h(Y_{2,i} | Y_{2,i}^{-1}, S_2^n, \tilde{H}_n^n) - h(Z_2^n)
\]

\[(a) \leq \sum_{i=1}^{n} E_{\tilde{H}_i} \{h(Y_{1,i} | S_{1,i}, \tilde{H}_i)\} - nh(Z_1) + \sum_{i=1}^{n} E_{\tilde{H}_i} \{h(Y_{2,i} | S_{2,i}, \tilde{H}_i)\} - nh(Z_2)
\]

\[(b) = \sum_{i=1}^{n} E_{\tilde{H}_i} \{\max_{0 \leq |\rho_i| \leq 1} h(Y_{1,G,i} | S_{1,G,i}, \tilde{H}_i)\} - nh(Z_1) + \sum_{i=1}^{n} E_{\tilde{H}_i} \{\max_{0 \leq |\rho_i| \leq 1} h(Y_{2,G,i} | S_{2,G,i}, \tilde{H}_i)\} - nh(Z_2)
\]

\[= \sum_{i=1}^{n} E_{\tilde{H}_i} \{\max_{0 \leq |\rho_i| \leq 1} \log (2\pi \text{cov}(Y_{1,G,i} | S_{1,G,i}, \tilde{H}_i))\} - \sum_{i=1}^{n} \log(2\pi)
\]

\[+ \sum_{i=1}^{n} E_{\tilde{H}_i} \{\max_{0 \leq |\rho_i| \leq 1} \log \left(\frac{|E\{Y_{1,G,i}^* | S_{1,G,i}, \tilde{H}_i\}|^2}{\text{var}(S_{1,G,i}) |\tilde{H}_i|} \right) \}
\]

\[+ \sum_{i=1}^{n} E_{\tilde{H}_i} \{\max_{0 \leq |\rho_i| \leq 1} \log \left(\frac{|E\{Y_{2,G,i}^* | S_{2,G,i}, \tilde{H}_i\}|^2}{\text{var}(S_{2,G,i}) |\tilde{H}_i|} \right) \}
\]

\[= \sum_{i=1}^{n} E_{\tilde{H}_i} \{\max_{0 \leq |\rho_i| \leq 1} \log \left(1 + |h_{21,i}|^2 + \frac{|h_{11,i}|^2 + |h_{31,i}|^2 + |h_{32,i}|^2(1 - |\rho_i|^2) + 2|h_{11,i}||h_{31,i}| |\rho_i| \cos(\theta_i)|}{1 + |h_{32,i}|^2} \right) \}
\]

\[+ \sum_{i=1}^{n} E_{\tilde{H}_i} \{\log \left(1 + |h_{32,i}|^2 + \frac{|h_{22,i}|^2}{1 + |h_{21,i}|^2} \right) \}
\]

\[\leq \sum_{i=1}^{n} E_{\tilde{H}_i} \{\log \left(1 + |h_{21,i}|^2 + \frac{|h_{11,i}|^2 + |h_{31,i}|^2 + |h_{32,i}|^2 + 2|h_{11,i}||h_{31,i}|}{1 + |h_{32,i}|^2} \right) \}
\]

\[+ \sum_{i=1}^{n} E_{\tilde{H}_i} \{\log \left(1 + |h_{32,i}|^2 + \frac{|h_{22,i}|^2}{1 + |h_{21,i}|^2} \right) \}
\]

\[\leq \sum_{i=1}^{n} \log \left(1 + |a_{21}|^2 + \frac{|a_{11}|^2 + |a_{31}|^2 + |a_{11}|^2 |a_{32}|^2 + 2|a_{11}| a_{31}^*}{1 + |a_{32}|^2} \right) + \sum_{i=1}^{n} \log \left(1 + |a_{32}|^2 + \frac{|a_{22}|^2}{1 + |a_{21}|^2} \right)
\]

