On The Cover Time and Mixing Time of Random Geometric Graphs

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Abstract

The cover time and mixing time of graphs has much relevance to algorithmic applications and has been extensively investigated. Recently, with the advent of ad-hoc and sensor networks, an interesting class of random graphs, namely *random geometric graphs*, has gained new relevance and its properties have been the subject of much study. A random geometric graph $\mathcal{G}(n, r)$ is obtained by placing n points uniformly at random on the unit square and connecting two points iff their Euclidean distance is at most r. The phase transition behavior with respect to the radius r of such graphs has been of special interest. We show that there exists a critical radius r_{opt} such that for any $r \geq r_{\text{opt}}$ $\mathcal{G}(n, r)$ has optimal cover time of $\Theta(n \log n)$ with high probability, and, importantly, $r_{\text{opt}} = \Theta(r_{\text{con}})$ where r_{con} denotes the critical radius guaranteeing asymptotic connectivity. Moreover, since a disconnected graph has infinite cover time, there is a phase transition and the corresponding threshold width is $O(r_{\text{con}})$. On the other hand, the radius required for rapid mixing $r_{\text{rapid}} = \omega(r_{\text{con}})$, and, in particular, $r_{\text{rapid}} = \Theta(1/poly(\log n))$. We are able to draw our results by giving a tight bound on the electrical resistance and conductance of $\mathcal{G}(n, r)$ via certain constructed flows.

Key words: Random Walks, Cover Time, Mixing Time, Random Graphs

1 Introduction

A random geometric graph (RGG) is a graph $\mathcal{G}(n, r)$ resulting from placing n points uniformly at random on the unit square [1] and connecting two points iff their Euclidean distance is at most r. While these graphs have traditionally been studied in relation to subjects such as statistical physics and hypothesis testing [29], random geometric graphs have gained new relevance with the advent of ad-hoc and sensor networks [14,30] as they are a model of such networks. Sensor networks have strict energy and memory constraints and in many cases are subject to high dynamics, created by failures, mobility and other factors. Thus, purely deterministic algorithms have disadvantages for such networks as they need to maintain data structures and have expensive recovery mechanism. Recently, questions regarding the random walk properties of such networks have been of interest especially due to the locality, simplicity, low-overhead and robustness to failures of the process [17,5,7]. In particular random walk techniques have been proposed for gossiping in random geometric graphs [23], for information collection and query answering [33,4] and even for routing [8,34].

Two important characteristics of random walks on a graph are mixing time and cover time. The *mixing time* of a graph G is the time taken by a simple random walk on G to sample a node according to the steady state distribution of G, which means sampling uniformly at random if G is regular. If the mixing time is poly-logarithmic in the number of nodes, then we say that G is *rapid*

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[1] We focus on the 2-dimensional case; see section 6 for discussion.

mixing. The *cover time* C_G of a graph G is the expected time taken by a simple random walk on G to visit all nodes in G. This property has much relevance to algorithmic applications [23,16,38,20,4], and methods of bounding the cover time of graphs have been thoroughly investigated [25,2,10,9,40,3]. Several bounds on the cover times of particular classes of graphs have been obtained with many positive results [10,9,21,22,11].

In ad-hoc and sensor networks, interference grows with increased communication radius. So, for a desirable property P of random geometric graphs, one wants to find a tight upper bound on the smallest radius r_P, that will guarantee that P holds with high probability. The radius r_P is called *critical radius* if P exhibits a sharp threshold, the difference between the smallest radius for which the property holds with high probability and the largest radius for which the property holds with low probability goes to zero as $n \to \infty$. The critical radius for connectivity, r_{con}, has been of special interest, and it has been shown that if $\pi r^2 \geq \pi r_{\text{con}}^2 = \frac{\log n + \gamma_n}{n}$ then $\mathcal{G}(n, r)$ is connected with probability going to one as $n \to +\infty$ iff $\gamma_n \to +\infty$ [28,19].

In this paper we study the existence of critical radii for properties of optimal cover time and rapid mixing. In particular, we study the existence of a radius r_{opt} that will guarantee with high probability that $\mathcal{G}(n, r)$ with $r \geq r_{\text{opt}}$ has *optimal cover-time* and a radius r_{rapid} that will guarantee with high probability that $\mathcal{G}(n, r)$ with $r \geq r_{\text{rapid}}$ is rapid mixing. Optimal cover time is cover time of $\Theta(n \log n)$ [15], the same order as the complete graph. We show that such thresholds do exist, and, surprisingly, the threshold for optimal cover time occurs at a radius $r_{\text{opt}} = \Theta(r_{\text{con}})$. On the other hand, $r_{\text{rapid}} = \omega(r_{\text{con}})$, and, in particular, the radius required for rapid mixing is $r_{\text{rapid}} = \Theta(1/poly(\log n))$.

1.1 Discussion of Our Results and Techniques

The main contribution of this paper is in giving new tight theoretical bounds on the cover time and sharp threshold width associated with cover time for random geometric graphs. Our main result can be formalized as follows:

Theorem 1.1 (Cover Time of RGG) *For $c > 1$, if $r^2 \geq \frac{c8 \log n}{n}$, then* w.h.p. [2] *$\mathcal{G}(n, r)$ has cover time $\Theta(n \log n)$. If $r^2 \leq \frac{\log n}{\pi n}$, then $\mathcal{G}(n, r)$ has infinite cover time with positive probability (bounded away from zero).*

Our result has important implications for applications. Corollaries to our result is that both the *partial cover time* [4], which is the expected time taken by a random walk to visit a constant fraction of the nodes, and the *blanket time* [39], which is the expected time taken by a random walk to visit all nodes with frequencies according to the stationary distribution, are optimal for random geometric graphs. This demonstrates both the efficiency and quality of random walk approaches and certain token-management schemes for some ad-hoc and sensor networks [12,23,4].

Another contribution is bounding the mixing-time and spectral gap of random geometric graphs:

Theorem 1.2 (Mixing Time of RGG) *Radius $r = \Omega(1/poly(\log n))$ is* w.h.p. *necessary and sufficient for $\mathcal{G}(n, r)$ to be rapidly mixing.*

A similar result was obtained independently to our earlier version [5] by [31,7]. Note that the bounds on the cover-time in Theorem 1.1 improve upon bounds

[2] Event $\mathcal{E}_n$ occurs with high probability if probability $P(\mathcal{E}_n)$ is such that $\lim_{n\to\infty} P(\mathcal{E}_n) = 1$.

on the cover time obtainable via Theorem 1.2 as cover time can be bounded by the spectral gap [9]. In particular, the spectral gap method and Theorem 1.2 only guarantees optimal cover time of $\mathcal{G}(n, r)$ for $r = \Theta(1)$.

The techniques we use to prove our results rely on two main features. First, we show that random geometric graphs are *geo-dense*, a term we define here which describes geometric graphs that have desirable properties of uniform node distribution across the unit square and regularity on the node degree. In particular, in geo-dense graphs every *bin* larger than a certain size has the number of nodes inside it proportional to its size. Second, we use different flow based arguments to prove our theorems. In both cases, bins are the building blocks in the flow constructions, and we use the fact that for certain size bins all the nodes inside it form a clique. In the proof of Theorem 1.1 we use a flow to bound the resistance R of the graph [13] which in turn bounds the cover time. In the proof of Theorem 1.2 we use a flow to bound the conductance of the graph [36], which in turn bounds the spectral gap and the mixing time.

The rest of the paper is organized as follows: The next subsection discusses related work. Section 2 covers preliminaries starting from Markov Chains, then known results on conductance and mixing time, and finally known results on resistance and cover time. Section 3 defines Geo-denseness and delineates the geo-denseness of random geometric graphs. In sections 4 and 5 we bound the mixing time and cover time of $\mathcal{G}(n, r)$ respectively. Finally, we conclude with section 6.

1.2 Related Work

There is a vast body of literature on cover times and on geometric graphs, and to attempt to summarize all of the relevant work would not do it justice. We have already mentioned some of the related results previously, however, here we would like to highlight the related literature that has been most influential to our result, namely that of Chandra *et al.* [10] and Doyle and Snell [13].

The work of Doyle and Snell [13] is a seminal work regarding the connection between random walks and electrical resistance. In particular, they proved that while the infinite 2-dimensional grid has infinite resistance, for any $d \geq 3$ the resistance of the d-dimensional grid is bounded from above, and these results were established to be sufficient in re-proving Pólya's beautiful result that a random walk on the infinite 2-dimensional grid is recurrent whereas a random walk on the infinite d-dimensional grid for any $d \geq 3$ is transient. In obtaining this result, essentially the authors bounded the power of a unit current flow from the origin out to infinity and found that the power diverges for the 2-dimensional case and converges for every dimension greater than two. The authors used a layering argument, namely partitioning nodes into disjoint contour layers based on their distance from the origin, and the rate of growth of consecutive layers can be seen as the crucial factor yielding the difference between the properties of the different dimensions. Later, Chandra *et al.* [10] proved the tight relation between commute time and resistance, and used that relationship to extend Doyle and Snell's result by bounding the cover time of the *finite* d-dimensional mesh by computing the power and resistance via an expanding contour layers argument. Together with the tight lower bound of Zuckerman [40], they showed that the 2-dimensional torus has cover time of

$\Theta(n \log^2 n)$, and for $d \geq 3$ the d-dimensional torus has an optimal cover time of $\Theta(n \log n)$.

While this paper deals with random geometric graphs there are striking similarities between $\mathcal{G}(n, r)$ and a more familiar family of random graphs, the *Bernoulli* graphs $\mathcal{B}(n, p)$ in which each edge is chosen independently with probability p [6]. For example, for critical probability $p_{\text{con}} = \pi r_{\text{con}}^2 = \frac{\log n + \gamma_n}{n}$, $\mathcal{B}(n, p)$ is connected with probability going to one as $n \to +\infty$ iff $\gamma_n \to +\infty$, and both classes of graphs have sharp thresholds for monotone properties [6]. Regarding cover time, Jonasson [21] and Cooper and Frieze [11] gave tight bounds on the cover time and an interesting aspect of our result is that we add another similarity and both classes of graphs have optimal cover time around the same threshold for connectivity. Yet, despite the similarities between $\mathcal{G}(n, r)$ and $\mathcal{B}(n, p)$, *Bernoulli* graphs are not appropriate models for connectivity in wireless networks since edges are introduced independently of the distance between nodes. In wireless networks the event of edges existing between i and j and between j and k is *not* independent of the event of an edge existing between k and i. There are other notable differences between $\mathcal{G}(n, r)$ and $\mathcal{B}(n, p)$ as well. For example, the proof techniques for the above results for $\mathcal{G}(n, r)$ are very different than the proof techniques for the respective results for $\mathcal{B}(n, p)$. Interestingly, whereas the proof of [11] for optimality of cover time in Bernoulli graphs of $\Theta(\log n)$ average degree depends on the property that Bernoulli graphs do *not* have small cliques (and, in particular that small cycles are sufficiently far apart), in the case of random geometric graphs the existence of many small cliques uniformly distributed over the unit square like bins, in other words *geo-denseness*, is essential in our analysis.

