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New Lower Bounds on the Error Probability for Signals over an AWGN Channel

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Abstract — **New lower bounds on the error probability when communicating one of M equally likely signals over an Additive White Gaussian Noise (AWGN) channel are proposed. The bounds are derived by improving on a recent lower bound on the probability of a union of events by de Caen, and applying it to this problem. The resulting bounds, specialized for linear codes, improve on the latest lower bound appearing in the current literature by Seguin.**

I. Introduction

Consider a finite family of events $\{A_i\}_{i\in I}$ in a probability space $(\Omega, \mathcal{F}, P)$. For each $x \in \Omega$ define $deg(x) = |\{i \in I : x \in A_i\}|$. The basis of our results is the following [1]

Theorem 1.

$$P\left(\bigcup_{i\in I} A_i\right) \geq \sum_{i\in I} \frac{\left(\sum_{x\in A_i} p(x)m_i(x)\right)^2}{\sum_{j\in I}\sum_{x\in A_i\cap A_j} p(x)m_i^2(x)} \qquad (1)$$

where $m_i(x)$ is any real function on Ω such that the sums in the r.h.s of (1) converge. Equality in (1) is achieved when

$$m_i(x) = m^*(x) = \frac{1}{deg(x)}. \qquad (2)$$

This bound does not depend only on the $P(A_i)$'s and $P(A_i \cap A_j)$'s. However, a proper choice of the function $m_i(x)$ may result in the same computational complexity while improving on de Caen's bound [2] (as well as Seguin's bound [3] derived from it), achieved by choosing $m_i(x) \equiv 1$. In this talk, we apply the bound in (1) to derive lower bounds on the decoding error probability when using BPSK modulation of a linear code $\mathcal{C}$ on the AWGN channel. The only knowledge on the code used is its distance distribution.

Let $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{M-1}$ be M equally likely signals of length N transmitted over the AWGN channel. Let $\mathbf{r} = \mathbf{s}_0 + \mathbf{n}$ be the received signal, where $\mathbf{s}_0$ is the signal transmitted and $\mathbf{n}$ is the Gaussian noise vector. Assuming the signals are derived from a linear code, the probability of decoding error using a maximum likelihood decoder is given by

$$P(\varepsilon) = P(\varepsilon|\mathbf{s}_0) = P\left(\cup_{i\neq 0}\varepsilon_{0i}|\mathbf{s}_0\right), \qquad (3)$$

where $\varepsilon_{0i} = \{\mathbf{r} \in \mathbb{R}^N : \|\mathbf{r} - \mathbf{s}_i\| < \|\mathbf{r} - \mathbf{s}_0\|\}$ and $\|\cdot\|$ is the Euclidean norm. To apply the bound in (1) to (3), an appropriate function $m_i(x)$ should be chosen. The choice for $m_i(x)$ ought to approximate its optimal value, as given in (2), using the available information on the code. Moreover, a mathematically endurable function is required for the sums in (1) to be feasible. Hence, $m_i(x)$ should be a function of the code as well as the channel's type.

II. Results

For a given $\mathbf{x} \in \mathbb{R}^N$, $deg(\mathbf{x}) = |\{i : \mathbf{x} \in \varepsilon_{0i}, i \neq 0\}|$. By choosing

$$m_i(\mathbf{x}|\mathbf{s}_0) = \exp\left(-[a\|\mathbf{x}\|^2 + b\langle\mathbf{x}, \mathbf{s}_0\rangle + c\|\mathbf{s}_0\|^2]\right) \qquad (4)$$

where a, b and c are parameters to be optimized, the resulting bound can be represented in terms of the error function $Q(\cdot)$ and the bivariate normal distribution alone. For an easier optimization problem we derive two simpler bounds. The first, denoted by *norm*, is obtained by substituting $b = -2a$ and $c = a$, i.e., assuming $deg(\mathbf{x})$ increases with $\|\mathbf{x} - \mathbf{s}_0\|^2$. In the second, denoted by *dot product*, $a = c = 0$, assuming $deg(\mathbf{x})$ increases with $\langle\mathbf{x}, \mathbf{s}_0\rangle$. Furthermore, when lower bounds are discussed, one can use only a subset of the code. We choose the subset $\mathcal{C}_d^*$, which includes all codewords of weight d_{min} and the zero codeword. The resulting bounds for the code BCH(63,24), together with Shannon's and Seguin's lower bounds and Poltyrev's upper bound, are given in Figure 1. In [1], it is shown that the new bounds, as well as analogous bounds derived for the BSC, provide tighter bounds on the error exponent than the de Caen based bounds. Hence, the bound in (1) provides a powerful framework for deriving lower bounds on the error probability.

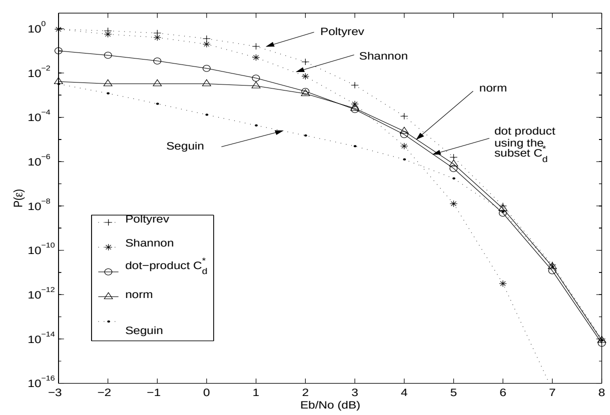

Figure 1: Bounds on the decoding error probability.

References

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