\[= n \log \left(1 + |a_{21}|^2 + \frac{|a_{11}|^2 + |a_{31}|^2 + |a_{11}|^2 |a_{32}|^2 + 2|a_{11}| a_{31}^*}{1 + |a_{32}|^2} \right) + n \log \left(1 + |a_{32}|^2 + \frac{|a_{22}|^2}{1 + |a_{21}|^2} \right)
\]

where (a) follows since conditioning reduces entropy, (b) follows since $Z_{1,i}$ and $Z_{2,i}$ are i.i.d. in time, (c) follows from [9, Lemma 2] which states that given a deterministic $\tilde{H}_i$, $h(Y_{k,i} | S_{k,i}, \tilde{H}_i)$ is maximized with $Y_{k,i}$ and $S_{k,i}, k \in \{1, 2\}$ distributed according to the zero-mean, jointly complex Normal distribution with the covariance matrix $\text{cov}(Y_{k,G,i} | S_{k,G,i}, \tilde{H}_i) = \text{cov}(Y_{k,i}, S_{k,i}, \tilde{H}_i), k \in \{1, 2\}$, (d) follows from a direct application of the formula for the
conditional covariance of jointly complex Normal RVs [12, Sec. VI], (e) follows since we have $0 \leq |\rho_i| \leq 1$ and $-1 \leq \cos(\theta_i) \leq 1$, and (f) follows since in the phase fading model, the magnitudes of the channel coefficients are constant and do not depend on the time index. Thus, since the expressions in step (e) are independent of the channel phases, then the expectation can be omitted. Observe that as $P_{e,k}^{(n)} \to 0$ for $n \to \infty$, then $e_{kn} \to 0$, $k \in \{1, 2\}$ as $n \to \infty$. Hence, we conclude that

$$R_{\text{sum}} \leq \log \left( 1 + |h_{21}|^2 + \frac{|a_{11}|^2 + |a_{31}|^2 + |a_{11}||a_{32}|^2 + 2|a_{11}|a_{31}^*|a_{32}|}{1 + |a_{32}|^2} \right) + \log \left( 1 + |a_{22}|^2 + \frac{|a_{22}|^2}{1 + |a_{21}|^2} \right)$$

$$= \log \left( 1 + \text{SNR}^\alpha + \frac{\text{SNR} + \text{SNR}^\beta + \text{SNR}^{1+\lambda} + 2\text{SNR}^{\frac{1+\beta}{2}}}{1 + \text{SNR}^\lambda} \right) + \log \left( 1 + \text{SNR}^\lambda + \frac{\text{SNR}}{1 + \text{SNR}^\alpha} \right)$$

$$\overset{(a)}{=} \log \left( \text{SNR}^{\max\{\alpha, 1, \beta - \lambda\}} \right) + \log \left( \text{SNR}^{\max\{\lambda, 1 - \alpha\}} \right),$$

where (a) follows since $\max\{1, \beta\} \geq \frac{1+\beta}{2}$. Therefore, the genie-aided GDoF upper bound is given by

$$\text{GDoF}^+_1 = \max\{\alpha, 1, \beta - \lambda\} + \max\{\lambda, 1 - \alpha\}. \quad (5)$$
B. An Upper Bound Based on the Cut-Set Theorem

Following along the lines of the cut-set theorem [13, Thm. 15.10.1], we obtain the following upper bounds on the achievable rates:

\[ n R_1 = H(M_1) \]
\[
\leq I(M_1; Y_1^n | M_2) + n \epsilon_{1n}
\]
\[
\leq I(M_1; Y_1^n, \hat{Y}^n | M_2) + n \epsilon_{3n}
\]
\[
= \sum_{i=1}^{n} h(Y_{1,i}, \tilde{H}_i | Y_{i-1}^{i-1}, \tilde{H}_i^{i-1}, M_2) - h(Y_{1,i}, \tilde{H}_i | Y_{i-1}^{i-1}, \tilde{H}_i^{i-1}, M_1, M_2) + n \epsilon_{1n}
\]
\[
= \sum_{i=1}^{n} h(\tilde{H}_i | Y_{i-1}^{i-1}, M_2) + h(Y_{1,i} | Y_{i-1}^{i-1}, \tilde{H}_i, M_2)
- h(\tilde{H}_i | Y_{i-1}^{i-1}, M_1, M_2) - h(Y_{1,i} | Y_{i-1}^{i-1}, \tilde{H}_i, M_1, M_2) + n \epsilon_{1n}
\]
\[
= \sum_{i=1}^{n} h(\tilde{H}_i) + h(Y_{1,i} | Y_{i-1}^{i-1}, \tilde{H}_i, M_2)
- h(\tilde{H}_i) - h(Y_{1,i} | Y_{i-1}^{i-1}, \tilde{H}_i, M_1, M_2) + n \epsilon_{1n}
\]
\[
\leq \sum_{i=1}^{n} h(Y_{1,i} | Y_{i-1}^{i-1}, \tilde{H}_i, M_2, X_{2,i}) - h(Y_{1,i} | Y_{i-1}^{i-1}, \tilde{H}_i, M_1, M_2, X_{1,i}, X_{2,i}, X_{3,i}) + n \epsilon_{1n}
\]
\[
= \sum_{i=1}^{n} I(X_{1,i}, X_{3,i} | Y_{i-1}^{i-1}, \tilde{H}_i, X_{2,i}) + n \epsilon_{1n}
\]
\[
= \sum_{i=1}^{n} E_{\tilde{H}_i} \left\{ I(X_{1,i}, X_{3,i} | Y_{i-1}^{i-1}, \tilde{H}_i) \right\} + n \epsilon_{1n}
\]
\[
\leq \sum_{i=1}^{n} E_{\tilde{H}_i} \left\{ \max_{0 \leq \rho_i \leq 1} I(X_{1G,i}, X_{3G,i} | Y_{1G,i} | X_{2G,i}, \tilde{H}_i) \right\} + n \epsilon_{1n}
\]
\[
\leq \sum_{i=1}^{n} E_{\tilde{H}_i} \left\{ \log \left( 1 + |h_{11,i}|^2 + |h_{31,i}|^2 \right) \right\} + n \epsilon_{1n}
\]
\[
= \sum_{i=1}^{n} \log \left( 1 + |a_{11}|^2 + |a_{31}|^2 \right) + n \epsilon_{1n}
\]
\[
= n \log \left( 1 + |a_{11}|^2 + |a_{31}|^2 \right) + n \epsilon_{1n},
\]

where (a) follows from Fano’s inequality, (b) follows since mutual information is nonnegative, (c) follows since channel coefficients are i.i.d. in time and independent from other channel variables, (d) follows since \(X_{2,i}\) is a function of \(M_2\) and since conditioning reduces entropy, (e) follows since the channel outputs at time \(i\) depend only
on the channel inputs and the channel coefficients at time \( i \), and (f) follows from [9, Eq. (A.10)]. Next, we have