Geo-denseness is also essential in our method of bounding the conductance of

$\mathcal{G}(n, r)$ to bound the mixing time. Previous work on the use of conductance to bound the mixing time of graphs has been primarily geared towards approximations for hard counting problems and has utilized large, sophisticated constructions of Markov Chains [36].

Another recent result with a bin-based analysis technique for random geometric graphs is that of Muthukrishnan and Pandurangan [27]. However, their technique uses large overlapping bins where the overlap is explicitly stated to be essential and there is no direct utilization of cliques.

In a recent related work Goel *et al.* [18] have proved that any monotonic property of random geometric graphs has a sharp threshold and have bounded the threshold width. While for general graphs optimality of cover time is not a monotonic property A, it follows from our result that optimality of cover time is monotonic for $\mathcal{G}(n, r)$ and has a threshold width of $O(r_{\text{con}})$. This is an order lower than the bounds obtained by Goel *et al.*, but supports their conjectured threshold width.

2 Preliminaries

2.1 Markov chains and the Simple Random Walk

The probabilistic rules by which a random walk operates are defined by the corresponding *Markov chain*. Let $\mathfrak{M}$ be a Markov chain over state space Ω and probability transition matrix P (i.e $P(x, y)$ is the probability to move from x at time t to y at time $t + 1$). In such terms, the stationary distribution of $\mathfrak{M}$,

if such exists, is then defined as the unique probability vector π such that

$$\pi P = \pi$$

A primary motivation in considering a random walk approach as opposed to a deterministic protocol is simplicity and locality of computation. So, if the random walk is currently at node q, then the simplest probabilistic rule by which to choose the next node is simply to choose a node uniformly at random from among the set of neighbors of q. We call the Markov chain $\mathfrak{M} = (\Omega, P)$ corresponding to such a random walk the *simple random walk*. Note that we may just as well define such $\mathfrak{M}$ by its underlying graph $G = (V, E)$. For such G, for any node $v \in V$, let $\delta(v)$ denote the degree of v, that is the number of neighbors of v in G and let $P(v, u) = \frac{1}{\delta(v)}$ for $(v, u) \in E$ and 0 otherwise. It is well known that the simple random walk $\mathfrak{M} = (\Omega, P)$ over a connected graph $G = (V, E)$ has a stationary distribution π such that, for any node $q \in V$ [24],

$$\pi(q) = \frac{\delta(q)}{2m} \tag{1}$$

where $m = |E|$. Further, when the underlying graph G is regular, that is when there is d such that for all q in $\mathfrak{M}$, $\delta(q) = d$, the stationary distribution is the uniform distribution [24]

$$\pi(q) = \frac{d}{2m} = \frac{1}{n} \ \forall q \in \Omega$$

where $n = |\Omega| = |V|$. It is also easy to confirm that the chain is *reversible*, that it satisfies the detailed balance condition with respect to π

$$Q(u, v) = \pi(v)P(v, u) = \pi(u)P(u, v) \ \forall v, u \in V$$

If P is also aperiodic (i.e, G is non-bipartite, which we assume true in our

case[3]) then the chain is *ergodic* and the distribution of the states at time t approaches π as $t \to \infty$, regardless of the starting state.

At stationary distribution, it is clear that the random walk has optimal load-balancing qualities for regular graphs G. Similarly, it is clear that the faster the random walk on a regular graph converges to stationarity, the greater its load-balancing qualities.

2.2 Mixing Time and the Spectral Gap $(1 - \lambda_1)$

The efficiency with which a random walk of $\mathfrak{M}$ may be used to sample over state space Ω with respect to stationary distribution π is precisely given by the rate at which the distribution of the states at time t converges to π as $t \to \infty$. In order to speak of convergence of probabilities, one must have a notion of distance over time. Let x be the state at time $t = 0$ and denote by $P^t(x, \cdot)$ the distribution of the states at time t. The *variation distance* at time t with respect to the initial state x is defined to be [35]

$$\Delta_x(t) = \max_{S \subseteq \Omega} |P^t(x, S) - \pi(S)|$$

When the state space is finite it can be verified that [32]:

$$\Delta_x(t) = \frac{1}{2} \sum_{y \in \Omega} |P^t(x, y) - \pi(y)|$$

The rate of convergence to stationary may be measured by the *mixing time*, the function [35]

$$\tau_x(\epsilon) = min\{t \mid \Delta_x(t') \leq \epsilon, \forall t' \geq t\}$$

[3] One odd length cycle is sufficient to guarantee that G is non-bipartite.

which intuitively is the minimum number of steps t required, starting from node x, to guarantee that for any node y the probability of being at y after t or more steps is at most ϵ away from the probability of being at y under the stationary distribution (i.e. $\pi(y)$). A chain $\mathfrak{M}$ is considered *rapidly mixing* iff $\tau_x(\epsilon)$ is $O(poly(\log(n/\epsilon)))$. For $\mathfrak{M}$ to be used for efficient sampling (according to its stationary distribution), we want $\mathfrak{M}$ to be rapidly mixing.

As the stationary distribution π is defined to be such that $\pi P = \pi$, it corresponds to the eigenvalue $1 = \lambda_0$ of P. Let the rest of the eigenvalues of P in decreasing order be: $1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq -1$. Since $\mathfrak{M}$ is ergodic $\lambda_{n-1} > -1$, and it is well known that the rate of convergence to π is governed by the second largest eigenvalue in absolute value $\lambda_{max} = \max\{\lambda_1, |\lambda_{n-1}|\}$, and in particular by the *spectral gap* $1 - \lambda_{max}$ [35]:

Proposition 2.1 *For an ergodic Markov chain, the quantity $\tau_x(\epsilon)$ satisfies*

(i) $\tau_x(\epsilon) \leq (1 - \lambda_{max})^{-1}(\ln \pi(x)^{-1} + \ln \epsilon^{-1})$

(ii) $\max_{x \in \Omega} \tau_x(\epsilon) \geq \frac{1}{2}\lambda_{max}(1 - \lambda_{max})^{-1} \ln(2\epsilon)^{-1}$

As we want the starting state of a random walk to be arbitrary, the statement above implies that a large spectral gap $(1 - \lambda_{max})$ is both a necessary and sufficient condition for rapid mixing. In practice the smallest eigenvalue is not important since by simply adding self-loop probabilities of $\frac{1}{2}$ ("staying" probability) at each node, we create a new chain that has the same stationary distribution, and its eigenvalues, $\{\lambda'_i\}$, are similarly ordered and satisfy $\lambda'_{n-1} > 0$ and $\lambda'_{max} = \lambda'_1 = \frac{1}{2}(1 + \lambda_1)$ [36]. This shows that it is sufficient to bound λ_1 to prove rapid mixing. A well-known method for bounding λ_1 to prove

rapid mixing when the underlying graph has a geometric interpretation is a *Conductance argument* [20]. This is the method we shall use, as random geometric graphs have a strong geometric interpretation.

2.3 Conductance

Intuitively, one would expect that when the graph that underlies the Markov chain $\mathfrak{M}$ doesn't have *bottlenecks*, the lower the probability of getting stuck in any particular set of states, and thus the more rapidly mixing $\mathfrak{M}$ is. The property of "no bottlenecks" is formalized in a continuous manner with the notion of *conductance*.

The *conductance* of a reversible Markov chain $\mathfrak{M}$ is defined by [36]

$$\Phi = \Phi(\mathfrak{M}) = \min_{S \subset \Omega, 0 < \pi(S) \leq 1/2} \frac{Q(S, \overline{S})}{\pi(S)}$$

where $\overline{S} = \Omega - S$, $\pi(S)$ is the probability density of S under the stationary distribution π, and $Q(S, \overline{S})$ is the sum of $Q(v, u)$ over all $(v, u) \in S \times \overline{S}$.

In graph-theoretic terms, the conductance of $\mathfrak{M}$ is the minimum over all subsets $S \subset \Omega$ of the ratio of the weighted flow across the cut $Cut(S, \overline{S})$ to the weighted capacity of S. The higher the conductance of $\mathfrak{M}$, there are fewer bottlenecks in $\mathfrak{M}$, and the more rapidly mixing $\mathfrak{M}$ is. This intuition is confirmed by the following theorem:

Theorem 2.2 ([35]) *The second eigenvalue λ_1 of a reversible Markov chain $\mathfrak{M}$ satisfies*

$$1 - 2\Phi \leq \lambda_1 \leq 1 - \frac{\Phi^2}{2}$$

The above Theorem along with Proposition 2.1 yield the following powerful

corollary bounding the mixing time via conductance:

Corollary 2.3 ([20]) *Let $\mathfrak{M}$ be a finite, reversible, ergodic Markov chain with loop probabilities $P(x,x) \geq \frac{1}{2}$ for all states x. Let Φ be the conductance of $\mathfrak{M}$. Then, for any initial state x, the mixing time of $\mathfrak{M}$ satisfies*

$$\tau_x(\epsilon) \leq 2\Phi^{-2}(\ln \pi(x)^{-1} + \ln \epsilon^{-1})$$

Regarding cover time, the best cases usually also correspond to dense, highly connected graphs. When connectivity decreases and bottlenecks exist in the graph, the cover time increases, therefore, intuitively, one would anticipate a relationship between the spectral gap $(1 - \lambda_1)$ and small cover time. In confirmation of this intuition, a bound for the cover time for regular graphs G that is based on the spectral gap $(1 - \lambda_1)$ is as follows:

Theorem 2.4 ([9,1]) *For regular graph $G = (V, E)$ with $n = |V|$ and second largest eigenvalue λ_1 the cover time of G is bounded as follows:*

$$C_G = O(n \log n/(1 - \lambda_1))$$

In the next subsection we present yet another way to bound the Cover Time: via resistance.

2.4 Bounding The Cover Time via Resistance

For a graph $G = (V, E)$ with $|V| = n, |E| = m$, the electrical network $\mathcal{E}(G)$ is obtained by replacing each edge $e \in E$ with a 1 Ohm resistor, and this is the network we analyze when we speak of the resistance properties of G. For $u, v \in V$ let R_{uv} be the *effective resistance* between u and v: the voltage

induced between u and v by passing a current flow of one ampere between them. Let R be the electrical resistance of G: the maximum effective resistance between any pair of nodes [13].

Let H_{uv} be the *hitting time*, the expected time for a random walk starting at u to arrive to v for the first time, and let C_{uv} be the *commute time*, the expected time for a random walk starting at u to first arrive at v and then return to u. Chandra *et al.* [10] proved the following equality for the commute time C_{uv} in terms of the effective resistance R_{uv}:

Theorem 2.5 ([10]) *For any two vertices u and v in G the commute time $C_{uv} = 2mR_{uv}$*

Using this direct relation between resistance and random walks and Matthews' theorem [25] the authors introduced the following bound on the cover time for any graph with n nodes and m edges, where R is the the electrical resistance of the graph:

$$mR \leq \textit{cover time} \leq O(mR \log n) \tag{2}$$

Let $H_{\max}$ be the maximum hitting time over all pairs of nodes in G. Since $H_{uv} \leq C_{uv}$ it follows that $H_{\max} \leq \max_{u,v \in V} C_{uv} = 2mR$. In [4] it has been shown that the partial cover time can be bounded by $H_{\max}$, so combining:

$$\textit{partial cover time} \leq O(mR) \tag{3}$$

Thus, by bounding the resistance R we may obtain tight bounds on the cover time C_G through (2) and on the partial cover time through (3).

A powerful method used to bound resistance is by bounding the power of a current flow in the network. The following definitions and propositions from

the literature [10,13,37] help to formalize this method.

Definition 2.6 (Power of a flow) *Given an electrical network (V, E, ρ), with resistance $\rho(e)$ for each edge e, a flow c from a* source *u to a* sink *v is a function from $V \times V$ to $\mathbb{R}$, having the property that $c(x, y) = 0$ unless $\{x, y\} \in E$, and c is anti-symmetric, i.e., $c(x, y) = -c(y, x)$. The net flow out of a node will be denoted $c(x) = \sum_{y \in V} c(x, y)$ and $c(x) = 0$ if $x \neq u, v$. The flow along an edge e is $c(e) = |c(u, v)|$. The power $P(c)$ in a flow is $P(c) = \sum_{e \in E} \rho(e)c^2(e)$. A flow is a current flow if it satisfies Kirchoff's voltage law, i.e., for any directed cycle $x_0, x_1, \ldots, x_{k-1}, x_0$, $\sum_{i=0}^{k-1} c(x_i, x_{i+1 \bmod k}) \cdot \rho(x_i, x_{i+1 \bmod k}) = 0$.*

Proposition 2.7 *[Thomson Principle [13,37]] For any electrical network (V, E, ρ) and flow c with only one source u, one sink v, and $c(u) = -c(v) = 1$ (i.e a **unit** flow), we have $R_{uv} \leq P(c)$, with equality when the flow is a current flow.*

Finally,

Proposition 2.8 *[Rayleigh's Short/Cut Principle [13]] Resistance is never raised by lowering the resistance on an edge, e.g. by "shorting" two nodes together, and is never lowered by raising the resistance on an edge, e.g. by "cutting" it. Similarly, resistance is never lowered by cutting a node, leaving each incident edge attached to only one of the two halves of the node.*

3 Geo-dense Geometric Graphs

Our analytical results for random geometric graph are based on the "nice" properties that those graphs have. These properties include the uniformity of nodes distribution and the regularity of node degree. In the following section

we will define this using the notion of *geo-dense* graph, that is, geometric graph (random or deterministic) with uniform node density across the unit square. In *geo-dense* graphs there are no large areas that fail to contain a sufficient number of nodes. We show that *random* geometric graphs are *geo-dense* and for radius $r_{\text{reg}} = \Theta(r_{\text{con}})$ all nodes have the same order degree *w.h.p.* To do so we introduce the notion of *bins*, equal size areas that partition the unit square. These bins will be our buildings block for future proofs as well.

A **geometric graph** is a graph $G(n, r) = (V, E)$ with $n = |V|$ such that the nodes of V are embedded into the unit square with the property that $e = (u, v) \in E$ if and only if $d(u, v) \leq r$ (where $d(u, v)$ is the Euclidean distance between points u and v).

Definition 3.1 *Let $G(n, r(n))$ be a geometric graph. For a constant $\mu \geq 1$, we say that such a graph is μ-**geo-dense** if every square bin of size $A \geq r^2/\mu$ (in the unit square) has $\Theta(nA)$ nodes.*

We can claim the following on geo-dense graphs:

Lemma 3.2 *Let $G(n, r)$ be a 2-geo-dense geometric graph with V the set of nodes and E the set of edges. Let $\delta(v)$ denote the degree, i.e. the number of neighbors, of $v \in V$. Then: (i) $\forall v \in V \;\; \delta(v) = \Theta(nr^2)$ and (ii) $m = |E| = \Theta(n^2 r^2)$.*

PROOF. (i). First note that the *geo-dense* property guarantees that if we divide the unit square into square bins of size $\frac{r}{\sqrt{2}} \times \frac{r}{\sqrt{2}}$ each, then the number of nodes in every bin will be $\Theta(nr^2)$. Since, for every bin, the set of nodes in the bin forms a clique, and every node $v \in V$ is in some bin, we have that $\delta(v) = \Omega(nr^2), \forall v \in V$. Similarly, when we divide the area into bins of size

$r \times r$ every node may be connected to the nodes of at most nine bins (that is its own bin and the bordering bins), and we have that $\delta(v) = \Theta(nr^2), \forall v \in V$. (ii) follows directly from (i). $\square$

3.1 Geo-dense Random Geometric Graphs

To prove the *geo-dense* property for $\mathcal{G}(n, r)$ we utilize the following lemma which seems to be folklore [39] although we include a proof in Appendix B since we have not found a reference including a proof of the *minimum* condition.

Lemma 3.3 (Balls in Bins) *For a constant $c > 1$, if one throws $n \geq cB \log B$ balls uniformly at random into B bins, then* w.h.p. *both the minimum and the maximum number of balls in any bin is $\Theta(\frac{n}{B})$.*

Following Balls in Bins Lemma we can now make the claim about the geo-density of $\mathcal{G}(n, r(n))$ precise:

Lemma 3.4 (Geo-density of $\mathcal{G}(n, r)$) *For constants $c > 1$ and $\mu \geq 1$, if $r^2 = \frac{c\mu \log n}{n}$ then* w.h.p. *$\mathcal{G}(n, r)$ is* μ-geo-dense*, that is, any bin area of size r^2/μ in $\mathcal{G}(n, r)$ has $\Theta(\log n)$ nodes* w.h.p.

PROOF. Let an area of r^2/μ be a bin. If we divide the unit square into such equal size bins we have $B = \frac{n}{c \log n}$ bins. For the result to follow we check that

Lemma 3.3 holds by showing that $n \geq c' B \log B$ for some constant $c' > 1$:

$$\begin{aligned} B \log B &= \frac{n}{c \log n} \log(\frac{n}{c \log n}) \\ &= \frac{n}{c \log n}(\log(n) - \log(c \log n)) \\ &= \frac{n}{c} - (\frac{n}{c \log n})(\log(c \log n)) \\ &\leq n/c \qquad \square \end{aligned}$$

Now combining the results of Lemmas 3.2 and 3.4 we can also claim the following about $\mathcal{G}(n, r(n))$:

Corollary 3.5 *For $c > 1$, if $r^2 \geq \frac{c2 \log n}{n}$, then* w.h.p. *$\forall v \in \mathcal{G}(n, r)$, $\delta(v) = \Theta(nr^2)$ and $m = |E| = \Theta(n^2 r^2)$.*

Recall that the critical radius for connectivity r_{con} is s.t. $\pi r_{\text{con}}^2 = \frac{\log n}{n}$. We have just showed that for $r_{\text{reg}} = \Theta(r_{\text{con}})$ *w.h.p.* $\mathcal{G}(n, r_{\text{reg}})$ will have the nice properties mentioned above. Note however, that even though $\mathcal{G}(n, r_{\text{reg}})$ is *geo-dense* in our terms, it is *not* a dense graph in graph theoretic terms (i.e a graph with $\Theta(n^2)$ edges), but is a sparse graph with expected number of $\Theta(n \log n)$ edges.

4 The Mixing Time of Random Geometric Graphs

In this section we demonstrate that for sufficiently large n, the conductance Φ of $\mathcal{G}(n, r)$ is $\Phi(\mathcal{G}(n, r)) = \Theta(r)$ with high probability, and we give a useful continuous approximation to Φ in Appendix D. Based on the conductance results, we show that for $\mathcal{G}(n, r)$ to be rapidly mixing, radius at least $r_{\text{rapid}} = \Theta(1/poly(\log n))$ is necessary and sufficient.

4.1 Bounding the Conductance of $\mathcal{G}(n,r)$

Let $\mathcal{G}(n,r)$ be a random geometric graph constructed as mentioned earlier. The main result of this section is as follows:

Theorem 4.1 (Conductance of RGG) *For* $c > 1$, *if* $r^2 \geq \frac{c4\log n}{n}$, *then* w.h.p.

$$\Phi(\mathcal{G}(n,r)) = \Theta(r)$$

¿From Theorem 4.1, Theorem 2.2 and Corollary 2.3 we obtain these bounds:

Corollary 4.2 *For* $c > 1$, *if* $r^2 \geq \frac{c4\log n}{n}$, *then* w.h.p. *the mixing time of* $\mathcal{G}(n,r)$ *is as follows:*

(1) $\tau_x(\epsilon) = O(r^{-2}(\ln n + \ln \epsilon^{-1}))$

(2) $1 - \lambda_1 = \Omega(r^2)$ *and* $1 - \lambda_1 = O(r)$

Together with Proposition 2.1 (ii) we also obtain the necessary condition:

Theorem 1.2 *Radius* $r = \Omega(1/poly(\log n))$ *is* w.h.p. *necessary and sufficient for* $\mathcal{G}(n,r)$ *to be rapidly mixing.*

Now we may begin the proof of the main result of this section:

PROOF. [of Theorem 4.1] Let $Cut(S,\overline{S})$ denote the cut size between S and $\overline{S}$ in $\mathcal{G}(n,r)$: the total number of edges crossing from S to $\overline{S}$.