\[
nR_1 = H(M_1) \\
\leq I(M_1; Y^n_1 | M_2) + n\epsilon_{1n} \\
\leq I(M_1; Y^n_1, Y^n_3, \hat{H}^n | M_2) + n\epsilon_{1n} \\
= \sum_{i=1}^{n} h(Y_{1,i}, Y_{3,i}, \hat{H}_1 | Y_{1,i}^{i-1}, Y_{3,i}^{i-1}, \hat{H}_1^{i-1}, M_2) - h(Y_{1,i}, Y_{3,i}, \hat{H}_1 | Y_{1,i}^{i-1}, Y_{3,i}^{i-1}, \hat{H}_1^{i-1}, M_1, M_2) + n\epsilon_{1n} \\
= \sum_{i=1}^{n} h(Y_{1,i}, Y_{3,i} | Y_{1,i}^{i-1}, Y_{3,i}^{i-1}, \hat{H}_1, M_2, X_{2,i}, X_{3,i}) \\
- h(Y_{1,i}, Y_{3,i} | Y_{1,i}^{i-1}, Y_{3,i}^{i-1}, \hat{H}_1, M_1, M_2, X_{2,i}, X_{3,i}) + n\epsilon_{1n} \\
\leq \sum_{i=1}^{n} h(Y_{1,i}, Y_{3,i} | X_{2,i}, X_{3,i}, \hat{H}_1) - h(Y_{1,i}, Y_{3,i} | X_{1,i}, X_{2,i}, X_{3,i}, \hat{H}_1) + n\epsilon_{1n} \\
= \sum_{i=1}^{n} I(X_{1,i}; Y_{1,i}, Y_{3,i} | X_{2,i}, X_{3,i}, \hat{H}_1) + n\epsilon_{1n} \\
= \sum_{i=1}^{n} E_{\hat{H}_1} \{ I(X_{1,i}; Y_{1,i}, Y_{3,i} | X_{2,i}, X_{3,i}, \hat{H}_1) \} + n\epsilon_{1n} \\
\leq \sum_{i=1}^{n} E_{\hat{H}_1} \{ \max_{0 \leq |\rho| \leq 1} I(X_{1G,i}; Y_{1G,i}, Y_{3G,i} | X_{2G,i}, X_{3G,i}, \hat{H}_1) \} + n\epsilon_{1n} \\
\overset{(a)}{=} \sum_{i=1}^{n} E_{\hat{H}_1} \{ \log \left( 1 + |h_{11,i}|^2 + |h_{13,i}|^2 \right) \} + n\epsilon_{1n} \\
= \sum_{i=1}^{n} \log \left( 1 + |a_{11}|^2 + |a_{13}|^2 \right) + n\epsilon_{1n} \\
= n \log \left( 1 + |a_{11}|^2 + |a_{13}|^2 \right) + n\epsilon_{1n}, \tag{7}
\]

where (a) follows from [9, Eq. (A.5)].
Following similar steps for $R_2$ we obtain

$$nR_2 = H(M_2)$$

$$\leq I(M_2; Y_2^n | M_1) + n\epsilon_2n$$

$$\leq I(M_2; Y_2^n, Y_3^n, \tilde{H}_3^n | M_1) + n\epsilon_2n$$

$$= \sum_{i=1}^{n} h(Y_{2,i}, Y_{3,i}, \tilde{H}_3 | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1) - h(Y_{2,i}, Y_{3,i}, \tilde{H}_3 | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1, M_2) + n\epsilon_2n$$

$$= \sum_{i=1}^{n} h(\tilde{H}_3 | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1) + h(Y_{2,i}, Y_{3,i} | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1)$$

$$- h(\tilde{H}_3 | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1, M_2) - h(Y_{2,i}, Y_{3,i} | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1, M_2) + n\epsilon_2n$$

$$\leq \sum_{i=1}^{n} h(Y_{2,i}, Y_{3,i} | X_{1,i}, X_{3,i}, \tilde{H}_3) - h(Y_{2,i}, Y_{3,i} | Y_2^{i-1}, Y_3^{i-1}, \tilde{H}_3^{i-1}, M_1, M_2, X_{1,i}, X_{2,i}, X_{3,i}) + n\epsilon_2n$$

$$= \sum_{i=1}^{n} I(X_{2,i} ; Y_{2,i}, Y_{3,i} | X_{1,i}, X_{3,i}, \tilde{H}_3) + n\epsilon_2n$$

$$= \sum_{i=1}^{n} I(X_{2,i} ; Y_{2,i} | X_{1,i}, X_{3,i}, \tilde{H}_3) + I(X_{2,i} ; Y_{3,i} | X_{1,i}, X_{3,i}, \tilde{H}_3) + n\epsilon_2n$$

$$\geq (a) \sum_{i=1}^{n} I(X_{2,i} ; Y_{2,i} | X_{1,i}, X_{3,i}, Y_{3,i}, \tilde{H}_3) + n\epsilon_2n$$

$$= \sum_{i=1}^{n} I(X_{2,i} ; Y_{2,i} | X_{1,i}, X_{3,i}, Z_{3,i}, \tilde{H}_3) + n\epsilon_2n$$

$$\geq (b) \sum_{i=1}^{n} I(X_{2,i} ; Y_{2,i} | X_{3,i}, \tilde{H}_3) + n\epsilon_2n$$

$$= \sum_{i=1}^{n} E_{\tilde{H}_3} \left\{ I(X_{2,i} ; Y_{2,i} | X_{3,i}, \tilde{H}_3) \right\} + n\epsilon_2n$$

$$\leq \sum_{i=1}^{n} E_{\tilde{H}_3} \left\{ \max_{0 \leq |\rho| \leq 1} I(X_{2G,i} ; Y_{2G,i} | X_{3G,i}, \tilde{H}_3) \right\} + n\epsilon_2n$$

$$= \sum_{i=1}^{n} E_{\tilde{H}_3} \left\{ \log \left( 1 + |h_{2,i}|^2 \right) \right\} + n\epsilon_2n$$

$$= \sum_{i=1}^{n} \log \left( 1 + |a_{22}|^2 \right) + n\epsilon_2n$$

$$= n \log \left( 1 + |a_{22}|^2 \right) + n\epsilon_2n \tag{8}$$

where (a) follows since the signal from Tx$_2$ is not received at the relay and since it is independent of the signal from Tx$_1$, the signal from the relay and the channel coefficients, and (b) follows since $X_{2,i}$ is independent of $X_{1,i}$ and $Z_{3,i}$, and since the signal from Tx$_1$ is not received at Rx$_2$. Hence, since for $n \to \infty$ we have $\epsilon_{kn} \to 0, k \in \{1, 2\}$,
by combining (6)-(8) we obtain
\[ R_{\text{sum}} \leq \max \left\{ \log \left( 1 + |a_{11}|^2 + |a_{31}|^2 \right), \log \left( 1 + |a_{11}|^2 + |a_{13}|^2 \right) \right\} + \log \left( 1 + |a_{22}|^2 \right) \]
\[ = \max \left\{ \log \left( 1 + \text{SNR} + \text{SNR}^\beta \right), \log \left( 1 + \text{SNR} + \text{SNR}^\gamma \right) \right\} + \log \left( 1 + \text{SNR} \right) \]
\[ = \max \left\{ \log \left( \text{SNR}^{\max\{1,\beta\}} \right), \log \left( \text{SNR}^{\max\{1,\gamma\}} \right) \right\} + \log \left( 1 + \text{SNR} \right). \]
Thus, we obtain the cut-set based GDoF upper bound:
\[ \text{GDoF}_2^+ = 1 + \min \left\{ \max\{1, \beta\}, \max\{1, \gamma\} \right\} = \max \left\{ 2, 1 + \min\{\beta, \gamma\} \right\}. \quad (9) \]
We conclude that an upper bound on the GDoF of the Z-ICR is given by the minimum of (5) and (9), presented in (2).