Since $\mathcal{G}(n,r)$ is *4-geo-dense* and "almost regular" *w.h.p.* by Lemma 3.4 and Corollary 3.5 we can observe that the minimum conductance is when we divide the area into two halves S and $\overline{S}$ with $\pi(S) \approx \pi(\overline{S}) \approx \frac{1}{2}$ and such that the length of the boundary between S and $\overline{S}$ is minimized. Similarly to the regular grid case (Appendix C), the separation satisfying this is with a separating line l parallel to one of the axis. Let $Cut_\Phi(S,\overline{S})$ be the above cut, the one that minimizes $\Phi(\mathcal{G}(n,r))$. For details on why such a separation yields the minimum ratio of weighted flow to capacity, we refer the reader to Appendix G. Next we bound $Cut_\Phi(S,\overline{S})$.

For the lower bound of $Cut_\Phi(S,\overline{S})$, partition the area into bins of size $\frac{r}{2\sqrt{2}} \times \frac{r}{\sqrt{2}}$ as in Figure 1 (A). By the *4-geo-dense* property *w.h.p.* the number of nodes in any bin is $\Theta(nr^2)$. Notice that the set of nodes in any two horizontally adjacent bins (such as B_0 and B_1 in Fig 1 (A)) forms a clique. Therefore, to lower bound $Cut_\Phi(S,\overline{S})$, we are only considering the crossing edges within each separate such clique along the dividing line l. Since there are at least $\frac{\sqrt{2}}{r}$ cliques along the dividing line l, and for each bin on the left side of l we have $\Omega(n^2r^4)$ such edges crossing to the right of l, we obtain the desired lower bound $Cut_\Phi(S,\overline{S}) = \Omega(r^3n^2)$.

For the upper bound partition the area into bins of size $r \times r$ as in Figure 1 (B). Note that for each edge (u,v) crossing l, v must be in some left bin B_0 adjacent to l, and so u must be in one of three possible bins B_1, B_2, B_3 that are on the right of l and touching B_0 as shown in the picture. To upper bound $Cut_\Phi(S,\overline{S})$, we consider the maximum number of crossing edges from any $r \times r$ sized bin B_0 in S to three $r \times r$ sized bins B_1, B_2 and B_3 in $\overline{S}$. As there are $\frac{1}{r}$ such bins as B_0, and from the *4-geo-dense* property, *w.h.p.* the number of

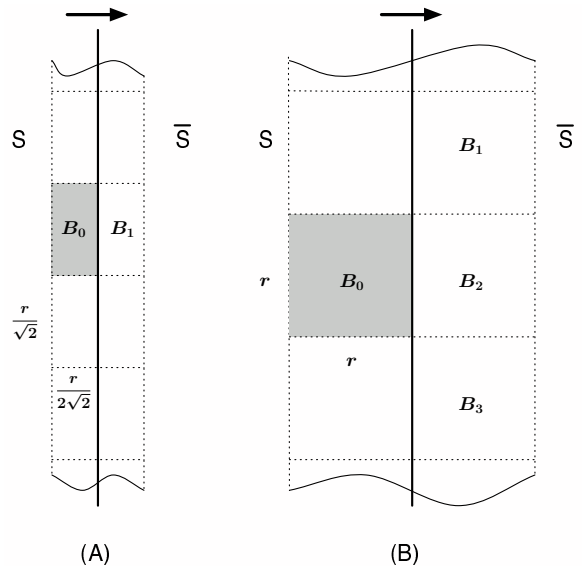

Fig. 1. (A) Lower bound for the Conductance in $\mathcal{G}(n,r)$. (B) upper bound for the Conductance in $\mathcal{G}(n,r)$

nodes in any bin is $\Theta(nr^2)$, we get the desired upper bound as follows:

$$Cut_\Phi(S,\overline{S}) = O(\frac{1}{r}\cdot nr^2\cdot 3nr^2) = O(r^3n^2)$$

So, combining the upper and lower bounds, we have that *w.h.p.* ,

$$Cut_\Phi(S,\overline{S}) = \Theta(r^3n^2)$$

And, thus, by corollary 3.5, equation (1), and the definition of $P(x, y)$ we complete the proof:

$$\begin{aligned}
\Phi(\mathcal{G}(n,r)) &= \min_{S\subset V, 0<\pi(S)\leq 1/2} \frac{Q(S,\overline{S})}{\pi(S)} \\
&= \min_{S\subset V, 0<\pi(S)\leq 1/2} \frac{1}{\pi(S)} \sum_{\substack{x\in S \\ y\in\overline{S}}} \pi(x)P(x,y) \\
&= \min_{S\subset V, 0<\pi(S)\leq 1/2} \Theta(2) \sum_{\substack{x\in S \\ y\in\overline{S}}} \Theta(\frac{1}{n})\Theta(\frac{1}{nr^2}) \\
&= Cut_\Phi(S,\overline{S}) \frac{1}{\Theta(r^2n^2)} \\
&= \Theta(r^3n^2)/\Theta(r^2n^2) \\
&= \Theta(r) \qquad \square
\end{aligned}$$

5 The Cover Time of Random Geometric Graphs

It follows from the previous section, Corollary 4.2 and Theorem 2.4 that we can bound the cover time of $\mathcal{G}(n, r)$ as follows:

Corollary 5.1 *For $c > 1$, if $r^2 \geq \frac{c4\log n}{n}$, then* w.h.p.

$$C(\mathcal{G}(n,r)) = O(r^{-2}n\log n)$$

If this bound on cover time were tight, then the only way to achieve optimal cover time for random geometric graphs would be by choosing a radius r that is constant irrespective of the network size n. Recalling that our definition of $\mathcal{G}(n, r)$ is normalized to a unit area, this would mean that only broadcast networks of constant hop diameter may have optimal cover and partial cover. Even the minimum radius required for rapid mixing, which is $r_{rapid} = \Theta(1/poly(\log n))$, is several orders lower than such a radius. However,

fortunately, the bounds given by Theorem 2.4 and correspondingly Corollary 5.1 are not tight. The method that we used to improve upon these results and to derive Theorem 1.1 is by bounding the electrical resistance of $\mathcal{G}(n, r)$, which bounds the cover time by (2). In turn, we bound the resistance R of $\mathcal{G}(n, r)$ by bounding the power of a unit flow as permitted by Thomson's Principle. For any pair of points u and v, we construct a flow c in such a manner that the power of the flow satisfies $P(c) = O(\frac{n}{m}) = O(\frac{1}{\delta_{\text{avg}}})$ where δ_{avg} denotes the average degree of a node in $\mathcal{G}(n, r)$. Since $R \leq P(c)$ the above flow together with (2) establish to be sufficient for $\mathcal{G}(n, r)$ to have optimal cover time.

To construct a flow from u to v, we partition the nodes into contour layers based on distance from u and expanding outward until the midpoint between u and v, then from the midpoint line onward contracting towards v in a mirror fashion. The idea of using contour layers that expand with distance from a point is similar to the layering ideas used by Chandra *et al.* [10] for meshes and originally by Doyle and Snell [13] for infinite grids. Layers in our case can be visualized as slices of an isosceles right triangle along the hypotenuse that connects u and v. The flow can thus be thought of as *moving through* consecutive layers, with the total flow on the edges connecting consecutive layers being 1. Just as the variance of a probability function is minimized for the uniform distribution, we minimize the power by allocating flow almost uniformly along the set of edges used between layer l and layer $l + 1$.

The construction of the above flow is based, as before, on the *geo-dense* property of random geometric graphs. To construct the flow we define *bins* as equal size areas that partition the unit square. These bins are used to construct our layered flow: Nodes in neighboring bins are in the same clique, and only edges between neighboring bins contribute to the flow.

Before proving Theorem 1.1 about *random* geometric graphs we are going to prove a more general theorem about geometric graphs.

Theorem 5.2 *A geometric graph $G(n,r)$ that is* 8-geo-dense *and has $r = \Theta(\frac{\log n}{n})$ has optimal cover time of $\Theta(n \log n)$, optimal partial cover time of $\Theta(n)$, and optimal blanket time of $\Theta(n \log n)$.*

Let $G(n,r)$ be a geometric graph that is *8-geo-dense*. We will prove Theorem 5.2 using the bound on the cover time from Eq. (2) and by bounding the resistance between any two points u, v in $G(n,r)$. Thus, since we showed that $m = |E| = \Theta(n^2 r^2)$, if the resistance R of $G(n,r)$ is $O(\frac{n}{m}) = O(\frac{1}{nr^2})$ then we are done.

Theorem 5.3 *The resistance R_{uv} between $u, v \in V$ is $\Theta(\frac{1}{nr^2} + \frac{\log(d(u,v)/r)}{n^2 r^4})$.*

PROOF. The proof of the upper bound will be by bounding the power of a unit flow c that we construct between u and v.

Let $T(u,v)$ be an isosceles right triangle such that the line (u,v) is the hypotenuse. It is clear that such a triangle which lies inside the unit square must exist. We divide our flow c into two disjoint flows c_1 and c_2 where c_1 carries a unit flow from u up to the line perpendicular to the *midpoint* of $d(u,v)$ in increasing layer size, and c_2 forwards the flow in decreasing layer size up to v which is the only sink. By symmetry we can talk only about c_1 since the construction of c_2 mirrors that of c_1 and $P(c) = P(c_1 + c_2) = 2P(c_1)$ since the flows are disjoint. To construct the flow in c_1 we divide the line

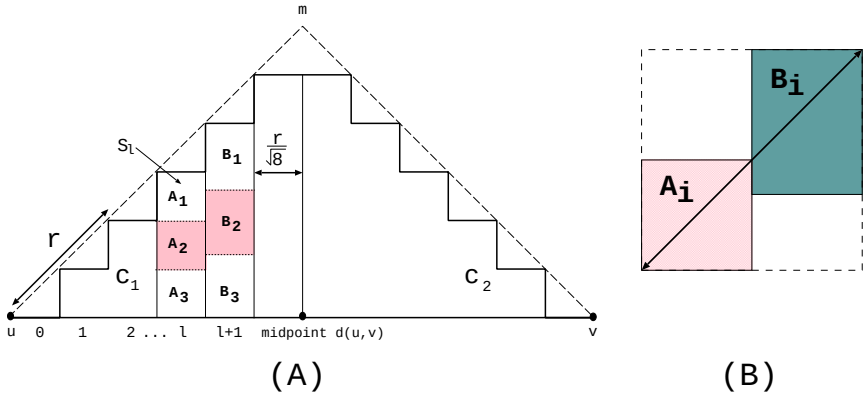

Fig. 2. $T(u,v)$ and the flow c between u and v in $G(n,r)$

$(u, midpoint(u,v))$ into $\frac{d(u,v)\sqrt{8}}{2r}$ segments of length $r/\sqrt{8}$, and number them from 0 to $d(u,v)\sqrt{2}/r - 1$ (see Fig 2 (A))[4].

Let S_l be the largest rectangle of width $r/\sqrt{8}$ included in the intersection of the area perpendicular to the l^{th} segment and $T(u,v)$. S_l will define the l^{th} layer in our flow. Note that the area of S_l is $lr^2/8$ and contains l equal size squares of area $r^2/8$, each of them containing $\Theta(nr^2)$ nodes by the *8-geo-dense* property. In particular every square contains at least $\alpha = \frac{nr^2}{c^*}$ nodes for a constant $c^* > 1$ independent of n. For each square, we choose a set of α nodes to participate in our flow construction.

Let $V_l \subseteq V$ be the set of nodes in layer l. $V_0 = u$, and for $l > 0$ a node v is in layer l if and only if it is in the set of participating nodes for a square located inside S_l. It follows that $|V_l| = \alpha l = \Theta(nr^2 l)$. Edges in our flow are only among edges $e = (x,y)$ s.t. $x \in V_l$ and $y \in V_{l+1}$, and all other edges have zero flow. In particular, the set of edges E_l that carries flow from layer l to layer $l+1$ in c_1 is defined as follows: For the case $l = 0$, E_0 contains all

[4] Assume for simplicity the expression divides nicely, if not, the proof holds by adding one more segment that will end at the midpoint and overlap with the previous segment.

the edges from u to nodes in V_1, noting that $|E_0| = |V_1| = \alpha = \Theta(nr^2)$ since $u \cup V_1$ is a clique (i.e the maximum $d(u, x), x \in V_1$ is r). This allows us to make the flow uniform such that each node in V_1 has incoming flow of $1/|V_1|$ and for each edge $e \in E_0$, $c_1(e) = 1/|E_0|$. For $l > 0$ (see again Fig. 2 (A)) we divide S_l into l equal squares $A_1, A_2, \ldots A_l$ each of size $r^2/8$. Let V_{A_i} be the set of participating nodes contained in the area A_i. We then divide S_{l+1} into l rectangles $B_1, B_2 \ldots B_l$. Each rectangle B_i is define such that V_{B_i} will contain exactly $\frac{l+1}{l}\alpha$ participating nodes and with B_i touching A_i for each i. Note that the area of B_i may vary for different i but is at least A_i and at most $2A_i$.

Now let $E_l = \{(x, y)|x \in V_{A_i} \text{ and } y \in V_{B_i}\}$. Note again that since, for each i, the maximum $d(x, y)$ between nodes in A_i and nodes in B_i is r (see Fig. 2 (B)), $V_{A_i} \cup V_{B_i}$ is a clique (as the worst case distance occurs between the first two layers). So, the number of edges crossing from A_i to B_i is $|V_{A_i}||V_{B_i}| = \alpha^2\frac{l+1}{l} = \Theta(n^2r^4)$ by *8-geo-dense* property. The clique construction allows us to easily maintain the uniformity of the flow such that into each node in V_{B_i} the total flow is $1/l|V_{B_i}|$, and each edge carries a flow of $1/E_l = 1/\alpha^2(l + 1) =$

$\Theta(1/n^2r^4l)$. All other edges have no flow. Now we compute the power of c:

$$\begin{aligned}
R_{uv} &\leq \sum_{e\in c} c(e)^2 = \sum_{e\in c_1} c_1(e)^2 + \sum_{e\in c_2} c_2(e)^2 = \\
&= 2\sum_{l=0}^{\sqrt{2}d(u,v)/r} \sum_{e\in E_l} c_1(e)^2 \\
&= 2\frac{1}{|E_0|} + 2\sum_{l=1}^{\sqrt{2}d(u,v)/r} \frac{|E_l|}{|E_l|^2} \\
&= 2\frac{1}{\alpha} + 2\frac{1}{\alpha^2}\sum_{l=1}^{\sqrt{2}d(u,v)/r} \frac{1}{l+1} \\
&= 2O(\frac{1}{nr^2}) + 2O(\frac{1}{n^2r^4})\sum_{l=1}^{\sqrt{2}d(u,v)/r} \frac{1}{l+1} \\
&= O(\frac{1}{nr^2} + \frac{\log(d(u,v)/r)}{n^2r^4})
\end{aligned}$$

To prove the lower bound we again follow in the spirit of [13] and use the "Short/Cut" Principle. We partition the graph into $\lfloor d(u,v)/r \rfloor + 1$ partitions by drawing $\lfloor d(u,v)/r \rfloor$ squares perpendicular to the line (u,v), where the first partition P_0 is only u itself and the l^{th} partition P_l is the area of the l^{th} square excluding the $(l-1)^{\text{th}}$ square area. The last partition contains all the nodes outside the last square including v (see Fig 3 (A)). We are shorting all vertices in the same partition (see Fig. 3 (B), and following the reasoning of the upper bound, let m_l be the number of edges between partition l and $l+1$. m_0 is $\Theta(nr^2)$ and for $l > 0$, $m_l = \Theta(n^2r^4l)$, so

$$\begin{aligned}
R_{uv} &\geq \sum_{l=0}^{\lfloor d(u,v)/r \rfloor} \frac{1}{m_l} \\
&= \Omega(\frac{1}{nr^2}) + \sum_{l=1}^{\lfloor d(u,v)/r \rfloor} \Omega(\frac{1}{n^2r^4l}) \\
&= \Omega(\frac{1}{nr^2} + \frac{\log(d(u,v)/r)}{n^2r^4}) \qquad \square
\end{aligned}$$

Corollary 5.4 *The resistance R of $G(n,r)$ is $\Theta(\frac{1}{nr^2} + \frac{\log(\sqrt{2}/r)}{n^2r^4})$.*

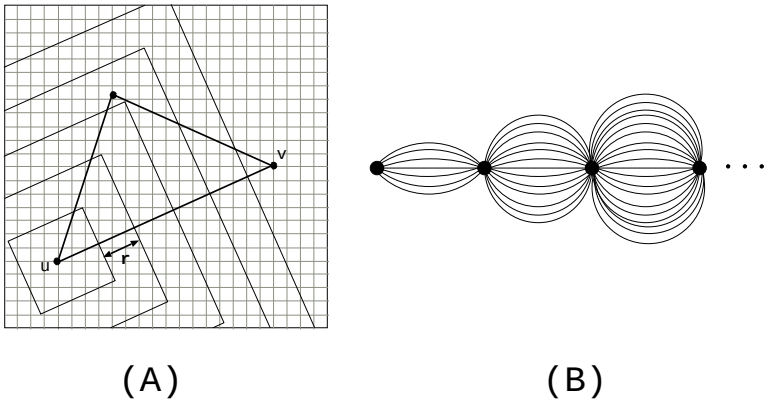

Fig. 3. Lower bound for R_{uv} on the $G(n,r)$

This follows directly from the fact that $\max d(u,v) \leq \sqrt{2}$. Now we can prove Theorem 5.2.

PROOF. [of Theorem 5.2] Remember that $m = \Theta(n^2r^2)$, so all we need is $R = O(n/m) = O(1/nr^2)$ and then the cover time bound will follow by (2) and partial cover time bound will follow from (3). In order to have $R = \Theta(\frac{1}{nr^2})$ we want that $\frac{\log(\sqrt{2}/r)}{n^2r^4} = O(\frac{1}{nr^2})$, which means $\frac{\log(1/r)}{nr^2} \leq \alpha$ for some constant α. Taking $r^2 = \frac{\beta \log n}{n}$, for a constant β, we get $\frac{\log(n/\beta \log n)}{\beta 2 \log n} = \frac{1}{2\beta} - \frac{\log(\beta \log n)}{2\beta \log n} \leq \frac{1}{2\beta}$. The optimality of the blanket time, B_G, will follow from Theorem 1 and Corollary 1 in [39] which proves that if $C_G = O(H_{\max} \log n)$ then $B_G = O(C_G)$ [5]. Recall that for any graph $H_{\max} = O(mR)$, in our case we have $H_{\max} = O(n)$ so the result follows. □

[5] Interestingly they conjectured that for all graphs $B_G = O(C_G)$ but could prove only special cases.

5.2 Cover Time and Resistance of $\mathcal{G}(n,r)$

After Proving Theorem 5.2, in order to prove Theorem 1.1 all we need to show is that for $c > 1$, $r^2 = \frac{c8\log n}{n}$ is sufficient to guarantee with high probability that $\mathcal{G}(n,r)$ is *8-geo-dense*. Note however that the second part of the theorem follows directly from [19] since if $\mathcal{G}(n,r)$ is disconnected with positive probability bounded away from zero when $r^2 \leq \frac{\log n}{\pi n}$, then it has infinite cover time with at least the same probability.

Now combining the results of Lemmas 3.3 and 3.4 we can prove Theorem 1.1

Theorem 1.1 *For $c > 1$, if $r^2 \geq \frac{c8\log n}{n}$, then* w.h.p. *$\mathcal{G}(n,r)$ has cover time $\Theta(n\log n)$. If $r^2 \leq \frac{\log n}{\pi n}$, then $\mathcal{G}(n,r)$ has infinite cover time with positive probability (bounded away from zero).*

PROOF. Clearly from Lemma 3.4 for $c > 1$, $r^2 = \frac{c8\log n}{n}$ satisfies the *8-geo-dense* property *w.h.p.* , and since r^2 is also $\Theta(\frac{\log n}{n})$ the result follows from Theorem 5.2. □

Corollary 5.5 *For $c > 1$, if $r^2 \geq \frac{c8\log n}{n}$, then* w.h.p. *$\mathcal{G}(n,r)$ has optimal partial cover time $\Theta(n)$ and optimal blanket time $\Theta(n\log n)$.*

6 Conclusions

We have shown that for a two dimensional random geometric graph $\mathcal{G}(n,r)$, if the radius r_{opt} is chosen just on the order of guaranteeing asymptotic connectivity then $\mathcal{G}(n,r)$ has optimal cover time of $\Theta(n\log n)$ for any $r \geq r_{\text{opt}}$.