IV. AN ACHIEVABLE GDOF

An achievable GDoF for the phase fading Z-ICR is presented in the following proposition:

**Proposition 1.** Consider the phase fading Z-ICR described in Section II. An achievable GDoF for this channel is given by
\[ \text{GDoF}_- = \min \left\{ \gamma, \max \left\{ (1 - \alpha)^+, (\beta - \alpha)^+ \right\} \right\} + (1 - \lambda)^+. \quad (10) \]

*Proof:* We consider the communications scheme used in [10, Thm. 1]. The transmitters use mutually independent codebooks generated according to an i.i.d. (in time) complex Normal distribution: \( X_k \sim \mathcal{CN}(0,1), k \in \{1,2,3\}. \) The achievability is based on the DF strategy at the relay, a backward decoding scheme at \( \text{Rx}_1, \) and a PtP decoding rule at \( \text{Rx}_2. \) This scheme leads to the following achievable rate region for the Z-ICR [10, Proposition 1]:
\[ R_1 \leq \min \left\{ I(X_1, X_3; Y_1|\hat{H}_1), I(X_1; X_3|X_3, \hat{H}_3) \right\} \quad (11a) \]
\[ R_2 \leq I(X_2; Y_2|\hat{H}_2). \quad (11b) \]
In [10, Thm. 1], the mutual information expressions (11) were evaluated for the mutually independent zero-mean
complex Normal channel inputs. The resulting expressions are:

\[
I(X_1, X_3; Y_1 | \hat{H}_1) = \mathbb{E}_{\hat{H}_1} \left\{ I(X_1, X_3; Y_1 | \hat{h}_1) \right\} \\
= \mathbb{E}_{\hat{H}_1} \left\{ \log \left( 1 + \frac{|h_{11}|^2 + |h_{31}|^2}{1 + |h_{21}|^2} \right) \right\} \\
= \log \left( 1 + \frac{a_{11}^2 + a_{31}^2}{1 + a_{21}^2} \right) \\
= \log \left( 1 + \frac{\text{SNR} + \text{SNR}^\beta}{1 + \text{SNR}^\alpha} \right) \\
= \log \left( \frac{\text{SNR}^{1-\alpha} + \text{SNR}^{\beta-\alpha}}{1 + \text{SNR}^\lambda} \right),
\]

(12)

\[
I(X_1; Y_3 | X_3, \hat{H}_3) = \mathbb{E}_{\hat{H}_3} \left\{ I(X_1; Y_3 | X_3, \hat{h}_3) \right\} \\
= \mathbb{E}_{\hat{H}_3} \left\{ \log(1 + |h_{13}|^2) \right\} \\
= \log \left( 1 + a_{13}^2 \right) \\
= \log \left( 1 + \text{SNR}^\gamma \right)
\]

(13)

\[
I(X_2; Y_2 | \hat{H}_2) = \mathbb{E}_{\hat{H}_2} \left\{ I(X_2; Y_2 | \hat{h}_2) \right\} \\
= \mathbb{E}_{\hat{H}_2} \left\{ \log \left( 1 + \frac{|h_{22}|^2}{1 + |h_{32}|^2} \right) \right\} \\
= \log \left( 1 + \frac{a_{22}^2}{1 + a_{32}^2} \right) \\
= \log \left( 1 + \frac{\text{SNR}}{1 + \text{SNR}^\lambda} \right).
\]

(14)

Combining (12)-(14), we obtain the achievable GDoF (10).

V. THE OPTIMALITY OF TREATING INTERFERENCE AS NOISE

Proposition 2. Consider the phase fading Z-ICR. If the interference is symmetric and weak in the sense that

\[
\lambda = \alpha, \quad \alpha \leq \frac{1}{2},
\]

(15a)

and it also holds that

\[
1 + \alpha \leq \beta \leq \gamma + \alpha,
\]

(15b)

then the optimal GDoF for the Z-ICR is

\[
\text{GDoF}_{\text{Opt}} = 1 + \beta - 2\alpha,
\]

(16)

and it is obtained with mutually independent, zero mean complex Normal channel inputs with the maximal allowed power.