Noting that $\mathcal{G}(n, r_{\text{opt}})$ still has a long diameter of $\Theta(\frac{1}{r_{\text{opt}}}) = \Theta(\sqrt{\frac{n}{\log n}})$, it is not surprising that it is not rapid mixing, a property which we have shown requires a radius of at least $r_{\text{rapid}} = \Theta(1/poly(\log n))$. Intuitively, this gap seems to indicate that although the partial cover is optimal, that is linear, the distribution of the uncovered nodes after the partial cover may be such that contiguous uncovered geometric regions may remain.

We present a similar proof bounding the cover time of 1-dimensional random geometric graphs in Appendix E. We find that the critical radius guaranteeing optimal cover time is $r_{\text{opt}} = \Omega(\frac{1}{\sqrt{n}})$ for such graphs, whereas the critical radius guaranteeing asymptotic connectivity is $r_{\text{con}} = \frac{\log n}{n}$. So, unlike the 2-dimensional case, we have $r_{\text{opt}} = \omega(r_{\text{con}})$.

Our proof techniques can be generalized to the d-dimensional random geometric graph $\mathcal{G}^d(n, r)$, yielding that for any given dimension d, $r_{\text{opt}} = \Theta(r_{\text{con}})$ with correspondingly optimal cover time. However, both grow exponentially with d which seems to be a consequence of a separation between average degree and minimum degree for higher dimensions rather than just an artifact of our method. Nevertheless, the case of dimension $d = 2$ is considered to be the hardest one [1]. This can intuitively be seen from the mesh results. The case for $d = 1$ (i.e the cycle) is easy to analyze. For $d > 2$ the cover time of the d-dimensional mesh is optimal [10], and we can show that for any k the cover time of the k-fuzz[6] is also optimal. On the other hand, as we show in Appendix F, the cover time of the $k-$fuzz in 2 dimensions (i.e. $G_k(n)$) for constant k is not optimal making this the most interesting case.

[6] For an integer k, let the **k-fuzz** of a graph G be the graph G_k obtained from G by adding an edge xy if x is at most k hops away from y in G.

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Appendix

A Optimal Cover Time is not Monotone

An immediate and well-known corollary to Rayleigh's Short/Cut Principle is that the Resistance R of a graph is monotone, as adding new edges can only decrease or not affect the resistance R. On the other hand, it is also well-known that, in general, cover time is not a monotone property of graphs. As a simple demonstrative example we can take the line of n nodes which has cover time of $O(n^2)$, and by adding edges we can create the lollipop graph which is known to have cover time of $O(n^3)$, and if we keep adding edges we will get the complete graph which has optimal cover time, $O(n \log n)$ [15]. One can wonder if this is still the case if the graph G already has cover time of $O(n \log n)$. In other words, can we create, by adding more edges, a graph G' which has cover time of $\omega(n \log n)$?

Lemma A.1 *Cover time of $O(n \log n)$ is not a monotone property of graphs.*

PROOF. The proof will be by counter example and by the lower bound for cover time given in Theorem 2 . Let G be the 3D grid of n nodes. It is known that G has cover time of $C_G = O(n \log n)$ [10]. We construct a graph G' be adding $O(n^2)$ edges to G is such a way that the resistance of the graph will not change: Let u_0 be the node at $(0, 0, 0)$ and u_n the node at $(\sqrt[3]{n}, \sqrt[3]{n}, \sqrt[3]{n})$. Make all the points at L_1 distance at most $\sqrt[3]{n}$ from u_0 a clique. The number of nodes in this clique is $\approx n/2$, and so the number of edges in this clique is $\approx n^2/8$, making the total number of edges in G' $m = \Theta(n^2)$. Since the minimum degree in G' is the same as in G, namely degree of 3 at u_n, the

resistance of $G' \geq \frac{1}{3}$, and by Theorem 2 we get $C_{G'} = \Omega(n^2)$. $\square$

B Proof of Lemma 3.3

Lemma 3.3 *For a constant $c > 1$, if one throws $n \geq cB \log B$ balls uniformly at random into B bins, then* w.h.p. *both the minimum and the maximum number of balls in any bin is $\Theta(\frac{n}{B})$*

PROOF. Let $n = cB \log B$ and note that when $n \to \infty$ then $B \to \infty$. In [26] it is proven that *w.h.p.* the maximum number of balls in any bin is $O(\frac{n}{B})$. Here we prove that *w.h.p.* the minimum number of balls in any bin is $\Omega(\frac{n}{B})$, namely, *w.h.p.* every bin has at least $\frac{\log B}{c^*}$ balls for a constant $c^* > 1$ to be determined below. Let X_i denote the number of balls in the i^{th} bin. Fix a bin, say the first bin, and consider $Pr[X_1 = \frac{\log B}{c'}]$ for a constant $c' > 1$:[7]

$$\begin{aligned} Pr[X_1 = \frac{\log B}{c'}] &= \binom{cB \log B}{\log B/c'} (\frac{1}{B})^{\frac{\log B}{c'}} (1 - \frac{1}{B})^{cB \log B - \frac{\log B}{c'}} \\ &\leq (ec'c)^{\frac{\log B}{c'}} e^{\frac{\frac{\log B}{c'} - cB \log B}{B}} \\ &= B^{\frac{1}{c'}} (c'c)^{\frac{\log B}{c'}} B^{\frac{1}{Bc'} - c} \\ &= B^{\frac{(\log c' + \log c)}{c'} + \frac{1}{c'} + \frac{1}{Bc'} - c} \end{aligned}$$

Since we want for this probability to be $\frac{1}{B^{1+\epsilon'}}$ for $\epsilon' > 0$ we need

$$c - (\frac{(\log c' + \log c)}{c'} + \frac{1}{c'} + \frac{1}{Bc'}) > 1$$

[7] by using $(1 - \frac{1}{n})^r \leq e^{-r/n}$, $\binom{n}{k} \leq (\frac{ne}{k})^k$ and $c' = e^{\log(c')}$.

Let $c = 1 + \epsilon$ where $\epsilon > 0$ can be arbitrary small constant and so we need c' s.t.

$$1 + \epsilon - (\frac{\log c' + \log(1+\epsilon)}{c'} + \frac{1}{c'} + \frac{1}{Bc'}) > 1 \tag{B.1}$$

Using $\log(1+\epsilon) < \epsilon$ the following c' will satisfy (B.1)

$$\frac{1}{c'-1} + \frac{1}{B(c'-1)} + \frac{\log c'}{c'-1} < \epsilon \tag{B.2}$$

So it is clear that there exists a constant c' that satisfies (B.2) for any constant $\epsilon > 0$. Then, let $c^* = c'$. Note easily that $Pr[X_1 = \frac{\log B}{c^*}] \geq Pr[X_1 = \frac{\log B}{c^*} - Q]$ for any $0 \leq Q \leq \frac{\log B}{c^*}$. Therefore, we have that for large enough B

$$\begin{aligned} Pr[X_1 \leq \frac{\log B}{c^*}] &\leq (\frac{\log B}{c^*}) Pr[X_1 = \frac{\log B}{c^*}] \\ &\leq (\frac{\log B}{c^*}) \frac{1}{B^{1+\epsilon'}} \end{aligned}$$

Finally to get the lower bound (minimum) for all bins, we use that the probability of the union of events is no more than their sum. Letting U denote the event that *some bin has less than* $\frac{\log B}{c^*}$ *balls*:

$$\begin{aligned} Pr[U] &\leq \sum_{i=1}^{B} Pr[X_i \leq \frac{\log B}{c^*}] \\ &= \sum_{i=1}^{B} Pr[X_1 \leq \frac{\log B}{c^*}] \\ &= B \frac{\frac{\log B}{c^*}}{B^{1+\epsilon'}} \\ &= \frac{\frac{\log B}{c^*}}{B^{\epsilon'}} = o(1) \end{aligned}$$

Therefore, with high probability every bin has at least $\frac{\log B}{c^*} = \Theta(\frac{n}{B})$ balls. Now, clearly, choosing $n > cB \log B$ can only increase the probability that every bin has at least $\frac{\log B}{c^*} = \Theta(\frac{n}{B})$ balls. So, we are done. $\square$

C Bounding the Conductance of the k dimensional Grid

To begin with a simple example of a conductance argument with similarities to the conductance argument for general random geometric graphs, we consider the case of the 2-dimensional grid which is a sub-class of the class of regular geometric graphs.

Let $M(2,n)$ denote the two dimensional grid of n nodes. Since the graph has a regular geometric structure, the minimum conductance occurs when we consider $min(|S|,|\overline{S}|)$ of maximum capacity, that is when $\pi(S) = \pi(\overline{S}) = \frac{1}{2}$ so that S has half of the nodes of $M(2,n)$. Furthermore, as there are many possible ways of separating the nodes of $M(2,n)$ into two halves S and $\overline{S}$, we need to consider the separation that gives the minimum flow across $Cut(S,\overline{S})$, which occurs when the length of the boundary between S and $\overline{S}$ is minimized (since every edge has the same weight due to regularity). The separation satisfying this is with a separating line l parallel to one of the axis. For details on why such a separation yields the minimum conductance, we refer the reader to Appendix G. Since there are $n^{\frac{1}{2}}$ edges crossing such a cut and each edge has weight $w = \frac{1}{4}$, the conductance of the two dimensional grid of n nodes is [8]

$$\begin{aligned}\Phi(M(2,n)) = 2Q(S,\overline{S}) &\approx 2\sum_{\substack{x\in S\\ y\in\overline{S}}} \frac{1}{n}\frac{1}{4}\\ &= 2n^{\frac{1}{2}}\frac{1}{4n} = (2n^{\frac{1}{2}})^{-1}\end{aligned}$$

This argument easily generalizes to the k dimensional grid $M(k,n)$, and we obtain the following by Theorem 2.2 and Corollary 2.3 above:

[8] We ignore the two nodes on the borders which have only 3 neighbors.

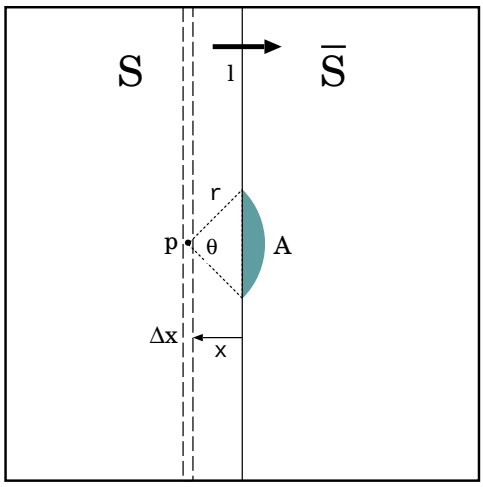

Fig. D.1. Approximating the Conductance in RGG

Lemma C.1 *For the k dimensional grid $M(k,n)$ of n nodes we have the following:*

(1) $\Phi(M(k,n)) \approx (kn^{\frac{1}{k}})^{-1}$

(2) $\frac{1}{2}(kn^{\frac{1}{k}})^{-2} \leq 1 - \lambda_1 \leq 2(kn^{\frac{1}{k}})^{-1}$

(3) $\tau_x(\epsilon) \leq 2k^2 n^{\frac{2}{k}}(\ln n + \ln \epsilon^{-1})$

D Continuous Approximation of Conductance

Following Fig. D.1, let l be the dividing line. A point p in S that is at distance $x < r$ from l neighbors the nodes in the gray area A in the Figure. The size of A is given by $\frac{1}{2}r^2(\theta - \sin\theta)$. (Observe that $\theta = 2\arccos(\frac{x}{r})$ and A is a function of x.) So p has an expected number of nA edges crossing to $\overline{S}$. Taking the integral over all the points in distance $0 \leq x \leq r$ and assuming that there are $n\Delta x$ nodes in the area $1 \cdot \Delta x$ we get that the expected number of edges

crossing from S to $\overline{S}$ is (ignoring the effect of the borders) [9]

$$\begin{aligned}
E[Cut(S,\overline{S})] \leq & \int_0^r nAndx \\
= & \int_0^r \frac{1}{2}r^2 n\Big[2\arccos(\frac{x}{r}) \\
& - \sin(2\arccos(\frac{x}{r})\Big]ndx \\
= & \frac{1}{2}r^2n^2\Bigg[-2r\sqrt{1-\frac{x^2}{r^2}} \\
& + \frac{2}{3}r(1-\frac{x^2}{r^2})^{\frac{3}{2}} + 2x\arccos(\frac{x}{r})\Bigg]_0^r \\
= & \frac{1}{2}r^2n^2(0-(-2r+\frac{2}{3}r) \\
= & \frac{2}{3}r^3n^2
\end{aligned}$$

To approximate the conductance we use the above upper bound on the cut size together with expected degree is $\pi r^2 n$ and by taking out part of the border effect as we take the integral over the area $(1-r)\cdot\Delta x$ (assuming $r << 1$)

$$\begin{aligned}
\Phi(\mathcal{G}(n,r)) = & \min_{S\subset V, 0<\pi(S)\leq 1/2} \frac{Q(S,\overline{S})}{\pi(S)} \\
\approx & \min_{S\subset V, 0<\pi(S)\leq 1/2} 2\sum_{\substack{x\in S \\ y\in\overline{S}}} \pi(x)P(x,y) \\
\approx & \min_{S\subset V, 0<\pi(S)\leq 1/2} 2\sum_{\substack{x\in S \\ y\in\overline{S}}} \frac{1}{n}\frac{1}{\pi r^2 n} \\
\approx & \min_{S\subset V, 0<\pi(S)\leq 1/2} 2E[Cut(S,\overline{S})]\frac{1}{n}\frac{1}{\pi r^2 n} \\
\approx & 2\frac{2}{3}n^2r^3(1-r)\frac{1}{\pi r^2 n^2} \\
= & \frac{4}{3\pi}r(1-r)
\end{aligned} \tag{D.1}$$

[9] Note that as $n\to\infty$ and $r\to 0$ the above bound tightens and approaches equality.

E Resistance Bounds for 1-dimensional RGGs

Let $\mathcal{G}^1(n,r)$ denote a 1-dimensional random geometric graph formed by placing n nodes uniformly at random on the line $[0,1]$. In this section we give the result for $\mathcal{G}^1(n,r)$. Note that for $\mathcal{G}^1(n,r)$ the critical radius for connectivity is $r_{\text{con}} = \frac{\log n}{n}$ [19]. As before we first prove a more general case.

Definition E.1 *Let $G^1(n,r)$ be a 1-dimensional geometric graph*[10]*, we say that such a graph is* ***geo-dense*** *if every 1-dimensional interval of length at least $A = \frac{r}{2}$ has $\Theta(nA) = \Theta(\frac{nr}{2})$ nodes as $n \to \infty$.*

Lemma E.2 *For any geo-dense 1-dimensional geometric graph $G = G^1(n,r)$,*

(1) The resistance of G is $\Theta(\frac{1}{nr} + \frac{1}{n^2r^3})$.

(2) The hitting time between two points u, v in G is $\Theta(n + \frac{d(u,v)}{r^2})$.

(3) The cover time C_G of G is such that $C_G = \Omega(n + \frac{1}{r^2})$ and $C_G = O((n + \frac{1}{r^2}) \log n)$.

PROOF. [of E.2] Consider any two points u, v in $G = G^1(n,r)$. Assume w.l.o.g. that v is to the right of u, namely that it has a higher coordinate than u. Let $d(u,v)$ be the distance between u and v.

For the upper bound, we use Thomson's Minimal Power Principle and bound the power of a unit flow between u and v. The idea is similar to the layering idea used for the 2-dimensional case, except that in 1-dimension there is no expansion of layers, and so each layer is a single bin.

[10] either random or deterministic

Partition the segment between u and v into $h = \frac{2d(u,v)}{r}$ smaller segments, say bins, of length $\frac{r}{2}$ each, thus with $\Theta(\frac{nr}{2})$ nodes in each bin by geo-dense property. And, number the bins as B_i with increasing index $0 \leq i \leq h$ from left to right, letting the corresponding node sets be denoted by V_{B_i}. Note that each V_{B_i} is a clique, and moreover that each pair of adjacent segments $V_{B_i} \cup V_{B_{i+1}}$ is also a clique. Now we construct our flow c as follows: For every edge of the form $e = (u, z), z \in V_{B_0}$, $c(e) = \frac{1}{|V_{B_0}|} = \Theta(\frac{2}{nr})$. For every edge of the form $e = (z, v), z \in V_{B_h}$, $c(e) = \frac{1}{|V_{B_h}|} = \Theta(\frac{2}{nr})$. For every pair of consecutive bins $V_{B_i}, V_{B_{i+1}}$, the total flow between the bins is 1, and the flow between the bins is allocated uniformly along the $|V_{B_i}||V_{B_{i+1}}| = \Theta(\frac{n^2r^2}{4})$ edges between the bins in the direction of increasing bin index. Letting E_l denote the set of edges between bin B_l and bin B_{l+1}, this means that $|E_l| = \Theta(\frac{n^2r^2}{4})$. Moreover, letting $E_0 = \{(u, z)|z \in V_{B_0}\}$ and $E_{h+1} = \{(z, v)|z \in V_{B_h}\}$, we have $|E_0| = |E_{h+1}| = \Theta(\frac{2}{nr})$. So, now we may bound the power and thus the resistance of c in a straightforward manner by utilizing the uniformity of the flow along the edges between consecutive layers, obtaining:

$$\begin{aligned} R_{uv} \leq P(c) = \sum_{e \in E} c^2(e) = \sum_{l=0}^{h+1} \sum_{e \in E_l} \frac{1}{|E_l|^2} = \\ = \sum_{l=0}^{h+1} \frac{1}{|E_l|} = \Theta(2\frac{2}{nr} + h\frac{4}{n^2r^2}) \\ = O(\frac{1}{nr} + \frac{d(u,v)}{n^2r^3}) \end{aligned}$$

For the lower bound, we use Rayleigh's Short/Cut Principle. Partition all nodes to the right of u into r length segments, which have $\Theta(nr)$ nodes each by geo-dense property, numbering the segments as B_i with increasing index i from left to right, and letting the corresponding node sets be denoted by V_{B_i}. Create a new graph G' by shorting nodes of G as follows: Short all nodes

to the left of u with $V_{B_0} \setminus \{u\}$ to create node b_0 in G'. Short all nodes of $\cup_{i \geq j-1} V_{B_i} \setminus \{v\}$ together to create node b_{j-1} in G'. Now for each $0 < i < j$, short all the nodes in V_{B_i} together to create node b_i in G'.

Note that in G, a node in segment B_i can only talk to nodes of segments B_{i-1}, B_i, or B_{i+1}, namely itself or adjacent segments. Note also that every consecutive set of nodes all within distance r of each other is a clique. So, the set of edges of G', not counting multi-edges yet, is

$$E' = \{(b_i, b_{i+1}) | 0 \leq i \leq j-2\} \cup \{(u, b_0), (b_{j-1}, v)\}$$

Moreover, the multiplicity of each edge (b_i, b_{i+1}) is $\Theta(n^2 r^2)$, and the multiplicity of edge (u, b_0) is the same as the multiplicity of edge (b_{j-1}, v) which is the same as $\Theta(nr)$. So summing the series of reciprocals of the edge multiplicities we get:

$$\begin{aligned} R_{uv}(G) \geq R_{uv}(G') &= \Omega(\frac{1}{nr} + (j-1)\frac{1}{n^2 r^2}) \\ &= \Omega(\frac{1}{nr} + \frac{d(u,v)}{r}\frac{1}{n^2 r^2}) \\ &= \Omega(\frac{1}{nr} + \frac{d(u,v)}{n^2 r^3}) \end{aligned}$$

So, combining the upper and lower bounds we get $R_{uv}(G) = \Theta(\frac{1}{nr} + \frac{d(u,v)}{n^2 r^3})$. By considering the maximum distance of 1 then, we get the resistance bound of our lemma. Then, since $m = \Theta(n^2 r)$, from Theorem 2.5 we get the hitting time result (as the commute time and hitting time are on the same order). And, finally by Equation (2), we get the cover time result. $\square$

It is a straightforward result of Balls in Bins lemma to show the following:

Corollary E.3 *If $r \geq c2\frac{\log n}{n}$ for some $c > 1$, then $\mathcal{G}^1(n, r)$ is* geo-dense w.h.p.