Proof: Consider (10) and note that if \(\alpha \leq \frac{1}{2}\) and \(1 + \alpha \leq \beta \leq \gamma + \alpha\), then it follows that \(\gamma \geq 1, \beta \geq 1,\) and \(\max \left\{ (1 - \alpha)^+, (\beta - \alpha)^+ \right\} = (\beta - \alpha)^+\). If it also holds that \(\lambda = \alpha\) then \(\text{GDoF}^\gamma = 1 + \beta - 2\alpha\).
Next, consider (2) and note that if $\lambda = \alpha$, $1 + \alpha \leq \beta$, and $\alpha \leq 0.5$, then $\max\{\alpha, 1, \beta - \lambda\} + \max\{\lambda, 1 - \alpha\} = 1 + \beta - 2\alpha$, and thus, (2) specializes as

$$GDoF^+ = \min\left\{\max\left\{2, 1 + \min\{\beta, \gamma\}\right\}, 1 + \beta - 2\alpha\right\}. $$

Additionally, if $\beta \geq 1$ and $\gamma \geq 1$, then $\max\left\{2, 1 + \min\{\beta, \gamma\}\right\} = 1 + \min\{\beta, \gamma\}$. Lastly, for $\beta \leq \gamma + \alpha$ we have $\beta \leq \min\{\beta, \gamma\} + 2\alpha$, i.e., $1 + \beta - 2\alpha \leq 1 + \min\{\beta, \gamma\}$, and thus, we obtain $GDoF^+ = 1 + \beta - 2\alpha$. We conclude that if (15) is satisfied, then (2) coincides with (10), characterizing the optimal GDoF for the phase fading Z-ICR.

Comment 2. Note that a necessary condition for the optimality of treating interference as noise to hold in the scenario of Proposition 2 is to have $\alpha \leq 0.5$, which corresponds to the weak interference regime. Specifically, if $\beta = \gamma = 2$, then optimality is achieved with $\alpha \leq 0.5$, and if $\beta = \gamma = 1.2$, optimality is achieved with $\alpha \leq 0.2$. This is depicted in Fig. 2.

Comment 3. Consider the phase fading Z-IC. In this model, one of the pairs is interfered while the second pair communicates over an interference-free channel. Since the message sets at the sources are independent, then an upper bound on the achievable sum-rate of the PF Z-IC is obtained from the cut-set theorem [13, Thm. 15.10.1]:

$$R_{PF \ Z-IC}^{sum} \leq \max_{f(x_1)f(x_2)} \left\{ I(X_1; Y_1|X_2, \widetilde H) + I(X_2; Y_2|\widetilde H) \right\}. $$

This corresponds to two interference-free PtP channels. Since the maximal achievable DoF for each link is 1, then the GDoF for this channel is upper bounded by 2. Comparing the upper-bound (17) with the achievable GDoF of the PF Z-ICR in (10), we note that if $1 \leq \beta \leq \gamma + \alpha$, then for the PF Z-ICR we have $GDoF_{Z-ICR}^- = (\beta - \alpha)^+ + (1 - \lambda)^+$. Hence, when $2 \leq (\beta - \alpha)^+ + (1 - \lambda)^+$ the relay node strictly increases the GDoF of the PF Z-IC even in scenarios in which the relay receives transmissions only from one of the sources, and the interference is treated as noise at the destinations. Figure 2 also depicts the GDoF upper bound of the Z-IC for comparison with that of Z-ICR. Observe that the GDoF for the Z-ICR is greater than that of the Z-IC for both $\beta = \gamma = 2$, $\alpha \leq 0.5$ and $\beta = \gamma = 1.2$, $\alpha \leq 0.2$. 

Fig. 2: The upper bound on the GDoF and the achievable GDoF of the Z-ICR.
VI. SUMMARY

The optimal GDoF of the phase fading Z-ICR was characterized for the case in which the relay receives only one of the sources. We showed that when interference is weak, the optimal GDoF can be achieved by using DF at the relay and by treating interference as noise at each receiver. We further showed that adding a relay node to the Z-IC strictly increases the achievable GDoF compared to the case without the relay.

REFERENCES