Note immediately last corollary that this yields negative results for $\mathcal{G}^1(n,r)$ in that optimality of cover time for one-dimensional geometric graphs requires a radius of order strictly greater than the order for connectivity. In particular:

Corollary E.4 *For any $r \geq c2\frac{\log n}{n}$, we have the following* w.h.p. *for $G = \mathcal{G}^1(n,r)$:*

(1) The resistance of G is $\Theta(\frac{1}{nr} + \frac{1}{n^2r^3})$.

(2) The hitting time between two points u, v in G is $\Theta(n + \frac{d(u,v)}{r^2})$.

(3) The cover time C_G of G is such that $C_G = \Omega(n + \frac{1}{r^2})$ and $C_G = O((n + \frac{1}{r^2})\log n)$.

As can be seen, the maximum hitting time between any two points is only optimal for $r = \Omega(\frac{1}{\sqrt{n}})$, and the cover time can only possibly be optimal for $r_{\text{opt}} = \Omega(\frac{1}{\sqrt{n}})$, so $r_{\text{con}} = o(r_{\text{opt}})$.

Definition E.5 *Let $G_1^1(n)$ denote the 1-dimensional grid of n nodes, and let $G_k^1(n)$ be the k-fuzz of $G_1^1(n)$.*

And, finally, since any 1-dimensional k-fuzz $G_1^1(n)$ is a type of regular and geo-dense geometric graph $G^1(n, \frac{k}{n})$:

Corollary E.6 *For any $1 \leq k \leq n$*

(1) The resistance of $G_k^1(n)$ is $\Theta(\frac{1}{k} + \frac{n}{k^3})$.

(2) The hitting time between two points u, v in $G_k^1(n)$ is $\Theta(n + \frac{n^2 d(u,v)}{k^2})$.

(3) The cover time C of $G_k^1(n)$ is such that $C = \Omega(n + \frac{n^2}{k^2})$ and $C = O((n + \frac{n^2}{k^2})\log n)$.

F Cover Time and Resistance of Deterministic Geometric Graphs

As an example of other applications of our results consider the following: for an integer k, let the k-**fuzz** [13] of a graph G be the graph G_k obtained from G by adding an edge xy if x is at most k hops away from y in G. In particular, let $G_1(n)$ denote the 2-dimensional grid of n nodes, and let $G_k(n)$ be the k-fuzz of $G_1(n)$. It is known that the cover time of $G_1(n)$ is $\Theta(n \log^2 n)$[10]; and so we ask what is the minimum k s.t. G_k has an optimal cover time of $\Theta(n \log n)$?

Definition F.1 *Let $\mathcal{D} = \mathcal{D}(n, r(n))$, where n is s.t. $\sqrt{n} \in \mathbb{Z}$ denote the class of r-**disk** geometric graphs, where the nodes of each instance of $\mathcal{D}(n, r)$ are placed on the unit square exactly as the 2-dimensional grid of n nodes. In other words, there is exactly one node at each position $(\frac{i}{\sqrt{n}}, \frac{j}{\sqrt{n}})$ $0 \leq i, j \leq \sqrt{n}, i, j \in \mathbb{Z}$. Since $\mathcal{D}(n, r(n))$ is a geometric graph there is an edge between two nodes iff their Euclidean distance is at most $r(n)$.*

Note the following:

Corollary F.2 *For $G_k(n)$, a $k-$fuzz of the 2-dimensional grid*

(1) $G_1(n) = \mathcal{D}(n, \frac{1}{\sqrt{n}})$ (i.e the 2-dimensional grid of n nodes).

(2) $G_k(n)$ is a super-graph of $\mathcal{D}(n, \frac{k}{\sqrt{2n}})$

(3) $G_k(n)$ is a sub-graph of $\mathcal{D}(n, \frac{k}{\sqrt{n}})$

Claim F.3 *For a constant k the resistance of $\mathcal{D}(n, \frac{k}{\sqrt{n}})$ is $\Theta(k^{-4} \log n)$.*

PROOF. It is clear that $\mathcal{D}(n, \frac{k}{\sqrt{n}})$ satisfies the **geo-dense** property,[11] so the result follows directly from Theorem 5.3.

[11] for large enough k

Theorem F.4 *For any constant k, the cover time of G_k is $\Theta(k^{-2}n\log^2 n)$.*

PROOF. The upper bound follows directly from Corollary F.2, Claim F.3, and equation (2). To prove the lower bound note that the resistance R_{uv} is $\Theta(k^{-4}\log(\frac{d(u,v)\sqrt{n}}{k}))$. Letting $d'(u,v) = d(u,v)\sqrt{n}$ denote the non-normalized distance (hop distance), we have that $R_{uv} = \Omega(k^{-4}\log(d'(u,v)))$ where $1 \leq d'(u,v) \leq \sqrt{2n}$. Now we can use the method of Zuckerman [40] (specifically in Lemma 2 and Theorem 4 of that paper). And, by noting that it is known that the commute time $C_{uv} = 2mR_{uv}$, we have that hitting time (E_uT_v in [40] notation) is $\Omega(k^{-2}n\log(d'(u,v))$. Then the result follows directly from the proof of Theorem 4 in [40].

Thus, we have the solution to our question:

Corollary F.5 *$G_k(n)$ has Cover Time of $\Theta(n\log n)$ if $k = \gamma_n$ and $\lim_{n\to\infty}\frac{\log n}{\gamma_n^2} \leq c$ for some constant c.*

G Minimum Conductance in the Unit Square

For any subset S of the nodes of Ω we shall also refer to the *conductance corresponding to S* Φ_S defined as follows:

$$\Phi_S = \frac{Q(S,\overline{S})}{\pi(S)} == \frac{1}{\pi(S)}\sum_{\substack{x\in S\\ y\in\overline{S}}}\pi(x)P(x,y) \tag{G.1}$$

In other words,

$$\Phi = \min_{S\subset\Omega, 0<\pi(S)\leq 1/2}\Phi_S \tag{G.2}$$

Lemma G.1 *Let $M(2,n)$ be the 2-dimensional grid of size n located in the unit square. The minimum conductance corresponds to S such that $\pi(S) = \pi(\overline{S}) = \frac{1}{2}$ with $S = \{s \mid s$ is in the left half of the unit square$\}$ up to constant factors.*

PROOF. Divide the unit square into n equal size bins and let each node of the grid sit at the middle of a bin. For a set of nodes X define the perimeter of X, P_X as the perimeter of the area that encloses all and only the bins with nodes in X. Define the boundary between X and $\overline{X}$, B_X, as the part of P_X that is strictly inside the unit square. Note that the nodes that have edges crossing from X to $\overline{X}$ are the nodes located in bins that contribute to B_X. Moreover the length of B_X exactly determines the number of edges that cross from X to $\overline{X}$. In particular, $B_X = \Theta(|Cut(X,\overline{X})|)$. Let $\phi(X) = \frac{B_X}{\pi(X)}$. Since we have a regular graph,[12] $\forall x \quad \pi(x) = \frac{1}{n}$ and $\forall x,y \quad P(x,y) = \frac{1}{4}$ so that $\Phi(X) = \phi(X)\frac{1}{4n}$. Clearly, for two sets X, X' with $\pi(X) = \pi(X')$ we have $\Phi(X) < \Phi(X')$ iff $B_X < B_{X'}$. Let D_α be a quarter of a disk with area α placed at the corner of the unit square. For any α, let X_α be the following unique set: $X_\alpha = \{x \mid x \in D_\alpha\}$.

The proof will be by contradiction. Assume that there is a set S^* which results in a strictly smaller order conductance than the set S in the lemma (recall that $\pi(S^*)$ is at most $\frac{1}{2}$). Note that for any capacity $\pi(X)$, B_X is minimized by convex X lying at a corner of the unit square, so as to only contribute a fraction of its perimeter to the boundary separating X and $\overline{X}$. Consider two

[12] Ignoring the negligible effect of nodes with degree 3 or 2.

cases: I. $\pi(S^*) \geq \frac{1}{\pi}$ [13] or II. $\pi(S^*) \leq \frac{1}{\pi}$. If $\pi(S^*) \geq \frac{1}{\pi}$ then B_{S^*} is at least 1 since a disk minimizes the ratio of perimeter to area. Note that the perimeter of $D_{\frac{1}{\pi}}$ that is strictly inside the unit square is 1 so $X_{\frac{1}{\pi}}$ will have $\pi(X_{\frac{1}{\pi}}) \approx \frac{1}{\pi}$ and $B_{X_{\frac{1}{\pi}}} \geq 1$. For case I then, $\pi(S^*) \leq \pi(S)$ and $B_{S^*} \geq B_S$ so $\phi(S^*) \geq \phi(S)$ and $\Phi(S^*) \geq \Phi(S)$. Contradiction.

If $\pi(S^*) \leq \frac{1}{\pi}$ then, again, the set $X_{\pi(S^*)}$ has $\pi(X_{\pi(S^*)}) = \pi(S^*)$ and $\phi(S^*) \geq \phi(X_{\pi(S^*)})$. Now, for X_α we have $\phi(X_\alpha) = \frac{B_{X_\alpha}}{\pi(X_\alpha)} \approx \frac{1}{\sqrt{\alpha}}$. So $\beta > \alpha \Longrightarrow \phi(X_\beta) < \phi(X_\alpha)$. Putting it together we get $\phi(S^*) \geq \phi(X_{\pi(S^*)}) \geq \phi(X_{\frac{1}{\pi}}) \geq \phi(S)$ and $\Phi(S^*) \geq \Phi(S)$. Contradiction. □

Note that the above proof depends only on the geo-denseness (i.e. regularity with respect to geometric area) of the grid and that we may substitute the bins defined in the above proof with bins defined in the proof of Theorem 4.1. Then we obtain the following corollary:

Corollary G.2 *The minimum conductance of $\mathcal{G}(n, r)$ with $r \geq c2\frac{\log n}{n}$ corresponds to S such that $\pi(S) = \pi(\overline{S}) = \frac{1}{2}$ with $S = \{s \mid s$ is in the left half of the unit square$\}$ w.h.p. up to constant factors.*

[13] Note that π refers to the area of the unit ball while $\pi(X)$ refers to the capacity of X.